CAPS Match 2023: Solutions and Marking schemes

ISTA, Austria
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Problem 1. Given an integer $n \geq 3$, determine the smallest positive number k such that any two points in any n-gon (or at its boundary) in the plane can be connected by a polygonal path consisting of k line segments contained in the n-gon (including its boundary). (David Hruška)

Solution. The following example shows that at least m segments are needed for any $2m$-gon, $m \geq 2$:

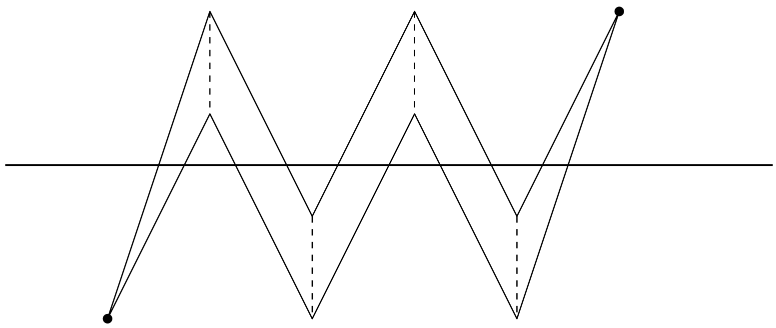


FIGURE 1. Example for $m = 5$

Indeed, a polygonal path connecting the marked vertices must intersect all the $m-1$ dashed line segments and hence it must at least m times cross the vertical line. Since no two of these intersections can belong to the same line segment (WLOG no segment lies at the vertical line), we conclude that the path must contain at least m line segments. It is clear that there is a $2m+1$-gon which needs at least m line segments as well. Hence, $k \geq \lfloor \frac{n}{2} \rfloor$ (obviously also for $n = 3$).

Now we prove that this number is always sufficient. Denote the given n-gon by P and the given points A and B. Fix a triangulation of P and a consider a triangle T containing A. Cutting P along the sides of T produces the triangle and at at most three disjoint polygons. Let us take the union of one containing B (if $B \in T$ we are done) with T and replace P by this new polygon. Doing the same for the point B we can assume that A and B lie in triangles with two sides belonging to the boundary of the r-gon for some $r \leq n$. Let us call the pairs of sides A-sides and B-sides, respectively. Then let us connect A by a line segment with the common vertex of the A-sides, analogously for B and B-sides. These two vertices can be connected by a polygonal path p starting at A and ending at B which has at most $\lfloor \frac{r}{2} \rfloor$ line segments – sides of the r-gon. Now we connect A with the second vertex of p and the penultimate vertex of p with B. That gives a polygonal path contained in P with the same number of vertices and hence also the desired upper bound.

Sketch of an alternative proof of the upper bound. Consider again a triangulation of P. The triangulation has $n-2$ triangles. We observe that if points A and B belong to different triangles, which is the only interesting case, they can be connected by a polygonal path with vertices in different triangles and with every segment except for the first one and the last one crossing a triangle (without having an endpoint in it) which is crossed by no other of the segments. It follows that the number k of segments used satisfies the inequality $(k+1)+(k-2) \leq n-2$ and hence $k \leq \lfloor \frac{n}{2} \rfloor$.

Problem 2. Let a_1, a_2, …, a_n be real numbers so that for every $k = 1, 2, \dots, n$ the following inequality holds:

$$n \cdot a_k \geq \sum_{i=1}^{k} a_i^2.$$

Prove that there exist at least $\frac{n}{10}$ indices k so that $a_k \leq 1000$.

(Sándor Kisfaludi-Bak & Karol Węgrzycki)

Solution.

Step 1. Sort the sequence.

