46^{th} Austrian Mathematical Olympiad

Beginners' Competition

June 9, 2015

1. Let a, b, c be integers with $a^3 + b^3 + c^3$ divisible by 18. Prove that abc is divisible by 6.

   (Karl Czakler)

2. Let x, y be positive real numbers with $xy = 4$.

   Prove that
   $$\frac{1}{x+3} + \frac{1}{y+3} \leq \frac{2}{5}.$$
   For which x and y does equality hold?

   (Walther Janous)

3. Anton chooses as starting number an integer $n \geq 0$ which is not a square. Berta adds to this number its successor $n+1$. If this sum is a perfect square, she has won. Otherwise, Anton adds to this sum, the subsequent number $n+2$. If this sum is a perfect square, he has won. Otherwise, it is again Berta's turn and she adds the subsequent number $n+3$, and so on.

   Prove that there are infinitely many starting numbers, leading to Anton's win.

   (Richard Henner)

4. Let k_1 and k_2 be internally tangent circles with common point X. Let P be a point lying neither on one of the two circles nor on the line through the two centers. Let N_1 be the point on k_1 closest to P and F_1 the point on k_1 that is farthest from P. Analogously, let N_2 be the point on k_2 closest to P and F_2 the point on k_2 that is farthest from P.

   Prove that $\angle N_1XN_2 = \angle F_1XF_2$.

   (Robert Geretschläger)

Working time: 4 hours.
Each problem is worth 8 points.