Limit Laws for Critical Dispersion on Complete Graphs

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Joint work with U. De Ambroggio, T. Makai and K. Panagiotou

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Dispersion Process - The Model

The dispersion process introduced by [Cooper et al., 2018]:

[Graph of a connected graph G with vertices and edges]
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- particles moving on the vertices of a connected graph $G$. 
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The dispersion process introduced by [Cooper et al., 2018]:

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- a particle is
  - happy if there are no other particles occupying the same vertex,
  - unhappy otherwise.
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- **initially**: $M \geq 2$ (unhappy) particles are placed on some vertex of $G$. 
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- **terminates** at the first time step at which all particles are happy.  
  \[ \Rightarrow \text{dispersion time!} \]
Dispersion Process - Quantities of Interest

In our setting:

- $G$: complete graph with $n$ vertices.
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- **\( U_t \):** number of unhappy particles at time \( t \in \mathbb{N}_0 \).

\[ U_0 = M \geq 2. \]

We are interested in...

- the dispersion time, i.e. the number of time steps until all particles become happy: 
  \[ T_{n, M} = \inf \{ t \in \mathbb{N}_0 : U_t = 0 \} \]
- the total number of jumps performed by the particles:
  \[ \sum_{t \geq 0} U_t = T_{n, M}. \]
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Dispersion Time - Phase Transition

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Dispersion Process

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\begin{array}{c|c|c|c|c}
|\varepsilon|^{-1} \ln(\varepsilon^2 n) & n^{1/2} & \varepsilon^{-1} \exp(\varepsilon^2 n) \\
-n^{-1/2}/4 & 0 & n^{-1/2}/4 \\
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- when $|\varepsilon| = \Theta(n^{-1/2})$: 

$T_{n,M}$ is smooth.
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  when $|\varepsilon| = o(1)$:

$$|\varepsilon|^{-1} \ln(\varepsilon^2 n) \quad n^{1/2} \quad \varepsilon^{-1} \exp(\varepsilon^2 n) \quad T_{n,M} \quad \varepsilon$$

- $-n^{-1/2}/4 \quad 0 \quad n^{-1/2}/4$

- when $|\varepsilon| = \Theta(n^{-1/2})$:
  $|\varepsilon|^{-1} \ln(\varepsilon^2 n) = \Theta(n^{1/2})$ and $\varepsilon^{-1} \exp(\varepsilon^2 n) = \Theta(n^{1/2})$. 
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  \[
  T_{n,M} \sim \begin{cases} 
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  \varepsilon^{-1} \exp(\varepsilon^2 n) & \text{when } |\varepsilon| = \Theta(n^{-1/2}).
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  ⇒ The transition in and out of the critical window where $|\varepsilon| = O(n^{-1/2})$ is smooth.
Dispersion Process - Main Results

We analyse the dispersion process within the critical window, i.e. when 
\[ M = n/2 + O(n^{1/2}). \]
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**Theorem [De Ambroggio, Makai, Panagiotou, S., 2024]**

Let $\alpha \in \mathbb{R}$ and $M = M(n) = n/2 + \alpha n^{1/2} + o(n^{1/2}) \in \mathbb{N}$. Then there exists a continuous and almost surely positive random variable $T_\alpha$ such that, as $n \to \infty$,

$$n^{-1/2} T_{n,M} \to T_\alpha \text{ in distribution.}$$

Furthermore, as $n \to \infty$,

$$(n \ln(n))^{-1} \sum_{t \geq 0} U_t \to \frac{2}{7} \text{ in distribution.}$$
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- Each of the \( M \sim n/2 \) particles performs on average typically \( \sim 4/7 \ln(n) \) jumps.
- We can say a lot about the distribution of \( T_\alpha \), e.g.

\[
\mathbb{E}[T_0] = \frac{\pi^{3/2}}{\sqrt{7}}, \text{ so that } \mathbb{E}[T_{n,M}] \sim n^{1/2} \cdot \frac{\pi^{3/2}}{\sqrt{7}}.
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- Main tool used in the proof: Diffusion Approximation.
Dispersion Process - Simulation

The number of unhappy particles $U_t$ fluctuates strongly:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$U_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td></td>
</tr>
<tr>
<td>3000</td>
<td></td>
</tr>
<tr>
<td>5000</td>
<td></td>
</tr>
<tr>
<td>7000</td>
<td></td>
</tr>
</tbody>
</table>

Three sample runs of the dispersion process with $n = 10^7$ and $M = n/2$, i.e. $\alpha = 0$. The trajectory is revealed only after $t' = 500$, where $U_{t'} \approx 10^4$ in all cases.
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Diffusion Approximation - Idea

- Approximate the behaviour of a discrete-time Markov chain *(here: number of unhappy particles)* by a (simpler) continuous-time Markov process with continuous paths.

\[
dX_t = b(X_t) \, dt + \sigma(X_t) \, dB_t, \quad t > 0, \quad X_0 = x \in \mathbb{R},
\]

where \( B \) is a Brownian motion and \( b, \sigma : \mathbb{R} \to \mathbb{R} \).