We may assume that the sequence $a_1, a_2, \dots, a_n$ is non-decreasing. Indeed, if $a_k > a_{k+1}$ then

$$n \cdot a_{k+1} \geq \sum_{i=1}^{k+1} a_i^2 \geq \sum_{i=1}^{k-1} a_i^2 + a_{k+1}^2$$

and

$$n \cdot a_k > n \cdot a_{k+1} \geq \sum_{i=1}^{k+1} a_i^2$$

which shows that swapping a_k with a_{k+1} produces a sequence which still satisfies the problem conditions.

Let M be the largest index such that $a_M \leq 1000$. We have to show that $M \geq \frac{1}{10}n$.

Step 2. Optimize.

We can replace a_k by 0 for $k < M$ and the given inequalities still hold. Next, replace a_{M+1} by 1000, and recursively, for $k = M+2, M+3, \dots, n$ replace a_k by the smallest number such that

$$n \cdot a_k \geq \sum_{i=1}^{k} a_i^2.$$

Clearly, the smallest such number exists and is equal to the smaller root of the quadratic equation

$$n \cdot x = \sum_{i=1}^{k-1} a_i^2 + x^2 = n \cdot a_{k-1} + x^2,$$

which happens to be equal to $\frac{n-\sqrt{n^2-4na_{k-1}}}{2}$. (Note in particular that $a_{k-1} < \frac{n}{4}$.) Hence, replacing a_k by the number $\frac{n-\sqrt{n^2-4na_{k-1}}}{2}$ we enforce equality $n \cdot a_k = \sum_{i=1}^{k} a_i^2$ and preserve inequalities $n \cdot a_{k'} \geq \sum_{i=1}^{k'} a_i^2$ for $k' > k$.

Let $f(x) = \frac{n-\sqrt{n^2-4nx}}{2}$. Then $a_{M+1+k} = f^k(1000)$ for $k = 1, 2, \dots, n$. (Here, f^k denotes the k-th iteration of f.)

Step 3. Finish.

Note that if $\frac{n}{4} > x$ then

$$f(x) = \frac{n - \sqrt{n^2 - 4nx}}{2} = \frac{n^2 - (n^2 - 4nx)}{2(n + \sqrt{n^2 - 4nx})} = \frac{2nx}{n + \sqrt{n(n-4x)}} > \frac{2nx}{n + \frac{n+(n-4x)}{2}} = \frac{nx}{n-x}.$$

Let $g(x) = \frac{nx}{n-x}$. Note that g is increasing on $(0, \frac{n}{4})$. Easy induction yields $g^k(x) = \frac{nx}{n-kx}$. Easy induction gives $f^k(1000) > g^k(1000) > 0$ for $k = 1, 2, \dots, n - M - 1$. In particular

$$0 < g^{n-M-1}(1000) = \frac{1000n}{n - 1000(n-M-1)} \implies n - M - 1 < \frac{n}{1000} \implies M > \frac{999}{1000}n - 1,$$

which is much stronger bound than we were asked to prove.

A variant of step 3. Once monotonicity of (a_k) and equalities are forced, we have for every $k = 1, 2, \ldots, n-1$:

$$na_{k+1} = na_k + a_{k+1}^2 \geq na_k + a_k a_{k+1}.$$

For each $k = M+1, \ldots, n-1$ we then have (as $a_k > 0$):

$$\frac{1}{a_k} - \frac{1}{a_{k+1}} \geq \frac{1}{n}.$$

Adding these inequalities up, we get (after telescoping)

$$\frac{1}{a_{M+1}} - \frac{1}{a_n} \geq \frac{n-M-1}{n}, \quad \text{so} \quad 1000 = a_{M+1} \leq \frac{n}{n-M-1}.$$

This means that $M \geq \frac{999}{1000}n - 1$.

