Limit Laws for Critical Dispersion on Complete Graphs

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Joint work with U. De Ambroggio, T. Makai and K. Panagiotou

AofA 2024

Dispersion Process - The Model

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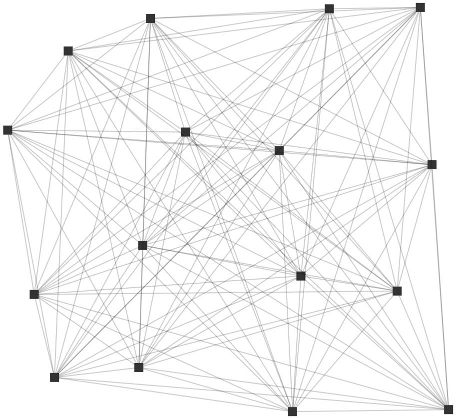

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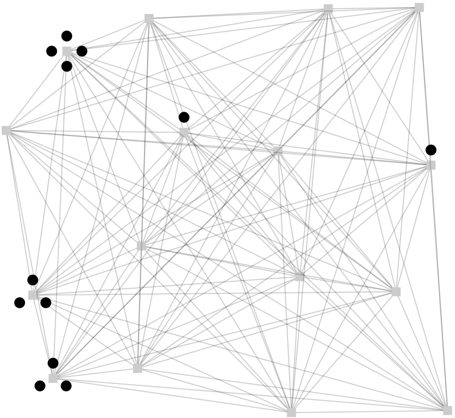

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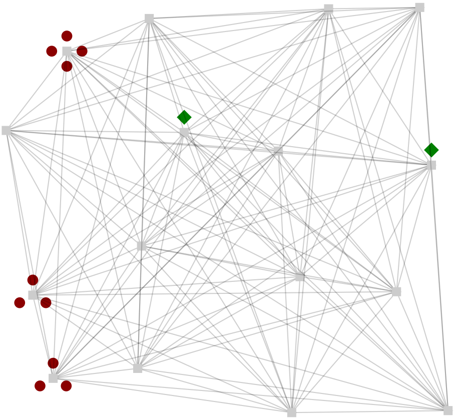
- particles moving on the vertices of a connected graph G.
- a particle is

  ◆ happy if there are no other particles occupying the same vertex,

  ● unhappy otherwise.



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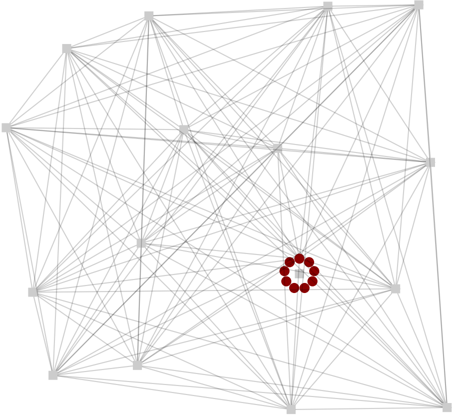

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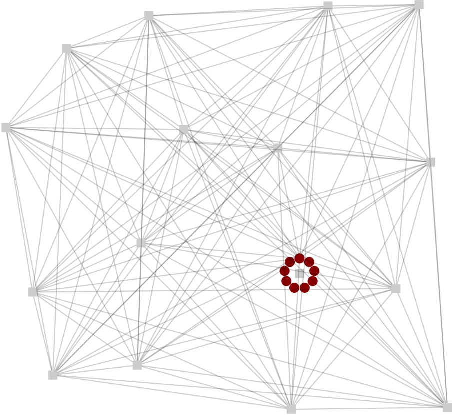

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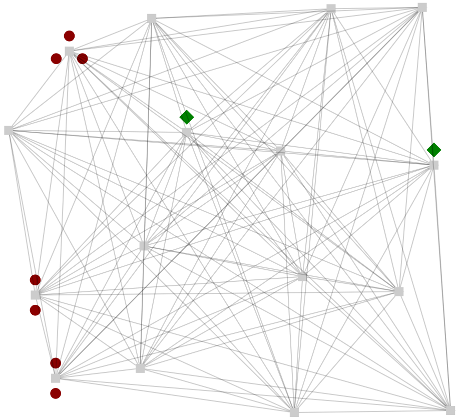

