Distinct fringe subtrees in binary search trees

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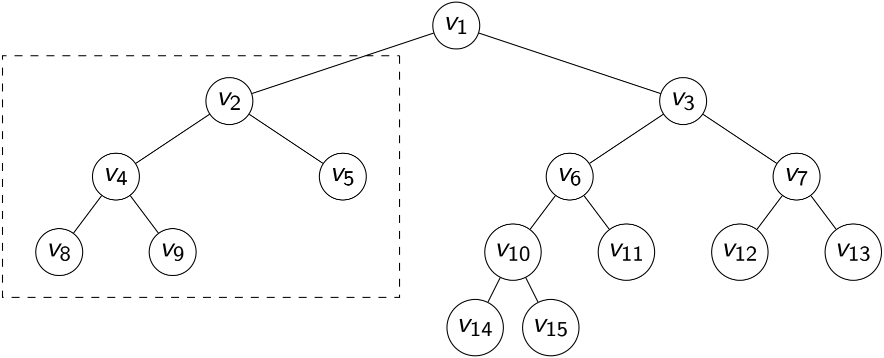
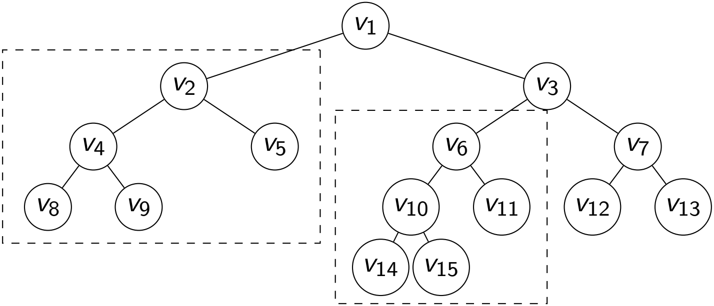
Fringe subtrees

A *fringe subtree* of a rooted tree is a subtree that consists of a vertex and all its descendants.



Identical and distinct fringe subtrees



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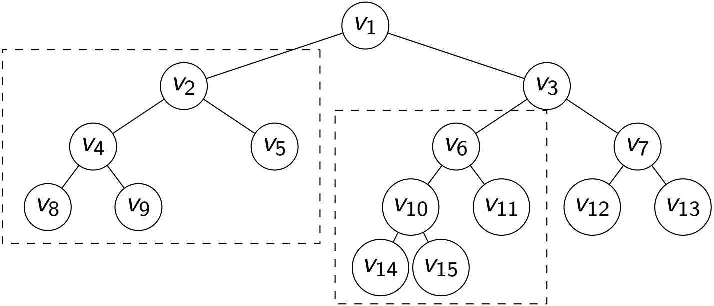

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There are five distinct equivalence classes of fringe subtrees:

v_1 $\quad$ v_3 $\quad$ v_2, v_6 $\quad$ v_4, v_7, v_{10} $\quad$ $v_5, v_8, v_9, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}$

Tree compression

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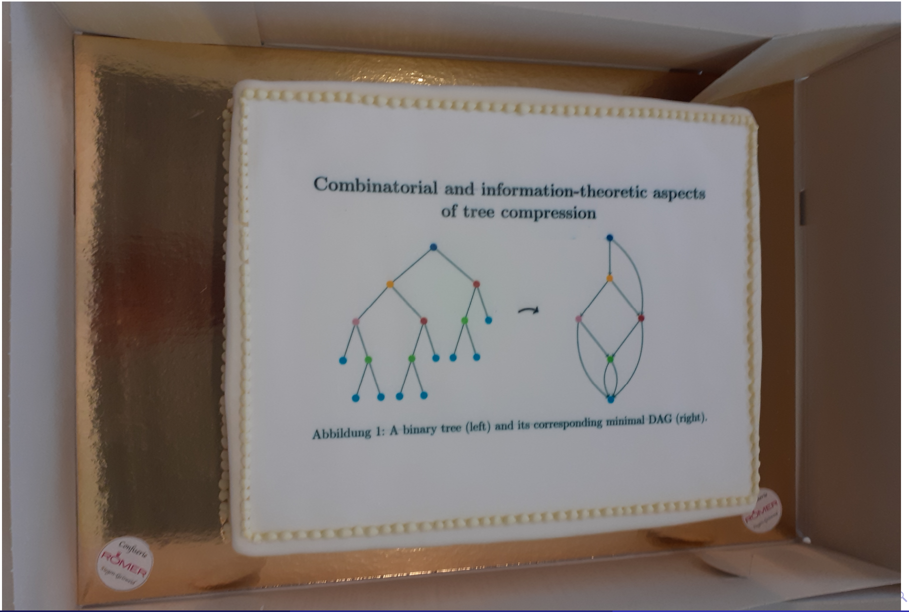