Problem 3. Given is a convex quadrilateral $ABCD$ with $\angle BAD = \angle BCD$ and $\angle ABC < \angle ADC$. Point M is the midpoint of segment AC. Prove that there exist points X and Y on the segments AB and BC, respectively, such that $XY \perp BD$, $MX = MY$ and $\angle XMY = \angle ADC - \angle ABC$. (Mykhailo Shtandenko)

Solution. Let P and Q be projections of point D onto the lines BA and BC respectively. We claim that $PM = MQ$ and $\angle PMQ = 2\angle PAD$. To simplify angle chasing, let's consider only the case when angles BAD and BCD are obtuse and therefore points P and Q will lie on the extensions of the segments BA and BC beyond A and C respectively (another case will be analogous with slightly different angle chasing). Let N, K be the midpoints of the segments AD and CD respectively. Then $PN = AN = ND = MK$ and $KQ = CK = KD = MN$. Moreover,

$$\begin{aligned}\angle MNP = \angle MNA + \angle ANP = \angle CDA + 2\angle ADP \\ = \angle CDA + 2\angle CDQ = \angle MKC + \angle CKQ = \angle MKQ,\end{aligned}$$

so triangles MNP and MKQ are congruent, thus $PM = MQ$. Now,

$$\begin{aligned}\angle PMQ &= \angle PMN + \angle NMK + \angle KMQ \\ &= \angle PMN + \angle ANM + \angle MPN = 180^\circ - \angle ANP = 2\angle PAD,\end{aligned}$$

so the claim is proved.

Now let the line which passes through P and is parallel to CD intersect segment BC at point Y (this line intersects segment BC because $\angle ABC < \angle CDA$), and analogously, let the line which passes through Q and is parallel to AD intersect segment AB at point X. We will prove that points X and Y are our desired points.

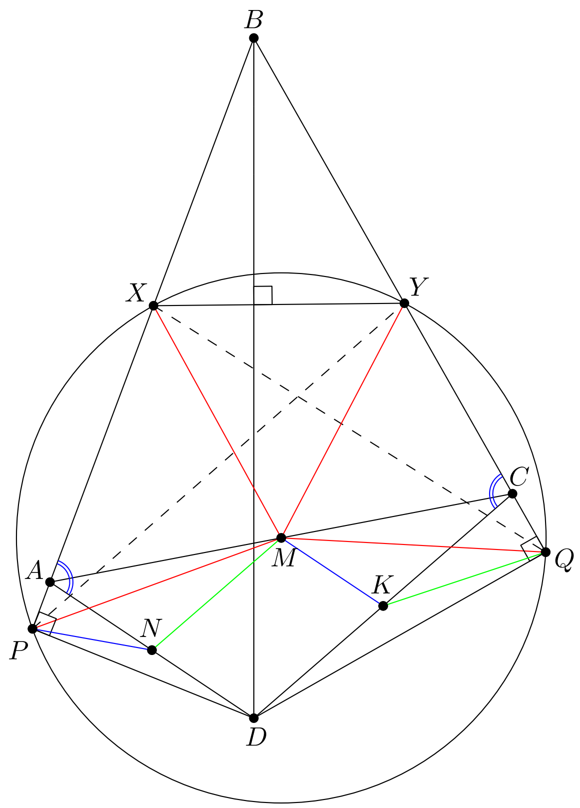


FIGURE 2.

Since $\angle PMQ = 2\angle PAD = 2\angle PXQ = 2\angle DCQ = 2\angle PYQ$, we deduce that points X and Y lie on the circle centered at M with radius $MP = MQ$. So $MX = MY$ and

$PXYQ$ is cyclic, also $PBQD$ is cyclic, therefore

$$\angle BXY = \angle BQP = 90^\circ - \angle PQD = 90^\circ - \angle PBD,$$

which means that $XY \perp BD$. Now,

$$\angle XMY = 2\angle XPY = 2(\angle PYQ - \angle ABC) = 2(180^\circ - \angle BCD) - 2\angle ABC$$
$$= 360^\circ - \angle BAD - \angle BCD = \angle ABC - \angle ABC = \angle ADC - \angle ABC,$$

as desired.

