Early typical vertices in subcritical random graphs of preferential attachment type

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AofA 2024

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- In the classical models the power of a vertex is measured by its current degree.
- Here we introduce a simpler model with this idea.
- A new vertex n attaches to earlier vertices $m \in \{1, \ldots, n-1\}$ with a probabilty propotional to m for a $0 < \quad < 1$.
- Then the expected degree of n behaves like:

$$\sum_{m=1}^{n-1} m \qquad \frac{n^{1}}{1}$$

Inhomogeneous random graph of preferential attachment type

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- Let $V_n = \{1, \ldots, n\}$ and choose the connection probability of two distinct vertices $i \neq j$ as

$$p_{ij} := \quad (i \vee j)^{\quad 1}(i \wedge j)^{\quad} \; ;$$

where $0 < \quad < 1$ parameterizes the strength of the preferences of early vertices and $\quad > 0$ is an edge density parameter.

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- To make the model sparse the proportionality factor should be of order n^{-1}.
- Let $V_n = \{1, \dots, n\}$ and choose the connection probability of two distinct vertices $i \neq j$ as

$$p_{ij} := \beta (i \vee j)^{\gamma - 1} (i \wedge j)^{-\gamma} ;$$

  where $0 < \gamma < 1$ parameterizes the strength of the preferences of early vertices and $\beta > 0$ is an edge density parameter.
- This is an inhomogeneous random graph with connection probabilities of preferential attachment type.

Inhomogeneous random graph of preferential attachment type

Theorem

In the inhomogeneous random graph of preferential attachment type there exists a giant component if and only if

$$\frac{1}{2} \quad \text{or} \quad > {}_c := \frac{1}{4} \quad \frac{}{2}.$$

This is a simplification of the main result in the Paper of Dereich and Mörters (2013).
We are working here in the *subcritical regime* , i.e. when $\; < \frac{1}{2}$ and $0 < \; < {}_c$.

Main result

Main Theorem

Let $S_n(i)$ be the size of the connected component of vertex $i \in V_n$ in the inhomogeneous random graph of preferential attachment type in the subcritical regime. If $o_n \in V_n$ is such that $\frac{o_n}{n} \to u \in (0,1]$, then

$$\lim_{u \downarrow 0} \lim_{n \to \infty} P\left(S_n(o_n) \quad u \quad x \right) = P\left(W \quad x\right);$$

for all $x > 0$, where

$$= \tfrac{1}{2} \qquad \sqrt{(\qquad \tfrac{1}{2})^2 + \;(2 \qquad 1)}.$$

and W is a positive random variable satisfying

$$P\left(W \quad x\right) = x^{(\; + = \;\;) + o(1)} \text{ as } x \to \infty.$$

For the inner limit we have

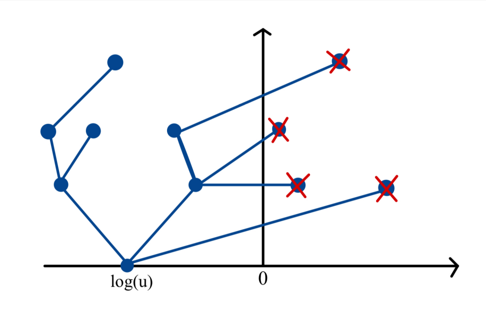
Proposition

If $o_n \in V_n$ is such that $\frac{o_n}{n} \to u \in (0, 1]$ and $x > 0$, then

$$\lim_{n \to \infty} \mathrm{P} \quad S_n(o_n) \quad u \quad x \; = \mathrm{P} \quad T(u) \quad u \quad x \; ;$$

where $T(u)$ is the number of particles in a killed branching random walk.

Branching random walk with killing



- The branching random walk is started in $\log u < 0$ and the displacements of the children of a vertex are given by an independent Poisson point process with intensity

$$(\mathrm{d}y) = \ \ (e^{\ \ y}\ \ {}_{y>0} + e^{(1\ \ \ )y}\ \ {}_{y<0})\,\mathrm{d}y\,:$$

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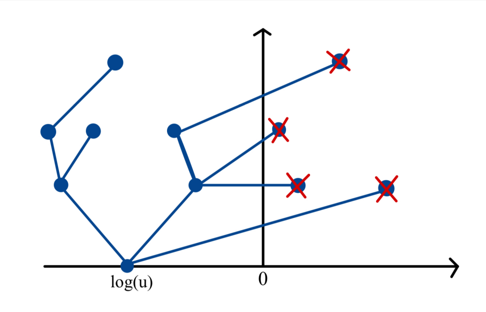

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$$(\mathrm{d}y) = \ (\mathrm{e}^{\ y}\ \ _{y>0} + \mathrm{e}^{(1\ \ )y}\ \ _{y<0})\,\mathrm{d}y :$$

- The process becomes extinct after a finite number of generations in the subcritical phase.

