46^{th} Austrian Mathematical Olympiad

National Competition (Final Round, part 1)

May 1, 2015

1. Let a, b, c, d be positive numbers. Prove that
$$(a^2+b^2+c^2+d^2)^2 \geq (a+b)(b+c)(c+d)(d+a).$$
When does equality hold?

(Georg Anegg)

2. Let ABC be an acute-angled triangle with $AC < AB$ and circumradius R. Furthermore, let D be the foot of the altitude from A on BC and let T denote the point on the line AD such that $AT = 2R$ holds with D lying between A and T. Finally, let S denote the mid-point of the arc BC on the circumcircle that does not include A.

   Prove: $\angle AST = 90\,^\circ$.

(Karl Czakler)

3. Alice and Bob play a game with a string of 2015 pearls.

   In each move, one player cuts the string between two pearls and the other player chooses one of the resulting parts of the string while the other part is discarded.

   In the first move, Alice cuts the string, thereafter, the players take turns.

   A player loses if he or she obtains a string with a single pearl such that no more cut is possible.

   Who of the two players does have a winning strategy?

(Theresia Eisenkölbl)

4. A *police emergency number* is a positive integer that ends with the digits 133 in decimal representation. Prove that every police emergency number has a prime factor larger than 7.

   (In Austria, 133 is the emergency number of the police.)

(Robert Geretschläger)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.