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Tree compression plays a role in (among other things)

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The number of distinct fringe subtrees is a measure for how much a tree is compressed by constructing the minimal DAG.

Prior results: simply generated trees

Theorem (Flajolet/Sipala/Steyaert 1990; Seelbach Benkner/W 2022)

Let X_n be the number of distinct fringe subtrees in a random tree with n vertices from a simply generated family (with some technical conditions). Then we have

$$\mathbb{E}(X_n) \sim \frac{Cn}{\sqrt{\log n}}$$

for some constant C. Moreover,

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For example, in the special case of uniformly random binary trees with n leaves ($n-1$ internal vertices), we have $\mathbb{E}(X_n) \sim \frac{2n}{\sqrt{\pi \log_4 n}}$.

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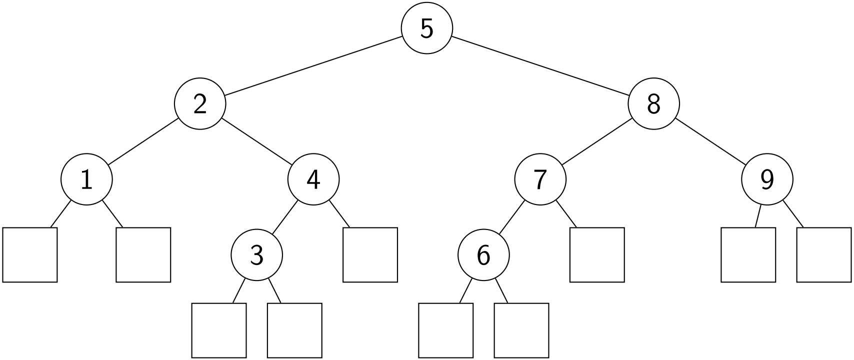
It is well known that this random tree model is also essentially equivalent to that of *binary increasing trees*, where vertex labels are increasing from the root to the leaves.

Binary search trees



Binary search tree resulting from the permutation $(5, 2, 8, 4, 1, 7, 9, 3, 6)$.

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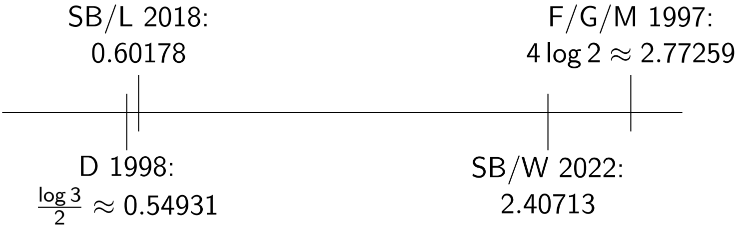

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Main theorem

Theorem

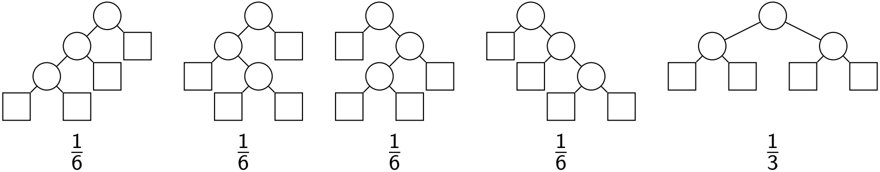
Let F_n be the number of distinct fringe subtrees in a random binary search tree with n internal vertices, and let c_1 be the constant $4\sum_{k\geq 1}\frac{\log k}{(k+1)(k+2)} \approx 2.40713$. We have

$$\mathbb{E}(F_n) \sim \frac{c_1 n}{\log n}$$

as $n \to \infty$. Moreover, we also have convergence in probability:

$$\frac{F_n}{n/\log n} \xrightarrow{p} c_1.$$

Binary search tree distribution



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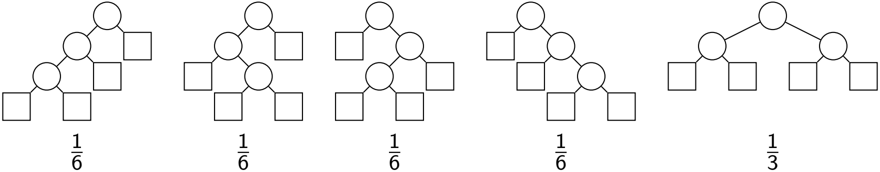

