CAPS Match 2023

ISTA, Austria

(First day – June 19, 2023)

1. Given an integer $n \geq 3$, determine the smallest positive number k such that any two points in any n-gon (or at its boundary) in the plane can be connected by a polygonal path consisting of k line segments contained in the n-gon (including its boundary).

2. Let a_1, a_2, …, a_n be real numbers so that for every $k = 1, 2, \ldots, n$ the following inequality holds:

$$n \cdot a_k \geq \sum_{i=1}^{k} a_i^2.$$

Prove that there exist at least $\frac{n}{10}$ indices k so that $a_k \leq 1000$.

3. Given is a convex quadrilateral $ABCD$ with $\angle BAD = \angle BCD$ and $\angle ABC < \angle ADC$. Point M is the midpoint of segment AC. Prove that there exist points X and Y on the segments AB and BC, respectively, such that $XY \perp BD$, $MX = MY$ and $\angle XMY = \angle ADC - \angle ABC$.

Time: 4 hours and 30 minutes.
Each problem is worth 7 points.

Language: English

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(Second day – June 20, 2023)

4. Let p, q and r be positive real numbers such that the equation

$$\lfloor pn \rfloor + \lfloor qn \rfloor + \lfloor rn \rfloor = n$$

is satisfied for infinitely many positive integers n.

(a) Prove that p, q and r are rational.

(b) Determine the number of positive integers c such that there exist positive integers a and b, for which the equation

$$\left\lfloor \frac{n}{a} \right\rfloor + \left\lfloor \frac{n}{b} \right\rfloor + \left\lfloor \frac{cn}{202} \right\rfloor = n$$

is satisfied for infinitely many positive integers n.

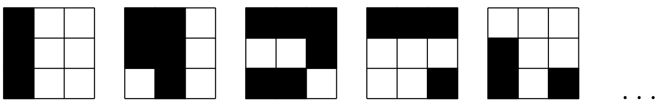
5. Let ABC be an acute-angled triangle with orthocenter H. Let D be the foot of the altitude from A to the line BC. Let T be a point on the circle with diameter AH such that this circle is internally tangent to the circumcircle of triangle BDT. Let N be the midpoint of segment AH. Prove that $BT \perp CN$.

6. Given is an integer $n \geq 1$ and an $n \times n$ board, whose all cells are initially white. Peter the painter walks around the board and recolors the visited cells according to the following rules. Each *walk* of Peter starts at the bottom-left corner of the board and continues as follows:

- if he is standing on a white cell, he paints it black and moves one cell up (or walks off the board if he is in the top row);
- if he is standing on a black cell, he paints it white and moves one cell to the right (or walks off the board if he is in the rightmost column).

Peter's walk ends once he walks off the board. Determine the minimum positive integer s with the following property: after *exactly* s walks all the cells of the board will become white again.

E.g. for $n = 3$ the states of the board after each of the initial five walks will be:



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