51^{st} Austrian Mathematical Olympiad

Junior Regional Competition—Solutions

6th June 2020

Problem 1. *Determine all pairs (a, b) of real numbers satisfying the inequality*

$$\frac{(1+a)^2}{1+b} \leq 1 + \frac{a^2}{b}$$

where $b \neq -1$ and $b \neq 0$. For which pairs (a, b) does equality hold?

(Walther Janous)

Answer. The inequality holds

(A) for all pairs (a, b) with $a \in \mathbb{R}$ and $b \in \mathbb{R}$ with $b < -1$ or $b > 0$, as well as

(B) for all pairs (a, a) with $a \in \mathbb{R} \setminus \{-1, 0\}$.

Equality holds exactly for the pairs in (B).

Solution. Equivalent transformations of the inequality yield:

$$\begin{aligned}
& 1 + \frac{a^2}{b} - \frac{(1+a)^2}{1+b} \geq 0 \\
\iff \quad & \frac{b^2 + b + a^2 + a^2 b - b - 2ab - a^2 b}{b(1+b)} \geq 0 \\
\iff \quad & \frac{a^2 - 2ab + b^2}{b(1+b)} \geq 0 \\
\iff \quad & \frac{(a-b)^2}{b(1+b)} \geq 0.
\end{aligned}$$

Therefore, equality holds if and only if $a = b$, thus for all pairs (a, a) with $a \in \mathbb{R} \setminus \{-1, 0\}$.

Let $a \neq b$. The inequality holds if and only if $b(1+b) > 0$, i.e. if and only if $b < -1$ or $b > 0$. Hence the strict inequality holds for all pairs (a, b) with $a \in \mathbb{R}$, $b \in \mathbb{R}$ with $b < -1$ or $b > 0$, and $a \neq b$.

(Walther Janous) □

Problem 2. *How many five-digit numbers exist with the property that the product of the digits of each number equals 900?*

(Karl Czakler)

Answer. 210

Solution. Note that $900 = 30^2 = 5^2 \cdot 3^2 \cdot 2^2$. Therefore, each number must contain the digit 5 exactly twice. The product of the remaining three digits is 36.

This yields the following five possibilities for the five digits:

(a) 5, 5, 9, 4, 1

(b) 5, 5, 9, 2, 2

(c) 5, 5, 6, 6, 1

(d) 5, 5, 6, 3, 2

(e) 5, 5, 4, 3, 3

There are 10 possible ways of allocating the two digits 5 within a five-digit number.

In the case that the remaining digits are pairwise distinct, we can arrange them in six different ways. In the case that exactly two of the remaining three digits coincide, we can arrange them in three different ways. Every arrangement gives a unique five-digit number. We conclude there exist

$$10 \cdot 6 + 10 \cdot 3 + 10 \cdot 3 + 10 \cdot 6 + 10 \cdot 3 = 210$$

five-digit numbers fulfilling the required property.

(Karl Czakler) □

Problem 3. *Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$ and $AB > CD$. Let E be the foot of the perpendicular from D onto the line AB and let M be the mid-point of the diagonal BD.*

Prove that the lines EM and AC are parallel.

(Karl Czakler)

Solution. The triangle BED is right angled and therefore E lies on the circle with the diameter BD and the center M, see Figure 1.

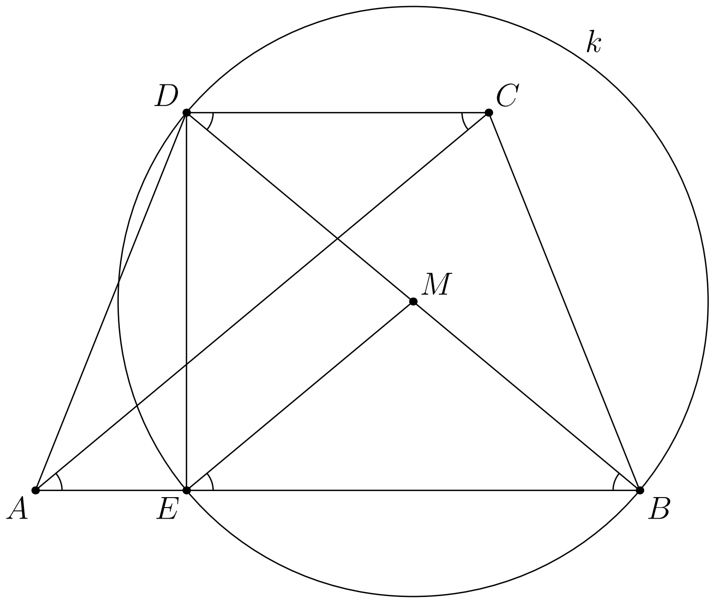

Figure 1: Problem 3

This yields

$$MB = ME = MD.$$

Thus, the triangle EMB is isosceles and therefore

$$\angle BEM = \angle MBE = \angle DBA.$$

As the trapezoid $ABCD$ is isosceles the triangles CAB and DBA are congruent and hence

$$\angle DBA = \angle BAC.$$

The equality $\angle BEM = \angle BAC$ yields $EM \parallel AC$.

(Karl Czakler) □

Problem 4. *Determine all positive integers a for which the equation*

$$7an - 3n! = 2020$$

has a solution n in positive integers.

(Note: For every positive integer n: $n! = 1 \cdot 2 \cdot \dots \cdot n$.)

(Richard Henner)

Answer. For $a = 68$ and $a = 289$ the equation has the solutions $n = 5$ and $n = 1$, respectively.

Solution. The left-hand side of the equation is divisible by n, and hence n has to be a divisor of 2020. Furthermore, n is smaller than 7, as for $n \geq 7$ the left-hand side of the equation is divisible by 7, whereas 2020 is not. Therefore, we have to check the cases that n equals $1, 2, 4$ or 5.

If $n = 1$, the given equation is equivalent to $7a = 2023 \iff a = 289$.

If $n = 2$ or $n = 4$, the equation is equivalent to $a = \frac{2026}{14}$ or $a = \frac{2092}{28}$, respectively. In both cases, the parameter a is not an integer.

Finally, if $n = 5$, the equation is equivalent to $a = \frac{2380}{35} = 68$.

(Richard Henner) □