⇒ Analyse the behaviour of the system using results from stochastic calculus.
Diffusion Approximation - Idea

- Approximate the behaviour of a discrete-time Markov chain (here: number of unhappy particles) by a (simpler) continuous-time Markov process with continuous paths.
- Scale time and space to obtain a continuous process.

\[
\begin{align*}
\frac{dX}{ds} &= b(X) ds + \sigma(X) dB_s, \\
X_0 &= x \in \mathbb{R},
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⇒ Analyse the behaviour of the system using results from stochastic calculus.
Diffusion Approximation - e.g. [Durrett, 1996]

Suppose that $Y_n(t) = Y_n t$ for $t \in \mathbb{N}_0$ is a discrete-time Markov chain with values in $S_n \subseteq \mathbb{R}$ (here: $Y_n t = n - 1/2 u_t$), $h: \mathbb{N} \to \mathbb{R}^+$ is a sequence with $\lim_{n \to \infty} h(n) = 0$ (here: $h(n) = n - 1/2$), $b, \sigma: \mathbb{R} \to \mathbb{R}$ are continuous functions such that the SDE $dX_s = b(X_s) ds + \sigma(X_s) dB_s$, $s > 0$, $X_0 = x \in \mathbb{R}$, has a (weakly) unique (weak) solution for all initial values $x \in \mathbb{R}$.

$Y_n 0 \to x \in \mathbb{R}$ as $n \to \infty$, for all $R < \infty$, $\lim_{n \to \infty} \sup_{x \in S_n} |x| \leq R ||| E [Y_n(t + 1) - x | Y_n(t) = x] / h(n) - b(x) ||| = 0$, $\lim_{n \to \infty} \sup_{x \in S_n} |x| \leq R ||| E [ (Y_n(t + 1) - x)^2 | Y_n(t) = x] / h(n) - \sigma^2(x) ||| = 0$, and $\lim_{n \to \infty} \sup_{x \in S_n} |x| \leq R ||| E [ |Y_n(t + 1) - x|^{p} | Y_n(t) = x] / h(n) = 0$ for some $p \geq 2$.

The time-scaled process $(Y_n(\lfloor s / h(n) \rfloor)) s \geq 0$ converges weakly to $X$ as $n \to \infty$. 

Diffusion Process

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is a discrete-time Markov chain with values in \( S^{(n)} \subseteq \mathbb{R} \) (here: \( Y_t^{(n)} = n^{-1/2} U_t \)),

\[ \text{with} \quad Y_0^{(n)} = x \in \mathbb{R} \]

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- $Y_0^{(n)} \rightarrow x \in \mathbb{R}$ as $n \rightarrow \infty$,
- for all $R < \infty$
  - $\lim_{n \rightarrow \infty} \sup_{x \in S(n), |x| \leq R} \mathbb{E}\left[ Y_{t+1}^{(n)} - x \mid Y_t^{(n)} = x \right] / h(n) - b(x) = 0$,
  - $\lim_{n \rightarrow \infty} \sup_{x \in S(n), |x| \leq R} \mathbb{E}\left[ (Y_{t+1}^{(n)} - x)^2 \mid Y_t^{(n)} = x \right] / h(n) - \sigma^2(x) = 0$,
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  - $\lim_{n \to \infty} \sup_{x \in S^{(n)}, |x| \leq R} \left[ \mathbb{E} \left[ Y_{t+1}^{(n)} - x \mid Y_t^{(n)} = x \right] / h(n) - b(x) \right] = 0$,
  - $\lim_{n \to \infty} \sup_{x \in S^{(n)}, |x| \leq R} \left[ \mathbb{E} \left[ (Y_{t+1}^{(n)} - x)^2 \mid Y_t^{(n)} = x \right] / h(n) - \sigma^2(x) \right] = 0$,
  - $\lim_{n \to \infty} \sup_{x \in S^{(n)}, |x| \leq R} \left[ \mathbb{E} \left[ |Y_{t+1}^{(n)} - x|^p \mid Y_t^{(n)} = x \right] / h(n) = 0 \right.$ for some $p \geq 2$.

