Phase transition for tree-rooted maps

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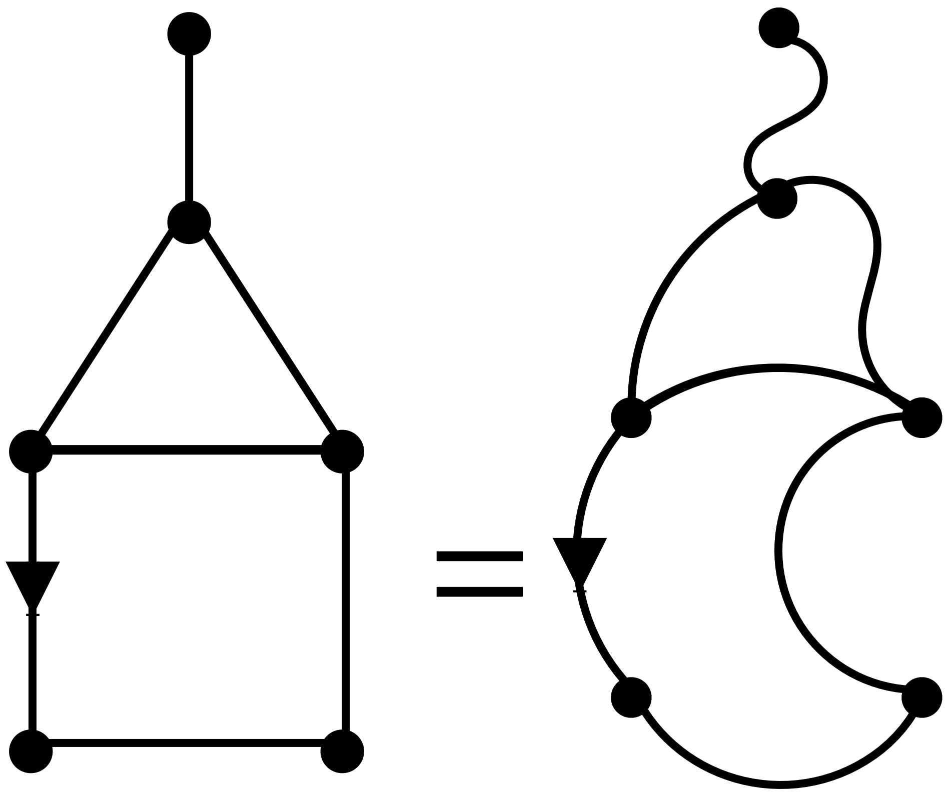
Joint work with Marie Albenque & Éric Fusy

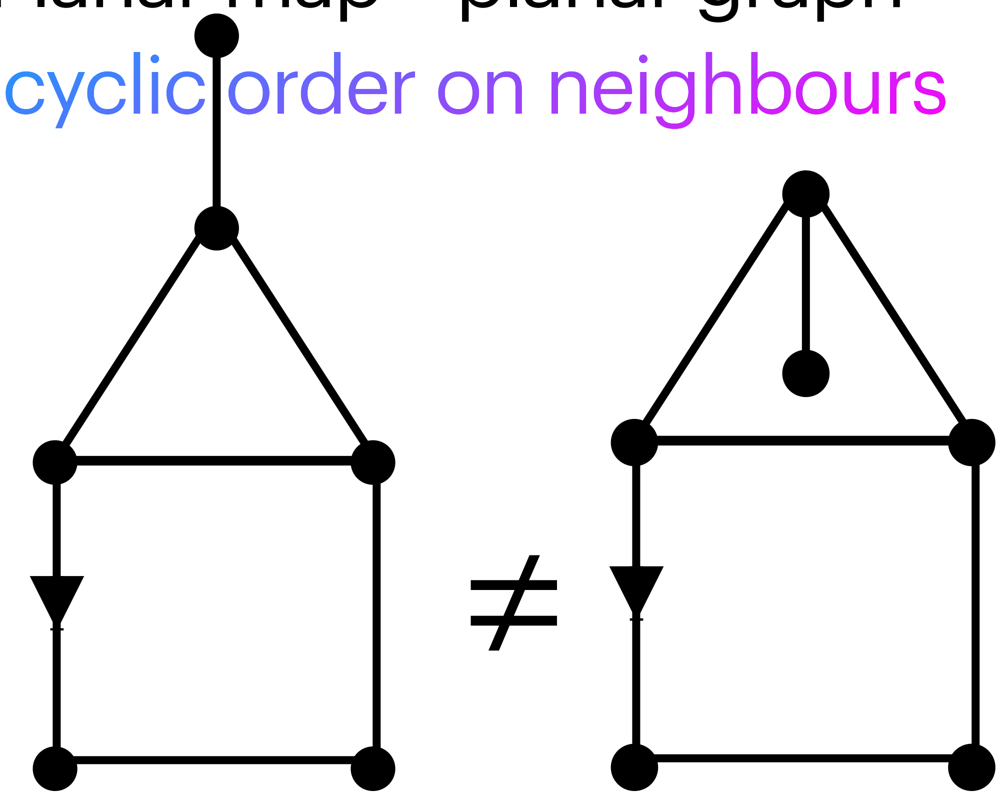
LIGM, Université Gustave Eiffel, France

Planar maps

Planar map $\mathfrak{m}$ = embedding on the sphere of a connected planar graph, considered up to homeomorphisms