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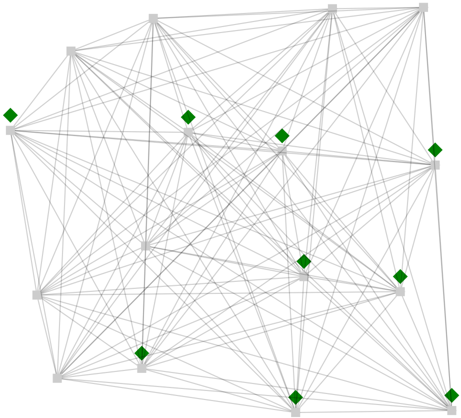
- terminates at the first time step at which all particles are happy. $\Rightarrow$ dispersion time!



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- ... the total number of jumps performed by the particles:

$$\sum_{t \geq 0} U_t = \sum_{t=0}^{T_{n,M}} U_t.$$

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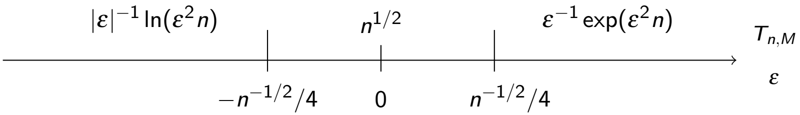

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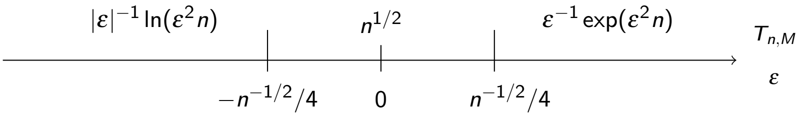

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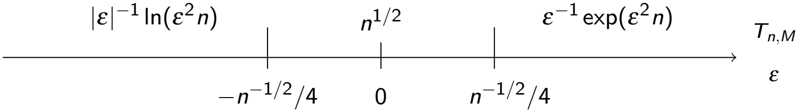

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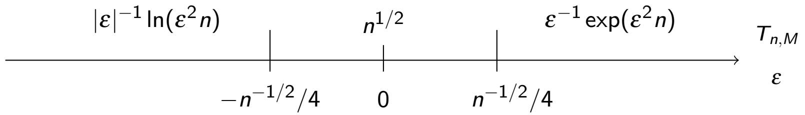

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  $|\varepsilon|^{-1}\ln(\varepsilon^2 n) = \Theta(n^{1/2})$ and $\varepsilon^{-1}\exp(\varepsilon^2 n) = \Theta(n^{1/2})$.

$\Rightarrow$ The transition in and out of the critical window where $|\varepsilon| = O(n^{-1/2})$ is smooth.

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Let $\alpha \in \mathbb{R}$ and $M = M(n) = n/2 + \alpha n^{1/2} + o(n^{1/2}) \in \mathbb{N}$. Then there exists a continuous and almost surely positive random variable T_α such that, as $n \to \infty$,

$$n^{-1/2} T_{n,M} \to T_\alpha \text{ in distribution.}$$

Furthermore, as $n \to \infty$,

$$(n \ln(n))^{-1} \sum_{t \geq 0} U_t \to \frac{2}{7} \text{ in distribution.}$$

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- Main tool used in the proof: Diffusion Approximation.

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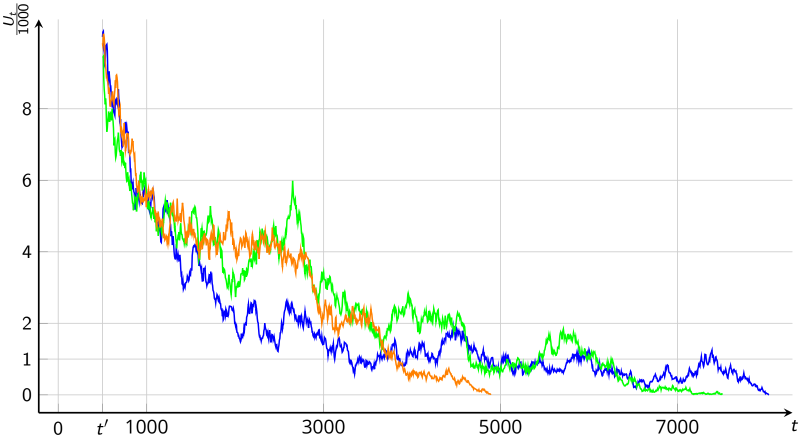
The number of unhappy particles U_t fluctuates strongly:



Three sample runs of the dispersion process with $n = 10^7$ and $M = n/2$, i.e. $\alpha = 0$.
The trajectory is revealed only after $t' = 500$, where $U_{t'} \approx 10^4$ in all cases.

