Tree walks and the spectrum of random graphs

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Motivations

Applications of the spectrum of random matrices / adjacency matrices of graphs

- wireless communication (MIMO)
- dimensionality reduction (Principal Component Analysis)
- clustering

Our main result. Information on the spectrum of a $G(n, p = c/n)$ graph as $n \to +\infty$ and c is large.

Tools

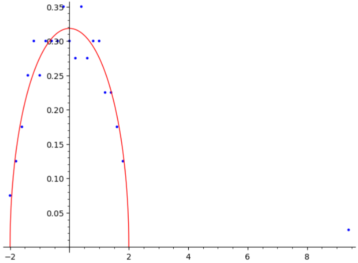
- exact enumeration of exhaustive walks on trees
- analytic combinatorics
- moment method

Wigner semicircle law

Random $G(n,p)$ graph. n vertices, each pair of distinct vertices is linked by an edge with probability p.

[Wigner 1958] Convergence of the empirical spectral distribution of $G(n,p)$

$$\mu_{n,p} := \frac{1}{n} \sum_{\lambda \in \mathrm{Sp}\left(\frac{\mathrm{Adj}(G(n,p))}{\sqrt{p(1-p)n}}\right)} \delta_\lambda \underset{n\to+\infty}{\longrightarrow} \frac{\sqrt{4-x^2}}{2\pi} \mathbb{1}_{x\in[-2,2]} dx$$



$\mu_{200,0.3}$

Sparse graphs

[Khorunzhy Shcherbina Vengerovsky 2004] Convergence for $G(n, p = c/n)$

$$\mu_{n,c/n} := \frac{1}{n} \sum_{\lambda \in \mathrm{Sp}\left(\frac{\mathrm{Adj}(G(n,c/n))}{\sqrt{c}}\right)} \delta_\lambda \underset{n\to+\infty}{\longrightarrow} \mu_c$$

Strange properties
Atomic part dense in $\mathbb{R}$
Continuous part if and only if $c > 1$ [Bordenave Sen Virág 2017]
Rank $(\mu_c(\{0\}))$ is known [Bordenace Lelarge 2010, Costello Tao Vu 2006]
μ_c known for regular graphs [McKay 1981] and trees [Bhamidi Evans Sen 2009]

We study μ_c as $c \to +\infty$ following [Bauer Golinelli 2001, Enriquez Ménard 2016]

Moment method

[Wigner 1958] $$m_{2\ell}(\mu_{n,p}) = \frac{1}{n} \sum_{\lambda \in \mathrm{Sp}\left(\frac{\mathrm{Adj}(G(n,p))}{\sqrt{p(1-p)n}}\right)} \lambda^{2\ell} \underset{n\to+\infty}{=} \mathrm{Cat}_\ell + O(n^{-1})$$

[Bauer Golinelli 2001, Enriquez Ménard 2016]

$$m_{2\ell}(\mu_{n,c/n}) = \frac{1}{n} \sum_{\lambda \in \mathrm{Sp}\left(\frac{\mathrm{Adj}(G(n,c/n))}{\sqrt{c}}\right)} \lambda^{2\ell} \underset{n\to+\infty}{=} \mathrm{Cat}_\ell + O(c^{-1})$$

Same main asymptotics, but different error terms!

From moments to tree walks

$$m_\ell(\mu_{n,c/n}) = \mathbb{E}\left(\frac{1}{n}\sum_{\lambda\in\mathrm{Sp}\left(\frac{A}{\sqrt{c}}\right)}\lambda^\ell\right) = \frac{c^{-\ell/2}}{n}\mathbb{E}(\mathrm{Tr}(A^\ell)) = \frac{c^{-\ell/2}}{n}\sum_{\substack{(v_1,\ldots,v_\ell)\in[n]^\ell \\ e \text{ distinct edges}}}(c/n)^e$$

Contribution of closed walks with m vertices, e edges, length ℓ bounded by

$$\frac{c^{-\ell/2}}{n}n^m\ell^\ell(c/n)^e = c^{e-\ell/2}\ell^\ell n^{m-e-1}$$

tends to 0 with n unless $e = m-1$, *i.e.* unless the walk spans a tree.
Thus, odd moments tend to 0.

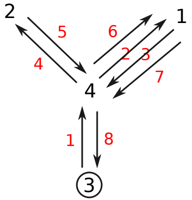
$w_{m,2\ell}$ = number of closed walks of length 2ℓ spanning a tree with m vertices

$$m_{2\ell}(\mu_c) = \sum_{m=1}^{\ell+1} c^{m-\ell-1}\frac{w_{m,2\ell}}{m!}$$

Tree walks

Tree walk of size m

- walk on the complete graph K_m
- starting and ending at the same vertex
- visiting each vertex
- and spanning a tree.