Second solution. Let E and F be projections of points A and C respectively onto the line BD. Let X be the point of intersection of MF and AB, and Y be the point of intersection of ME and BC. We will prove that these points X and Y are the desired points.

Using Pappus' Theorem for triples of points A, M, C and F, B, E, we have: $X = AB \cap MF, Y = ME \cap CB$, therefore point of intersection of the lines AE and CF lies on the line XY. But since $AE \parallel CF$, we have that $XY \perp BD$. Clearly, projection of M onto the line EF will coincide with the midpoint of the segment EF, so $ME = MF$. However, then $\angle XYM = 90^\circ - \angle YEF = 90^\circ - \angle EFM = \angle YXM$, so $XM = YM$. Now we are left with computing the angle between lines ME and MF.

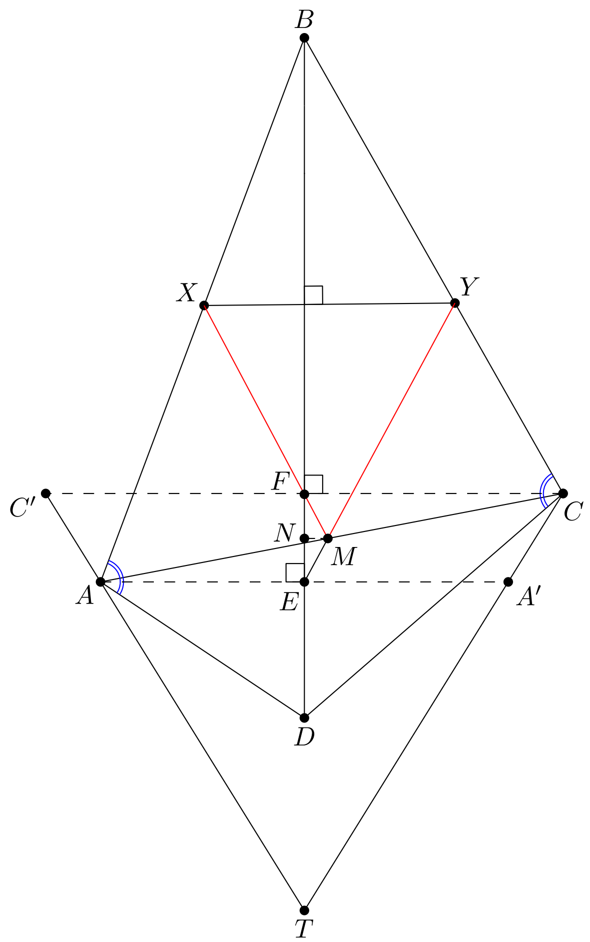


Figure 3.

To this end, consider point A' which is symmetric to A with respect to line BD, and analogously, let point C' be symmetric to C with respect to line BD. Then MF, ME are midlines of the triangles ACC' and ACA', respectively, so the angle between these two lines equals to the angle between lines $A'C$ and $C'A$. Let T be the point of intersection of these two lines. Then, due to symmetry, T lies on BD. Now, since $\angle BA'D = \angle BAD = \angle BCD$, quadrilateral $BCA'D$ is cyclic, so $\angle CTB = \angle BDA' - \angle DA'T =$

$\angle BDA - \angle DBC$. Analogously, $\angle ATB = \angle BDC - \angle ABD$, so summing last two equalities gives us $\angle CTA = \angle CDA - \angle ABC$, exactly what we wanted to prove.

Problem 4. Let p, q and r be positive real numbers such that the equation

$$\lfloor pn \rfloor + \lfloor qn \rfloor + \lfloor rn \rfloor = n$$

is satisfied for infinitely many positive integers n.

(a) Prove that p, q and r are rational.

(b) Determine the number of positive integers c such that there exist positive integers a and b, for which the equation

$$\left\lfloor \frac{n}{a} \right\rfloor + \left\lfloor \frac{n}{b} \right\rfloor + \left\lfloor \frac{cn}{202} \right\rfloor = n$$

is satisfied for infinitely many positive integers n.