$\Rightarrow$ The time-scaled process $(Y_{\lfloor s/h(n) \rfloor}^{(n)})_{s \geq 0}$ converges weakly to $X$ as $n \to \infty$. 
Logistic Branching Process - Absorption Time

The limiting process we get for \( M = n/2 + \alpha n^{1/2} + o(n^{1/2}) \in \mathbb{N}, \alpha \in \mathbb{R} \) is a logistic branching process:
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$$dX_s = \left(2\alpha X_s - \frac{7}{4} X_s^2\right) ds + \sqrt{X_s} dB_s, \quad s > 0.$$
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- Properties of $X$ are well-studied (e.g. [Lambert, 2005]).
- Initial value: start at $X_0 = \infty$ possible.

We obtain that $n^{-1/2} T_{n,M}$ converges in distribution to the absorption time of $X$:

$$T_\alpha = \inf\{s \geq 0 \mid X_s = 0\}.$$
Three sample runs of the dispersion process with \( n = 10^7 \) and \( M = n/2 \), i.e. \( \alpha = 0 \).
The trajectory is revealed only after \( t' = 500 \), where \( U_{t'} \approx 10^4 \) in all cases.
Summary - Outlook

For $\alpha \in \mathbb{R}$ and $M = n/2 + \alpha n^{1/2} + o(n^{1/2})$:

- **Diffusion approximation**: Approximate the (scaled) number of unhappy particles by a logistic branching process.

  $\Rightarrow$ As $n \to \infty$, $n^{-1/2} T_{n,M}$ converges in distribution to the absorption time $T_{\alpha}$ of the logistic branching process.
Summary - Outlook

For $\alpha \in \mathbb{R}$ and $M = n/2 + \alpha n^{1/2} + o(n^{1/2})$:

- **Diffusion approximation**: Approximate the (scaled) number of unhappy particles by a logistic branching process.

  $\Rightarrow$ As $n \to \infty$, $n^{-1/2} T_{n,M}$ converges in distribution to the absorption time $T_\alpha$ of the logistic branching process.

- Infer properties of $T_\alpha$ from results from stochastic calculus, e.g. expressions for all moments.
Summary - Outlook

For $\alpha \in \mathbb{R}$ and $M = n/2 + \alpha n^{1/2} + o(n^{1/2})$:

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Diffusion approximation indicates a **stronger statement for the total number of jumps** than $(n \ln(n))^{-1} \sum_{t \geq 0} U_t \xrightarrow{d} \frac{2}{7}$:
For $\alpha \in \mathbb{R}$ and $M = n/2 + \alpha n^{1/2} + o(n^{1/2})$:

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  \[ \Rightarrow \text{As } n \to \infty, \; n^{-1/2} T_{n,M} \text{ converges in distribution to the absorption time } T_\alpha \text{ of the logistic branching process.} \]

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Diffusion approximation indicates a stronger statement for the total number of jumps than $(n \ln(n))^{-1} \sum_{t \geq 0} U_t \xrightarrow{d} \frac{2}{7}$:

- There is a continuous random variable $A_\alpha$ such that, as $n \to \infty$,

  \[
n^{-1} \left( \sum_{t \geq 0} U_t - \frac{2}{7} n \ln n \right) \xrightarrow{d} A_\alpha.
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Summary - Outlook

For $\alpha \in \mathbb{R}$ and $M = n/2 + \alpha n^{1/2} + o(n^{1/2})$:

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Diffusion approximation indicates a stronger statement for the total number of jumps than $(n \ln(n))^{-1} \sum_{t \geq 0} U_t \overset{d}{\to} \frac{2}{7}$:

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  $$n^{-1} \left( \sum_{t \geq 0} U_t - \frac{2}{7} n \ln n \right) \overset{d}{\to} A_\alpha.$$

- Distribution of $A_\alpha$ can be described via the properties of $\int_0^\infty X_s ds$. 
Dispersion Process - Simulation

The number of unhappy particles $U_t$ fluctuates strongly:

Three sample runs of the dispersion process with $n = 10^7$ and $M = n/2$, i.e. $\alpha = 0$. The trajectory is revealed only after $t' = 500$, where $U_{t'} \approx 10^4$ in all cases.
Thank you!
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