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$$dX_s = b(X_s)ds + \sigma(X_s)dB_s, \quad s > 0, \quad X_0 = x \in \mathbb{R},$$

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$\Rightarrow$ Analyse the behaviour of the system using results from stochastic calculus.

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$\Rightarrow$ The time-scaled process $(Y_{\lfloor s/h(n)\rfloor}^{(n)})_{s\ge 0}$ converges weakly to X as $n \to \infty$.

Logistic Branching Process - Absorption Time

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- Properties of X are well-studied (e.g. [Lambert, 2005]).

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The limiting process we get for $M = n/2 + \alpha n^{1/2} + o(n^{1/2}) \in \mathbb{N}$, $\alpha \in \mathbb{R}$ is a logistic branching process:

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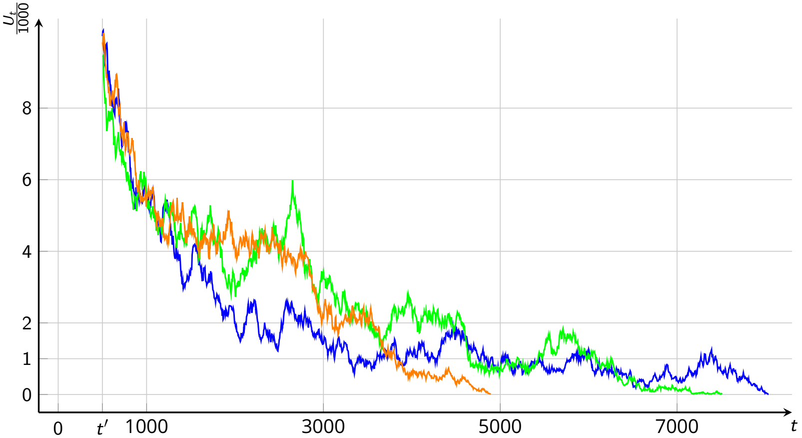
We obtain that $n^{-1/2} T_{n,M}$ converges in distribution to the absorption time of X:

$$T_\alpha = \inf\{s \geq 0 \mid X_s = 0\}.$$

Logistic Branching Process - Paths



Three sample runs of the dispersion process with $n = 10^7$ and $M = n/2$, i.e. $\alpha = 0$. The trajectory is revealed only after $t' = 500$, where $U_{t'} \approx 10^4$ in all cases.

Summary - Outlook

For $\alpha \in \mathbb{R}$ and $M = n/2 + \alpha n^{1/2} + o(n^{1/2})$:

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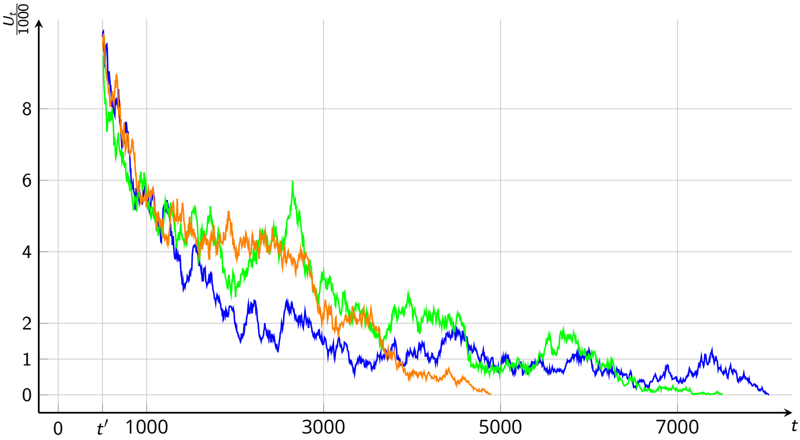
- Distribution of A_α can be described via the properties of $\int_0^\infty X_s ds$.

Dispersion Process - Simulation

The number of unhappy particles U_t fluctuates strongly:



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Thank you!

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