(Walther Janous)

Solution. We will first prove that $(p + q + r = 1 \wedge p, ,q, ,r \in \mathbb{Q})$ is an equivalent statement for the above.

(a) From

$$n = \lfloor pn \rfloor + \lfloor qn \rfloor + \lfloor rn \rfloor \leq pn + qn + rn$$

for some positive integer n we infer $p + q + r \geq 1$.

Let us write $p+q+r = 1+t$ for some $t \geq 0$. Then $pn+qn+rn = n+tn$, hence

$$(pn - \lfloor pn \rfloor) + (qn - \lfloor qn \rfloor) + (rn - \lfloor rn \rfloor) = tn \tag{1}$$

holds for infinitely many positive integers n. If $t \neq 0$, then the right hand side of (1) would achieve arbitrarily large values whereas the left hand side is bounded above by 3 — contradiction. Thus $t = 0$ and $p + q + r = 1$. Now equation (1) assures that $pn = \lfloor pn \rfloor$, $qn = \lfloor qn \rfloor$, $rn - \lfloor rn \rfloor$ holds for infinitely many positive integers n. In particular, $p = \lfloor pn \rfloor / n$, $q = \lfloor qn \rfloor / n$, $r = \lfloor rn \rfloor / n$ (for these n) are all rational numbers.

(b) Next, we observe that if p, q, r are rational numbers with common denominator N then equation (1) is fulfilled for all integer multiples of N, and hence $(p+q+r = 1$ and p, q, $r \in \mathbb{Q})$ is also a sufficient condition for the given statement.

Hence, for the second part we need to determine the number of positive integers c such that

$$\frac{1}{a} + \frac{1}{b} + \frac{c}{202} = 1 \quad \Longleftrightarrow \quad \frac{1}{a} + \frac{1}{b} = \frac{202 - c}{202}$$

can be solved with a, $b \in \mathbb{Z}_{>0}$. Wlog. we may assume $a \leq b$. Furthermore, we see that $1 \leq c \leq 201$ and write $k := 202 - c$, $d := \gcd(a, b)$, $A := a/d$, $B := b/d$ to arrive at the equivalent equation

$$\frac{1}{dA} + \frac{1}{dB} = \frac{k}{202} \quad \Longleftrightarrow \quad 202(A + B) = kdAB \quad \Longleftrightarrow \quad \frac{202(A + B)}{AB} = kd.$$

Considering the second equation, we observe that $A \mid 202 \cdot B$ and $B \mid 202 \cdot A$. Since A and B are coprime, both A and B need to be divisors of 202. The product AB of two coprime divisors of 202 is again a divisor of 202, so that the left hand side of the last equation is an integer. It follows that k has to be a divisor of $202(A + B)/(AB)$. Conversely, if $A \mid 202$, $B \mid 202$, $\gcd(A, B) = 1$, $A \leq B$ and k is a divisor of $202(A + B)/(AB)$, then $a = dA$ and $b = dB$ with $d := 202(A + B)/(kAB)$ fulfill the desired equation.

Checking all possible values for A, B and k gives[1] $k \in D(404) \cup D(303) \cup D(204) \cup D(203) \cup D(103)$ (corresponding to $(A, B) = (1, 1)$, $(1, 2)$, $(1, 101)$,

[1]Here, $D(n)$ denotes the set of all positive divisors of a positive integer n.

