50^{th} Austrian Mathematical Olympiad

Junior Regional Competition—Solutions

18th June 2019

Problem 1. *Determine all pairs (x, y) of integers with $x + y \neq 0$ which are satisfying*

$$\frac{x^2 + y^2}{x + y} = 10.$$

(Walther Janous)

Answer. $(x, y) \in \{(-2, 4),\ (-2, 6),\ (0, 10),\ (4, -2),\ (4, 12),\ (6, -2),\ (6, 12),\ (10, 0),\ (10, 10),\ (12, 4),\ (12, 6)\}$

Solution. An equivalent form of the given equation is

$$\begin{aligned} & x^2 + y^2 = 10x + 10y \\ \iff \quad & x^2 - 10x + y^2 - 10y = 0 \\ \iff \quad & (x - 5)^2 + (y - 5)^2 = 50 \end{aligned}$$

with $x + y \neq 0$. We therefore have to solve the equation $a^2 + b^2 = 50$ for integers a and b.

As $\max(a^2, b^2) \leq 50$ we get $\max(|a|, |b|) \leq 7$. Furthermore we obtain $\min(a^2, b^2) \leq 25$ which yields $\min(|a|, |b|) \leq 5$. Analyzing the individual cases, we obtain $(a, b) \in \{(\pm1, \pm7), (\pm7, \pm1), (\pm5, \pm5)\}$ as the only possible solutions. If $(a, b) \in \{(\pm1, \pm7), (\pm7, \pm1)\}$, we get that $x - 5 = \pm1$ and $y - 5 = \pm7$ (or x and y swapped), which yields $x \in \{4, 6\}$ and $y \in \{-2, 12\}$ (or $y \in \{4, 6\}$ and $x \in \{-2, 12\}$). The case $(a, b) = (\pm5, \pm5)$ gives $x - 5 = \pm5$ and $y - 5 = \pm5$ which is equivalent to $x \in \{0, 10\}$ and $y \in \{0, 10\}$. The pair $(x, y) = (0, 0)$ is the only one violating $x + y \neq 0$. Therefore, we get the (eleven) different pairs listed in the answer.

(Walther Janous) □

Problem 2. *Let $ABCD$ be a square. The equilateral triangle BCS is constructed on the exterior of the side BC. Let N denote the midpoint of the line segment AS and let H be the midpoint of the side CD.*

Prove: $\angle NHC = 60°$.

(Karl Czakler)

Solution. Let P be the midpoint of BS, see Figure 1. Since triangles $\triangle SNP$ and $\triangle SAB$ are similar with factor 2, the segment NP is parallel to AB and half the length of the segment AB. Therefore $NPCH$ is a parallelogram. As NP and BC are orthogonal and PC and BS are orthogonal, we have $\angle NPC = \angle CBP = 60°$. Thus we obtain $\angle NHC = \angle NPC = 60°$.

(Karl Czakler) □

Problem 3. *Alice and Bob play a game that allows the playing numbers 19 and 20 and the two possible starting numbers 9 and 10. Alice chooses her playing number and assigns the remaining playing number to Bob while Bob independently chooses the starting number.*

Alice adds her playing number to the starting number, Bob adds his playing number to the sum, then Alice again adds her playing number to this new sum and so on. The game lasts till the number 2019 is reached or exceeded.

A player who obtains exactly 2019 wins. If 2019 is exceeded, the game ends in a draw.

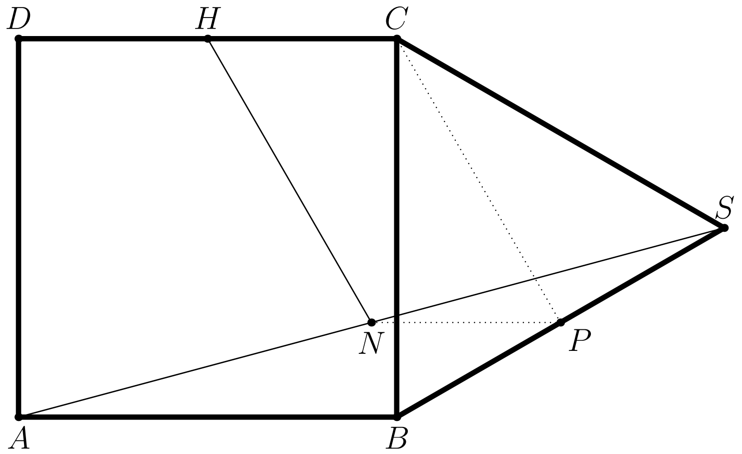

Figure 1: Problem 2

- *Show that Bob cannot win.*
- *Which starting number does Bob have to choose in order to prevent Alice from winning?*

(Richard Henner)

Solution. Let s be the starting number and a Alice's playing number and b Bob's playing number. Furthermore, let n be the number of rounds of the game (i.e., the number of times that Bob adds his number).

In order for Bob to win, the equation $s+39n = 2019$ must have an integer solution for n. But neither $s = 9$ nor $s = 10$ satisfy this condition, as neither 2010 nor 2009 is divisible by 39. Hence Bob cannot win. In fact, $51 \cdot 39 = 1989$ and $52 \cdot 39 = 2028$.

In order for Alice to win, the equation $s + 39n + a = 2019$ has to be satisfied for some n. As $28 \leq s + a \leq 30$ by definition, this can only work for $n = 51$, which implies $s + a = 30$. Therefore, we have $s = 10$ and $a = 20$.

We conclude that Bob has to choose 9 as starting number in order not to lose the game.

(Richard Henner) □

Problem 4. *Let p, q, r and s be prime numbers satisfying*

$$5 < p < q < r < s < p + 10.$$

Prove that the sum of these four prime numbers is divisible by 60.

(Walther Janous)

Solution. The four prime numbers have to fulfill $p > 5$ and $s < p + 10$ and hence they must be among the five consecutive odd numbers p, $p + 2$, $p + 4$, $p + 6$ and $p + 8$.

As we have to choose 4 out of the five numbers p, $p+2$, $p+4$, $p+6$, $p+8$, we have to omit exactly one of these numbers. If we omit one of the numbers p, $p + 2$, $p + 6$ or $p + 8$, three subsequent odd numbers remain, one of which has to be divisible by 3, which is excluded. Therefore, we have to omit $p + 4$.

Hence the four prime numbers have to be p, $q = p + 2$, $r = p + 6$ and $s = p + 8$.

Exactly one of the five consecutive integers p, $p + 2$, $p + 4$, $p + 6$ and $p + 8$ is divisible by 5. By construction, none of the chosen number p, q, r, s can be divisible by 5. This implies that $p + 4$ is divisible by 5. So $p + 4$ is divisible by 15.

The fact that

$$p + q + r + s = p + (p + 2) + (p + 6) + (p + 8) = 4p + 16 = 4(p + 4)$$

yields that the sum is divisible by 60.

Remark. The quadruple $(11, 13, 17, 19)$ indeed shows that the number 60 cannot be replaced by a greater one.

(Walther Janous) □