Coupling to the killed branching random walk

- To couple the graph and the branching random walk, we map labels from $\{1, \dots, n\}$ to positions in $(-\infty, 0]$. We do this using the following map

$$\phi_n : \{1, \dots, n\} \to (-\infty, 0] \ , \quad i \mapsto -\sum_{j=i+1}^{n} \frac{1}{j} \, .$$

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- Note that the youngest vertex is mapped to the origin, and older vertices are placed to the left with decreasing intensity.

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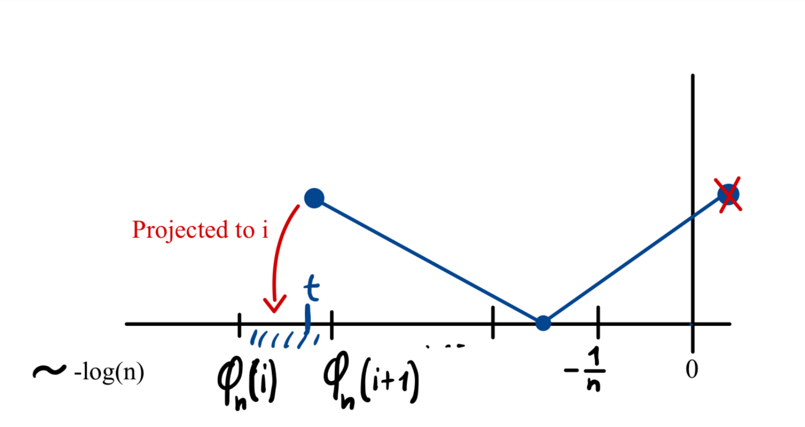
- Note that the youngest vertex is mapped to the origin, and older vertices are placed to the left with decreasing intensity.
- Observe that typical vertices are placed to the left and strong vertices a placed to the right.

Second step: Coupling to the killed branching random walk



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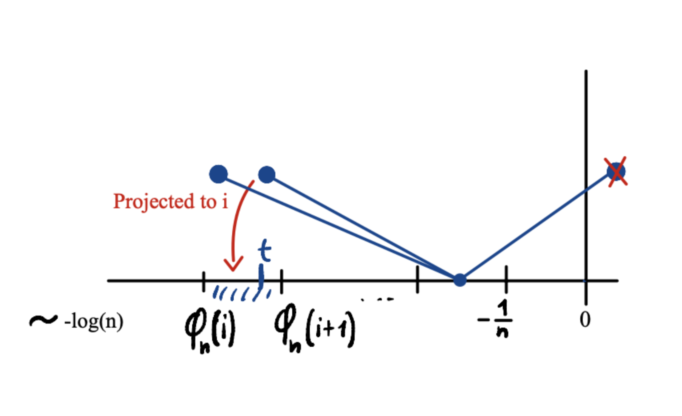
The coupling fails if



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- This effect allows to couple the Poisson offspring of the projected tree with the Bernouli offspring of the (tree like) Graph.

Coupling to the killed branching random walk

- This effect allows to couple the Poisson offspring of the projected tree with the Bernouli offspring of the (tree like) Graph.

$\Rightarrow$ There exists a coupling such that the projected tree coincides with the connected component of a typical vertex with high probabilty.

For the outer limit we have:

Proposition

Under the conditions of the main Theorem, for every $x > 0$,

$$\lim_{u \downarrow 0} \mathrm{P}\left(T(u) \quad xu \quad \right) = \mathrm{P}\left(W \quad x\right);$$

where W is a positive random variable satisfying

$$\mathrm{P}\left(W \quad x\right) = x^{(\ + = \ )+o(1)} \text{ as } x \to \infty.$$

The proof is based on the work of Aidekon et al

Thank you for your attention.

Outlook

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- Our aims is to find the size of the largest component in the subcritical phase.
- In rank one models it is known that the size is of the order of the largest degree, in our language of order n
- We heuristically derive a conjecture: Suppose we were allowed to let $n \to \infty$ and $u \to 0$ simultaneously.
- At best we could be allowed $u \quad \frac{c}{n}$. Then our hypothetic result would give that the most powerful vertices would have a connected component of size n

Branching random walks without killing

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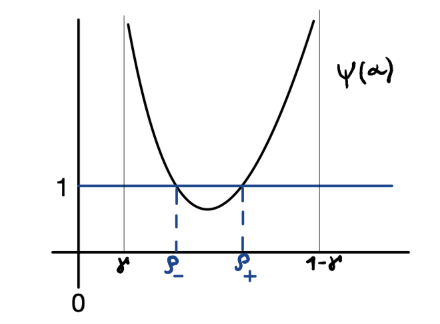
- Define

$$(\quad) = \mathbb{E} \sum_{x \in \Pi} e^{\quad x}.$$



The Nerman Setup

- Let be a point process on $[0; 1)$. We denote by $ = \mathrm{E}[ ]$ the intensity measure of the point process.