$(1,202)$, $(2,101)$), which together with the condition $c > 0 \iff k < 202$ yields the 15 possibilities

$$
\begin{aligned}
& k \in \{1,2,3,4,6,7,12,17,29,34,51,68,101,102,103\} \\
\iff \quad & c \in \{201,200,199,198,196,195,190,185,173,168,151,134,101,100,99\}.
\end{aligned}
$$

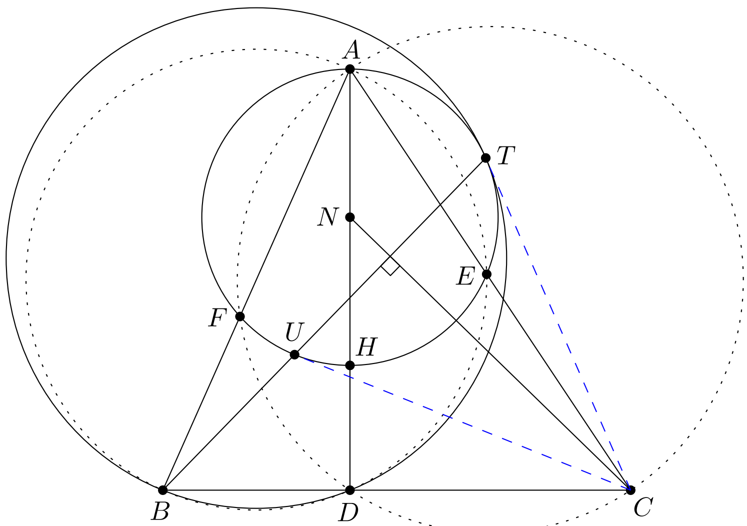
Problem 5. Let ABC be an acute-angled triangle with orthocenter H. Let D be the foot of the altitude from A to the line BC. Let T be a point on the circle with diameter AH such that this circle is internally tangent to the circumcircle of triangle BDT. Let N be the midpoint of segment AH. Prove that $BT \perp CN$.

(Michal Pecho)

Solution. Let ω be the circle with diameter AH and E the foot of the altitude from B to AC. Notice that E lies on ω. The power of the point C with respect to circle $(BDEA)$ is $CB \cdot CD = CE \cdot CA$, which is the power of the point C with respect to circles (BDT), so C lies on their radical axis which is their common tangent at T. Therefore $\angle CTN = 90°$.

Let U be the second intersection of circles ω and $(TNDC)$. As $\angle CUN = 90°$, CU is tangent to circle ω and we know that CT is also tangent to ω, hence $TU \perp CN$.

Let F be the foot of the altitude from C to AB. Notice that AF is the radical axis of circles ω and $(AFDC)$, CD is the radical axis of circle $(TUDC)$ and $(AFDC)$, hence B is the radical center of circles ω, $(AFDC)$ and $(TUDC)$, therefore B also lies on the radical axis TU of circles ω and $(TUDC)$, which is perpendicular to CN. Hence the problem is finished.

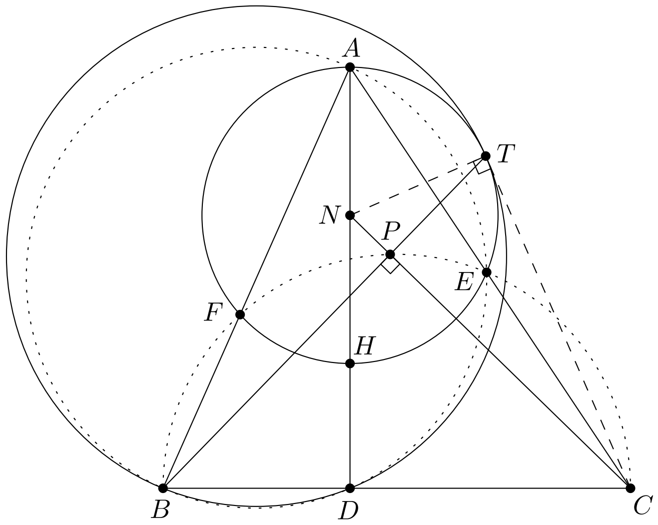


Another solution. Let P be the intersection of CN with the circle with diameter BC and let T' be the second intersection of ray $BP^{\rightarrow}$ with the circle with diameter AH. We will prove that circles (AHT') and (BDT') are tangent at point T' by showing that CT' is their common tangent.