The probability that a random binary search tree has a specific shape T can be expressed as

$$p(T) = \prod_{v} \frac{1}{N_v},$$

where the product is over all internal vertices and N_v is the number of *internal vertices* in the fringe subtree rooted at v.

Shape functional

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Theorem (Fill 1996)

Let the random variable L_n be defined by $L_n = -\log p(\mathcal{T}_n)$, where $\mathcal{T}_n$ is a random binary search tree of size n (n external vertices). We have

$$\mathbb{E}(L_n) = \mu n + O(\log n),$$

where $\mu = \sum_{k=1}^{\infty} \frac{2\log k}{(k+1)(k+2)}$. Moreover, $\mathbb{V}(L_n) = \sigma^2 n + O(1)$ for a constant $\sigma^2 > 0$, and the centred and normalised random variable $\frac{L_n - \mu n}{\sigma\sqrt{n}}$ converges in distribution to a standard normal distribution.

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- So if $p_k = \sum_{B \in \mathfrak{S}_k} p(B)$ is the probability that a random binary search tree has a shape that belongs to some subset $\mathfrak{S}_k$ of the set $\mathfrak{B}_k$ of all binary trees of size k, then the expected number of fringe subtrees whose shape belongs to $\mathfrak{S}_k$ is $\frac{2p_k n}{k(k+1)}$.

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- Moreover, the number in the previous statement is concentrated around its mean.

Proof sketch: lower bound

- Focus on "large" trees whose size at least $k_0 := \frac{1}{\mu}(\log n + (\log n)^{3/4})$, where $\mu = \sum_{k=1}^{\infty} \frac{2 \log k}{(k+1)(k+2)}$.

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- For $k \geq k_0$, choose $\mathfrak{S}_k$ to be the subset of $\mathfrak{B}_k$ consisting of those trees B for which $p(B) \leq \exp(-\mu k + k^{2/3})$, or equivalently $-\log p(B) \geq \mu k - k^{2/3}$.

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- Show that most trees belonging to $\mathfrak{S}_k$ for some $k \geq k_0$ only occur at most once as a fringe subtree with high probability.
- So the number of fringe subtrees whose size is at least k_0 provides an asymptotic lower bound:

$$F_n \gtrsim \sum_{k \geq k_0} \frac{2n}{k(k+1)} \sim \frac{2n}{k_0} \sim \frac{2\mu n}{\log n}.$$

Proof sketch: upper bound

- Split into "small", "medium" and "large" fringe subtrees:
  - Small: $k \leq k_1 := \frac{1}{2}\log_4 n$;
  - Medium: $k_1 < k \leq k_2 := \frac{1}{\mu}(\log n - (\log n)^{3/4})$, with μ as before.
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  - a majority of trees with "large" shape functional—their probability to occur is too low for them to contribute asymptotically;
  - and a minority of trees with "small" shape functional—there are not enough of those to contribute asymptotically.
- Bound the contribution of large fringe subtrees by their total number (ignoring whether they are distinct or not).

Entropy

The expected value of the shape functional can also be thought of as the entropy of the shape of a random binary search tree $\mathcal{T}_n$:

$$\mathbb{E}(L_n) = \mathbb{E}(-\log p(\mathcal{T}_n)) = -\sum_{B \in \mathfrak{B}_n} p(B) \log p(B).$$

So the growth constant for the number of distinct fringe subtrees is directly connected to the growth constant for this entropy.

Generalisations

The method is fairly general and also works for other types of random trees and notions of distinctness, provided that we have two ingredients available:

- information on the distribution of the number of fringe subtrees of a given size,
- information on the distribution of a suitable analogue of the shape functional.

Thank you!