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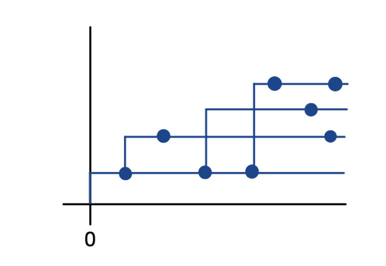

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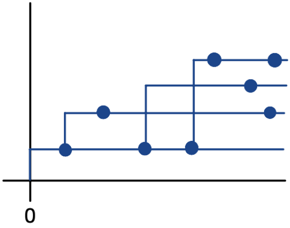

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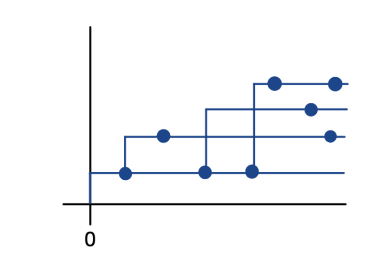

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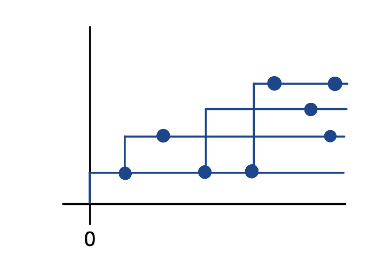

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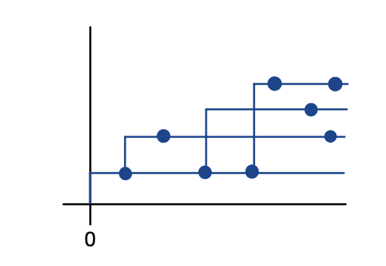

The *Crump-Mode-Jagers* or *general branching process*.

Convergence of the total number of particles

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- In our case it suffices to use a characteristic X_x, which may depend on $_x$ but is independent for each particle and distributed like some real valued X.
- We sum X_x over all particles born before time t,

$$Z_t^X := \sum_{x \in T;\ _x < t} X_x.$$

The following result is the work of Nerman formulated in our set-up.

Theorem

Suppose that $\mathrm{E}[X] < \infty$, then

$$e^{-\lambda t} Z_t^X \to W \text{ in probability, as } t \to \infty;$$

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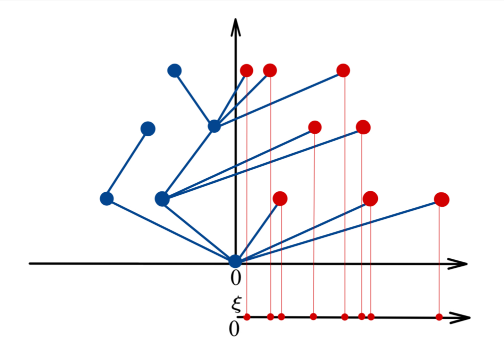
Suppose that $\mathrm{E}[X] < \infty$, then

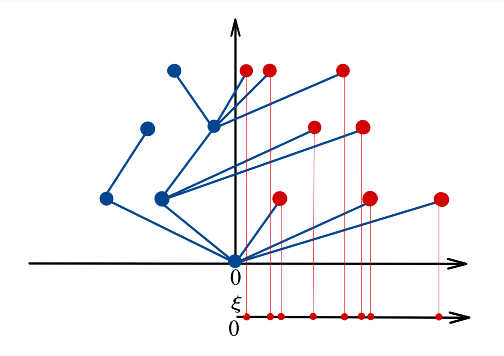
$$e^{\quad t} Z_t^X \to W \text{ in probability, as } t \to \infty;$$

- In order to use this to prove our main result we have to derive a suitable and X from .

Convergence of the total number of particles



- We let be the point process of locations of the frozen (non-branching) particles



- We let be the point process of locations of the frozen (non-branching) particles
- We let X 1 be the total number of branching particles including the particle at the origin

Convergence of the total number of particles

This leads to the following Theorem:

Theorem

We have $\mathrm{E}[X] < 1$ and satisfies the conditions above for the Malthusian parameter $=$. Moreover, for $t =$ $\log u$, we have

$$Z_t^X \overset{d}{=} T(u):$$

Proof.

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Proof.

- Shifting all particle positions by $t = \log u$ the killed branching random walk $T(u)$ becomes a branching random walk $T^{0}(u)$ started at the origin, with displacements given by , with a killing barrier at $t = \log u$.

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- the number of branching particles is the characteristic X_x

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$\square$

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- It also holds that $\int_0^{\infty} e^{\quad t} \;\, (dt) = 1$.
- To complete the proof we have

$$\lim_{n \to \infty} \mathrm{P}\;\; S_n(o_n) \quad u \quad\; x \;\; = \mathrm{P}\;\; T(u) \quad u \quad\; x \;\; ;$$

$$\lim_{u \downarrow 0} \mathrm{P}\;\; T(u) \quad xu \qquad = \mathrm{P}\,(W \quad x)\,;$$

and therefore

$$\lim_{u \downarrow 0} \lim_{n \to \infty} \mathrm{P}\;\; S_n(o_n) \quad u \quad\; x \;\; = \mathrm{P}\,(Y \quad x)\,;$$

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- The first one holds as $(\quad) = 1$ and $'(\quad) < 0$.
- For the second condition it suffices to check $E[W_1\ ] < \infty$ for some $\quad > 1$.