Early typical vertices in subcritical random graphs of preferential attachment type

Peter Mörters , Nick Schleicher AofA 2024

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- In the classical models the power of a vertex is measured by its current degree.
- Here we introduce a simpler model with this idea.
- A new vertex n attaches to earlier vertices m ∈ {1,...n−1} with a probability proportional to m^{-γ} for a 0 < γ < 1.
- Then the expected degree of n behaves like:

$$\sum_{m=1}^{n-1} m^{-\gamma} \sim \frac{n^{1-\gamma}}{1-\gamma}$$

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- Let V_n = {1,..., n} and choose the connection probability of two distinct vertices i ≠ j as

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• This is an inhomogeneous random graph with connection probabilities of preferential attachment type.

Theorem

In the inhomogeneous random graph of preferential attachment type there exists a giant component if and only if

$$\gamma \geq \frac{1}{2}$$
 or $\beta > \beta_c := \frac{1}{4} - \frac{\gamma}{2}.$

This is a simplification of the main result in the Paper of Dereich and Mörters (2013).

We are working here in the subcritical regime , i.e. when $\gamma < \frac{1}{2}$ and 0 $< \beta < \beta_c.$

Main result

Main Theorem

Let $S_n(i)$ be the size of the connected component of vertex $i \in V_n$ in the inhomogeneous random graph of preferential attachment type in the subcritical regime. If $o_n \in V_n$ is such that $\frac{o_n}{n} \to u \in (0, 1]$, then

$$\lim_{u \downarrow 0} \lim_{n \to \infty} \mathbb{P}\left(S_n(o_n) \ge u^{-\rho_-}x\right) = \mathbb{P}\left(W \ge x\right) \,,$$

for all x > 0, where

$$\rho_{\pm} = \frac{1}{2} \pm \sqrt{(\gamma - \frac{1}{2})^2 + \beta(2\gamma - 1)}.$$

and W is a positive random variable satisfying

$$\mathbb{P}(W \ge x) = x^{-(
ho_+/
ho_-) + o(1)}$$
 as $x \to \infty$.

For the inner limit we have

Proposition

If $o_n \in V_n$ is such that $\frac{o_n}{n} \to u \in (0,1]$ and x > 0, then

$$\lim_{n\to\infty}\mathbb{P}\left(S_n(o_n)\geq u^{-\rho_-}x\right)=\mathbb{P}\left(T(u)\geq u^{-\rho_-}x\right),$$

where T(u) is the number of particles in a killed branching random walk.

Branching random walk with killing



 The branching random walk is started in log u < 0 and the displacements of the children of a vertex are given by an independent Poisson point process with intensity

$$\pi(\mathrm{d} y) = \beta(\mathrm{e}^{\gamma y} \mathbb{1}_{y>0} + \mathrm{e}^{(1-\gamma)y} \mathbb{1}_{y<0}) \,\mathrm{d} y \,.$$

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• The process becomes extinct after a finite number of generations in the subcritical phase.

Coupling to the killed branching random walk

 To couple the graph and the branching random walk, we map labels from {1, · · · , n} to positions in (-∞, 0]. We do this using the following map

$$\phi_n: \{1,\ldots,n\} \to (-\infty,0] \quad , \quad i \mapsto -\sum_{j=i+1}^n \frac{1}{j} \; .$$

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- Note that the youngest vertex is mapped to the origin, and older vertices are placed to the left with decreasing intensity.
- Observe that typical vertices are placed to the left and strong vertices a placed to the right.

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The coupling fails if



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- ⇒ There exists a coupling such that the projected tree coincides with the connected component of a typical vertex with high probabilty.

For the outer limit we have:

Proposition

Under the conditions of the main Theorem, for every x > 0,

$$\lim_{u\downarrow 0} \mathbb{P}\left(T(u) \ge xu^{-\rho_{-}}\right) = \mathbb{P}\left(W \ge x\right) \,,$$

where W is a positive random variable satisfying

$$\mathbb{P}(W \ge x) = x^{-(\rho_+/\rho_-)+o(1)}$$
 as $x \to \infty$.

The proof is based on the work of Aidekon et al

Thank you for your attention.

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- In rank one models it is known that the size is of the order of the largest degree, in our language of order n^{γ}
- We heuristically derive a conjecture: Suppose we were allowed to let n → ∞ and u → 0 simultaneously.
- At best we could be allowed $u \approx \frac{c}{n}$. Then our hypothetic result would give that the most powerful vertices would have a connected component of size n^{ρ_-}

Branching random walks without killing

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- Define

$$\psi(\alpha) = \mathbb{E}\Big[\sum_{x\in\Pi} \mathrm{e}^{-\alpha\tau_x}\Big].$$



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The Crump-Mode-Jagers or general branching process.

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- In our case it suffices to use a characteristic X_x, which may depend on ξ_x but is independent for each particle and distributed like some real valued X.
- We sum X_x over all particles born before time t,

$$Z_t^X := \sum_{x \in \mathcal{T}, \sigma_x < t} X_x.$$

The following result is the work of Nerman formulated in our set-up.

Theorem

Suppose that $\mathbb{E}[X] < \infty$, then

$$e^{-\alpha t}Z_t^X \to W$$
 in probability, as $t \to \infty$,

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• In order to use this to prove our main result we have to derive a suitable ξ and X from π .



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- We let X ≥ 1 be the total number of branching particles including the particle at the origin

This leads to the following Theorem:

Theorem

We have $\mathbb{E}[X] < \infty$ and ξ satisfies the conditions above for the Malthusian parameter $\alpha = \rho_-$. Moreover, for $t = -\log u$, we have

$$Z_t^X \stackrel{d}{=} T(u).$$

Proof.

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We have $\mathbb{E}[X] < \infty$ and ξ satisfies the conditions above for the Malthusian parameter $\alpha = \rho_{-}$. Moreover, for $t = -\log u$, we have

$$Z_t^X \stackrel{d}{=} T(u).$$

Proof.

 Shifting all particle positions by t = -log u the killed branching random walk T(u) becomes a branching random walk T'(u) started at the origin, with displacements given by π, with a killing barrier at t = -log u.



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- the number of branching particles is the characteristic X_x

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- It also holds that $\int_0^\infty e^{\rho_- t} \xi(dt) = 1.$
- To complete the proof we have

$$\lim_{n\to\infty}\mathbb{P}\left(S_n(o_n)\geq u^{-\rho_-}x\right)=\mathbb{P}\left(T(u)\geq u^{-\rho_-}x\right).$$

$$\lim_{u\downarrow 0} \mathbb{P}\left(T(u) \ge x u^{-\rho_{-}}\right) = \mathbb{P}\left(W \ge x\right),$$

and therefore

$$\lim_{u \downarrow 0} \lim_{n \to \infty} \mathbb{P}\left(S_n(o_n) \ge u^{-\rho_-}x\right) = \mathbb{P}\left(Y \ge x\right) \,.$$



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(*i*)
$$\log(\psi(\rho_{-})) - \frac{\rho_{-}\psi'(\rho_{-})}{\psi(\rho_{-})} > 0$$
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$$\log(\psi(\rho_{-})) - \frac{\rho_{-}\psi'(\rho_{-})}{\psi(\rho_{-})} > 0$$
,
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- The first one holds as $\psi(\rho_{-}) = 1$ and $\psi'(\rho_{-}) < 0$.
- For the second condition it suffices to check $\mathbb{E}[W_1^{\alpha}] < \infty$ for some $\alpha > 1$.