Let E and F be the feet of the altitudes from B and C in triangle ABC, respectively. Points E, F are also intersection points of circles with diameters AH and BC. By angle-chasing, we get $\angle FBE = \angle NEF = 90° - \angle BAC$, so NE is tangent to the circle with diameter BC.

We have $NP \cdot NC = NE^2 = NT^2$ and $PT' \perp NC$, so $\angle NT'C = 90°$, which implies that CT' is tangent to the circle with diameter AH.

Points B, D, E, A lie on a circle, so $CT'^2 = CE \cdot CA = CD \cdot CB$, i.e. CT' is tangent to the circle (BDT'), hence $T' = T$ and $BT \perp CN$, as desired.



Another solution (via trig). Note that (for the standard angle naming) $AD = BD\tan\beta$, $HD = BD\cot\gamma$, $AN = HN = TN = BD\frac{\tan\beta-\cot\gamma}{2}$, $DN = BD\frac{\tan\beta+\cot\gamma}{2}$, $CD = AD\cot\gamma = BD\tan\beta\cot\gamma$, $BC = BD(1+\tan\beta\cot\gamma)$. The power of C wrt (AHT) is

$$CN^2 - HN^2 = BD^2\left((\tan\beta\cot\gamma)^2 + \left(\frac{\tan\beta+\cot\gamma}{2}\right)^2 - \left(\frac{\tan\beta-\cot\gamma}{2}\right)^2\right) = \cdots = CD\cdot CB,$$

which is same as power of C wrt (BDT). It means that CT is tangent to (AHT) and (BDT), so

$$CT^2 = BD^2\left(\tan^2\beta\cot^2\gamma + \tan\beta\cot\gamma\right).$$

To prove $BT \perp CN$ it is enough to verify that $BN^2 - BC^2 = TN^2 - TC^2$, and both of these expressions are readily seen to be equal to

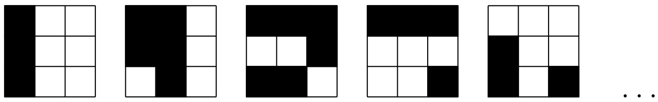
$$BD^2\left(\frac{1}{4}\tan^2\beta + \frac{1}{4}\cot^2\gamma - \tan^2\beta\cot^2\gamma - \frac{3}{2}\tan\beta\cot\gamma\right).$$

Problem 6. Given is an integer $n \geq 1$ and an $n \times n$ board, whose all cells are initially white. Peter the painter walks around the board and recolors the visited cells according to the following rules. Each *walk* of Peter starts at the bottom-left corner of the board and continues as follows:

- if he is standing on a white cell, he paints it black and moves one cell up (or walks off the board if he is in the top row);
- if he is standing on a black cell, he paints it white and moves one cell to the right (or walks off the board if he is in the rightmost column).

Peter's walk ends once he walks off the board. Determine the minimum positive integer s with the following property: after *exactly* s walks all the cells of the board will become white again.

E.g. for $n = 3$ the states of the board after each of the initial five walks will be:



(Łukasz Bożyk)

Solution. *Answer:* the highest power of 2 which is a divisor of $2(2n-2)!$, i.e. $2^{1+\nu_2((2n-2)!)}$, or equivalently: two to the power $n + \sum_{i=1}^{\infty} \left\lfloor \frac{n-1}{2^i} \right\rfloor$, or equivalently: the smallest s such that $\frac{s}{2^{2n-1}}\binom{2n-2}{n-1}$ is an integer.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Extend the board to $\mathbb{N} \times \mathbb{N}$ (the full quadrant of the checkered plane), with the bottom-left corner $(0, 0)$. The coloring after a fixed number of walks extends naturally to the infinite board as well (each walk is considered to have infinitely many steps $\uparrow/\rightarrow$).

