55^{th} Austrian Mathematical Olympiad

National Competition—Preliminary Round

27th April 2024

1. Let α and β be real numbers with $\beta \neq 0$. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(\alpha f(x) + f(y)) = \beta x + f(y)$$

holds for all real x and y.

(Walther Janous)

2. Let h be a semicircle with diameter AB. The two circles k_1 and k_2, $k_1 \neq k_2$, touch the segment AB at the points C and D, respectively, and the semicircle h from the inside at the points E and F, respectively. Prove that the four points C, D, E and F lie on a circle.

(Walther Janous)

3. Let $n \geq 3$ be an integer. A *circle dance* is a dance that is performed according to the following rule: On the floor, n points are marked at equal distances along a large circle. At each of these points is a sheet of paper with an arrow pointing either clockwise or counterclockwise. One of the points is labeled „Start“. The dancer starts at this point. In each step, he first changes the direction of the arrow at his current position and then moves to the next point in the new direction of the arrow.

   a) Show: Each circle dance visits each point infinitely often.

   b) How many different circle dances are there? Two circle dances are considered to be the same if they differ only by a finite number of steps at the beginning and then always visit the same points in the same order. (The common sequence of steps may begin at different times in the two dances.)

(Birgit Vera Schmidt)

4. A positive integer is called *powerful* if all exponents in its prime factorization are ≥ 2.

   Prove that there are infinitely many pairs of powerful consecutive positive integers.

(Walther Janous)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.