Excess of an edge visited $2k$ times $= k - 1$.

Simple edge: excess 0.

Excess of a tree walk $\xi(W) = \frac{\text{length}}{2} - \text{edges}$

Tree walks of excess 0 $\overset{\text{bij}}{\longleftrightarrow}$ labeled Catalan trees

Generating functions

Generating function of tree walks $$W(v,z)=\sum_{\ell,m\geq 0} w_{m,2\ell}\frac{v^m}{m!}z^\ell$$

Generating function by excess $$W_\xi(z)=\sum_{\ell\geq 0}\frac{w_{\ell-\xi+1,2\ell}}{(\ell-\xi+1)!}z^\ell$$

Ordinary moment generating function (linked to the Stieltjes transform)

$$M_{\mu_c}(z)=\sum_{\ell\geq 0} m_{2\ell}(\mu_c)z^{2\ell}=\frac{1}{c}W\left(c,\frac{z^2}{c}\right)=\sum_{\xi\geq 0}c^{-\xi}W_\xi(z^2)$$

Theorem 1. (conjectured by [Bauer Golinelli 2001]) Set $C(z)=\frac{1-\sqrt{1-4z}}{2z}$. There exist polynomials $K_{\xi,s}(x)$ such that

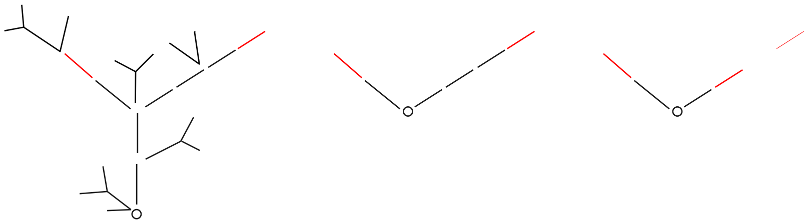
$$W_\xi(z)=C(z)\sum_{s=0}^{2\xi-2}\frac{K_{\xi,s}(zC(z)^2)}{(1-zC(z)^2)^{s+1}}.$$

Proof of Theorem 1

Inspired by the enumeration of connected graphs with fixed excess (# edges - # vertices) by [Wright 1977].

Kernel walks. Obtained by

- iteratively removing simple edges at the leaves (including the root)
- merging consecutive simple edges sharing a degree 2 vertex



Excess is conserved.
Finite number of kernel walks of a given excess.

Proof of Theorem 1

Polynomial generating function

$$K_\xi(u, v, z)$$

simple edges (u), vertices (v), half-length (z)

The number of vertices is fixed by the excess and the length

$$K_\xi(u, 1, z)$$

Replace each simple edge with a sequence and add a sequence of simple edges before the root

$$\frac{1}{1-z} K_\xi\left(\frac{1}{1-z}, 1, z\right)$$

Add a $W_0(z) = C(z)$ tree walk at the start of the walk and after each step

$$W_\xi(z) = \frac{C(z)}{1 - zC(z)^2} K_\xi\left(\frac{1}{1-zC(z)^2}, 1, zC(z)^2\right)$$

Expressing $K_\xi(u,v,z)$

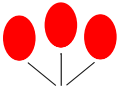
Superreduced walks. Kernel walks with no simple edge. GF $S(v,z)$

$$K(u,v,z) = \frac{1}{1-uz(K(u,v,z)-v)} S\left(v, \frac{z}{(1-uz(K(u,v,z)-v))^2}\right) - uvz(K(u,v,z)-v)$$

$K_\xi(u,v,z) = [y^{\xi-1}]K(u,y^{-1}v,yz)$ is expressed from $S(v,z)$ by Lagrange inversion.

Catalytic variable x: number of times the walk leaves the root
(idea already found by [Bauer Golinelli 2001] on a different tree walk family)

$$S(x,v,z) = v \exp\left(\mathcal{L}_{t=1}\left(D(t,xz)S(t,v,z)\right)\right)$$



where $D(t,x) = \sum_{k\geq 1} \frac{x^{k+1}}{(k+1)!} \frac{t^k}{k!}$.