For a fixed $s \geq 0$ define the function $f_s \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ recursively as follows:

- $f_s(x, y) = 0$ if $x < 0$ or $y < 0$;
- $f_s(0, 0) = s$;
- for $x \geq 0$ and $y \geq 0$: $f_s(x, y) = \left\lceil \frac{f_s(x, y-1)}{2} \right\rceil + \left\lfloor \frac{f_s(x-1, y)}{2} \right\rfloor$. $\qquad (*)$

We will prove that the color of the cell (x, y) after precisely s walks is precisely $f_s(x, y)$ mod 2, where 1 is black and 0 is white.

We proceed by induction on $x + y$ (for a fixed s). If $x = y = 0$, then the cell (x, y) has been repainted exactly s times, which agrees with the definition of f_s.

Suppose that $x + y \geq 1$ and that the statement holds for all cells whose sum of coordinates is $< x + y$. Consider the cell (x, y). It has changed its color precisely the number of times the painter visited it, and this number (by inductive assumption) is precisely equal to $\lceil \frac{1}{2} f_s(x, y-1) \rceil$ from the cell $(x, y-1)$ (that is the amount of moves $(x, y-1) \to (x, y)$) plus $\lfloor \frac{1}{2} f_s(x-1, y) \rfloor$ from the cell $(x-1, y)$ (that is the amount of moves $(x-1, y) \to (x, y)$). This finishes the proof by induction.

Suppose that after exactly s walks the entire $n \times n$ board is white, i.e. each of the numbers $f_s(x, y)$ for $x < n$ and $y < n$ is even. It means that the floors and the ceilings in the recursive definition can be omitted (for the considered range of x's and y's) and we obtain the standard recurrence relation for binomial coefficients (the Pascal's triangle) with an extra division by 2 with each increment of the sum $x + y$. It is easily proved by induction that

$$f_s(x, y) = \frac{s}{2^{x+y}} \binom{x+y}{x} \qquad (\circ)$$

for $x < n$ and $y < n$. Conversely: if numbers $\frac{s}{2^{x+y}}\binom{x+y}{x}$ are even integers for all $0 \le x, y < n$, then they are values of f_s (which is again proved by induction on $x + y$ restricted to the $n \times n$ square). It follows that the entire board is white after $s > 0$ walks if and only if for each pair $0 \le x, y < n$ the following inequality holds:

$$\nu_2(s) + \nu_2\left(\binom{x+y}{x}\right) \ge x + y + 1.$$

Let $s = 2^{2n-1-\nu_2(\binom{2n-2}{n-1})}$; it is the smallest positive integer such that $f_s(n-1, n-1)$ is even. We will prove that for this choice of s the remaining values of f_s in the $n \times n$ square are even as well, and the proof will be concluded. To this end it is enough to show that all the values $f_s(x, n-1)$ (in the row $y = n - 1$) are even (all the remaining terms can be uniquely restored from them by the given recurrence formula, and will be some linear combinations of even numbers with integer coefficients).

Suppose that $\nu_2(\binom{2n-2}{n-1}) = k$. The proof will be finished if we show that for each $i = 0, 1, \ldots, k$ we have

$$\nu_2\left(\binom{2n-2-i}{n-1}\right) \ge k - i.$$

But this is clearly true since

$$\binom{2n-2-i}{n-1} = \binom{2n-2}{n-1} \cdot \frac{(n-1)(n-2)\ldots(n-i)}{(2n-2)(2n-3)(2n-4)\ldots(2n-1-i)}$$

and each even term of the product in the denominator of the fraction has its corresponding half in the numerator, so it could eat at most one 2 from the prime factorization, hence

$$\nu_2\left(\binom{2n-2-i}{n-1}\right) \ge \nu_2\left(\binom{2n-2}{n-1}\right) - \left\lceil \frac{i}{2} \right\rceil = k - \left\lceil \frac{i}{2} \right\rceil \ge k - i.$$