Computations

We compute the first terms of the GF $S(v, z)$ of superreduced walks,
then the GF $K_\xi(u, v, z)$ of kernel walks of small excess ξ,
then the GF $W_\xi(z)$ of tree walks of excess ξ.

$$
\begin{aligned}
W_1(z) &= \frac{z^2 C(z)^5}{1 - zC(z)^2} \\
W_2(z) &= C(z)\left(\frac{z^3 C(z)^6 + 4z^4 C(z)^8 - 6z^5 C(z)^{10} + 2z^6 C(z)^{12}}{\left(1 - zC(z)^2\right)^3}\right) \\
W_3(z) &= z^4 C(z)^9 \left(\frac{1 + 16zC(z)^2 + 11z^6 C(z)^{12} + 95z^4 C(z)^8}{\left(1 - zC(z)^2\right)^5}\right) \\
&\quad - z^4 C(z)^9 \left(\frac{54z^5 C(z)^{10} + 62z^3 C(z)^6 + 5z^2 C(z)^4}{\left(1 - zC(z)^2\right)^5}\right), \\
&\vdots
\end{aligned}
$$

Extending (and correcting) results from [Enriquez Ménard 2016].

Surprising combinatorial identity

Recall $$M_{\mu_c}(z) = \sum_{\xi \geq 0} c^{-\xi} W_\xi(z^2)$$
and $[c^{-i}]M_{\mu_c}(\sqrt{z})$ is a rational function in $zC(z)^2$.

Extending [Enriquez Ménard 2016], we compute $p(x)$ polynomial of degree 5 such that
$$[c^{-i}]M_{\mu_c}\left(\sqrt{\frac{z}{p(1/c)}}\right)$$
is a polynomial in $zC(z)^2$ for all $0 \leq i \leq 5$.

Does not work if we change a coefficient of $W_\xi(z)$!

Conjecture. Degree of $p(x)$ could be extended. Divergent series.

A better rescalling

Set $$\tilde{\mu}_c := \lim_{n\to+\infty} \frac{1}{n} \sum_{\lambda\in \mathrm{Sp}\left(\frac{\mathrm{Adj}(G(n,c/n))}{\sqrt{cp(1/c)}}\right)} \delta_\lambda,$$ then for all ℓ

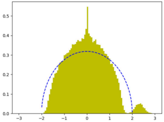
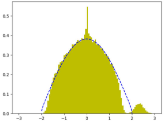
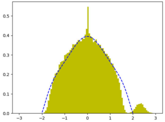
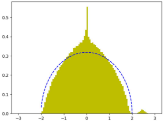
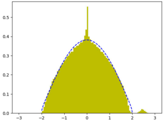
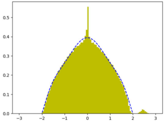
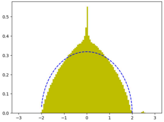
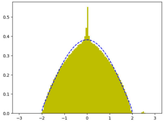
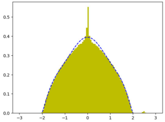
$$m_\ell(\tilde{\mu}_c) = m_\ell(\sigma + c^{-1}\sigma_1 + \cdots + c^{-5}\sigma_5) + O(c^{-6})$$

where σ is the semicircle law and the (σ_j) are a signed measures of mass 0 with explicit densities.

Maybe for every bounded continuous function φ

$$\int \varphi \, d\tilde{\mu}_c = \int \varphi \, d(\sigma + c^{-1}\sigma_1 + \cdots + c^{-5}\sigma_5) + O(c^{-6})$$

$$
\begin{aligned}
f_1(z) &= \frac{1}{2\pi}\big(1 - z^2\big)\sqrt{4 - z^2}\ \mathbb{1}_{(-2,2)}(z),\\
f_2(z) &= \frac{1}{2\pi}\big(1 - 6z^2 + 5z^4 - z^6\big)\sqrt{4 - z^2}\ \mathbb{1}_{(-2,2)}(z),\\
f_3(z) &= \frac{1}{2\pi}\ \big(9 - 140z^2 + 358z^4 - 299z^6 + 98z^8 - 11z^{10}\big)\ \sqrt{4 - z^2}\ \mathbb{1}_{(-2,2)}(z),\\
f_4(z) &= \frac{1}{2\pi}\big(56 + 1602z^2 - 8625z^4 + 16004z^6\\
&\qquad - 13447z^8 + 5624z^{10} - 1143z^{12} + 90z^{14}\big)\sqrt{4 - z^2}\ \mathbb{1}_{(-2,2)}(z),\\
f_5(z) &= \frac{1}{2\pi}\big(442 - 17946z^2 + 171911z^4 - 574676z^6 + 904447z^8\\
&\qquad - 768354z^{10} + 373181z^{12} - 103622z^{14} + 15298z^{16} - 931z^{18}\big)\sqrt{4 - z^2}\ \mathbb{1}
\end{aligned}
$$

sample size	$f_0(z)$	$f_0(z) + \frac{1}{5} f_1(z)$	$f_0(z) + \frac{1}{5} f_1(z) + \frac{1}{25} f_2(z)$
n=40 N=2500			
n=200 N=500			
n=1000 N=100			

Thank you!