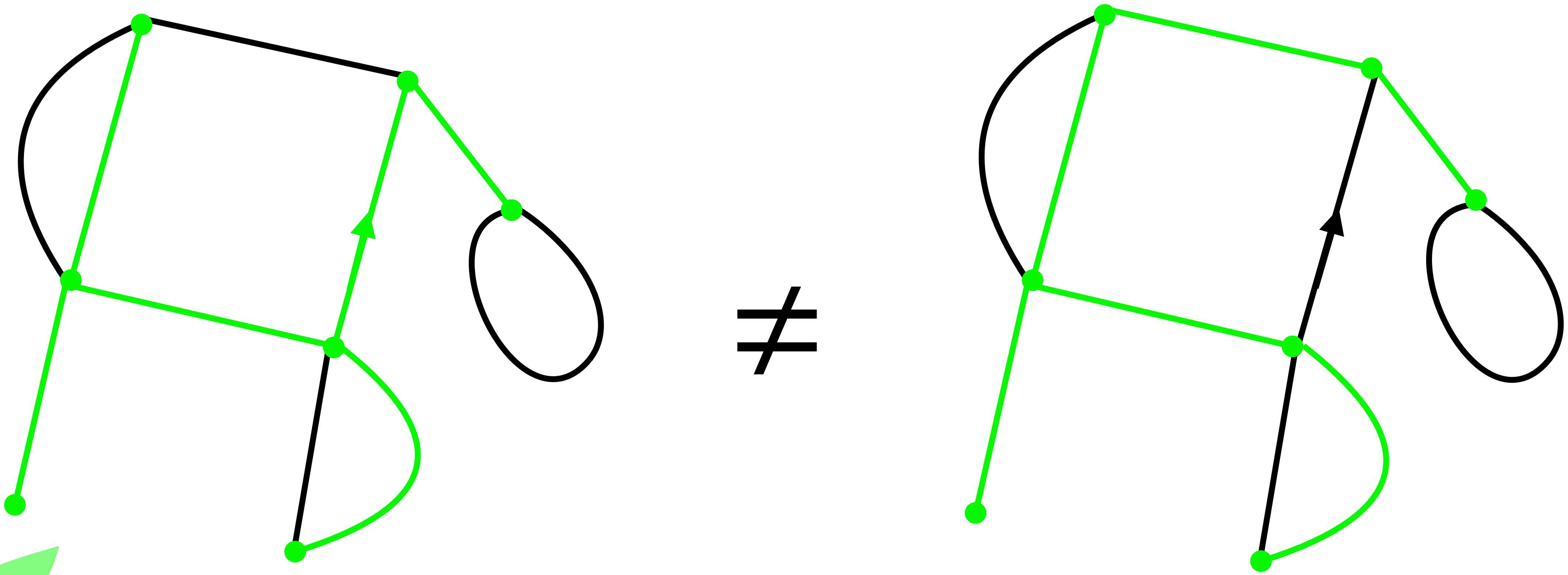
Planar map = planar graph + cyclic order on neighbours

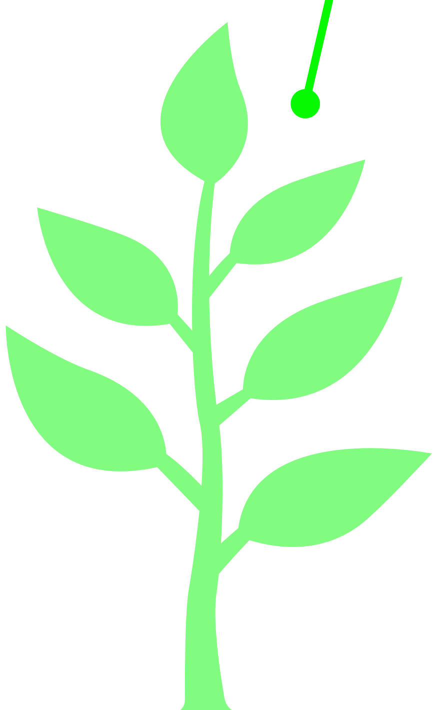


- Rooted planar map = map endowed with a marked oriented edge (represented by an arrow);
- Size $|\mathfrak{m}|$ = number of edges;
- Corner (does not exist for graphs !) = space between two consecutive edges around a vertex (trigonometric order).

Tree-rooted maps

= (rooted planar) maps endowed with a spanning tree.

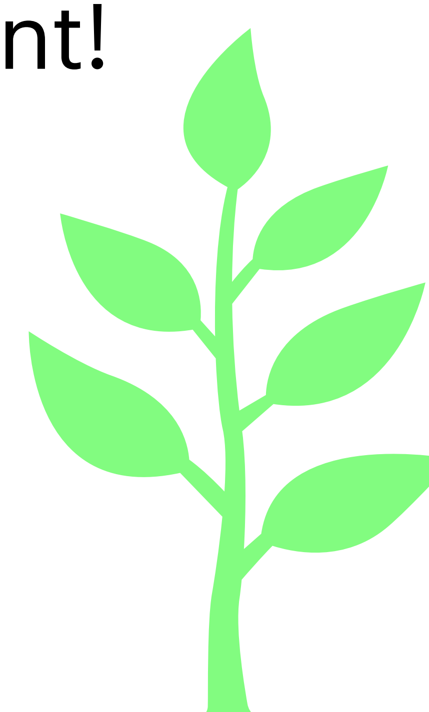




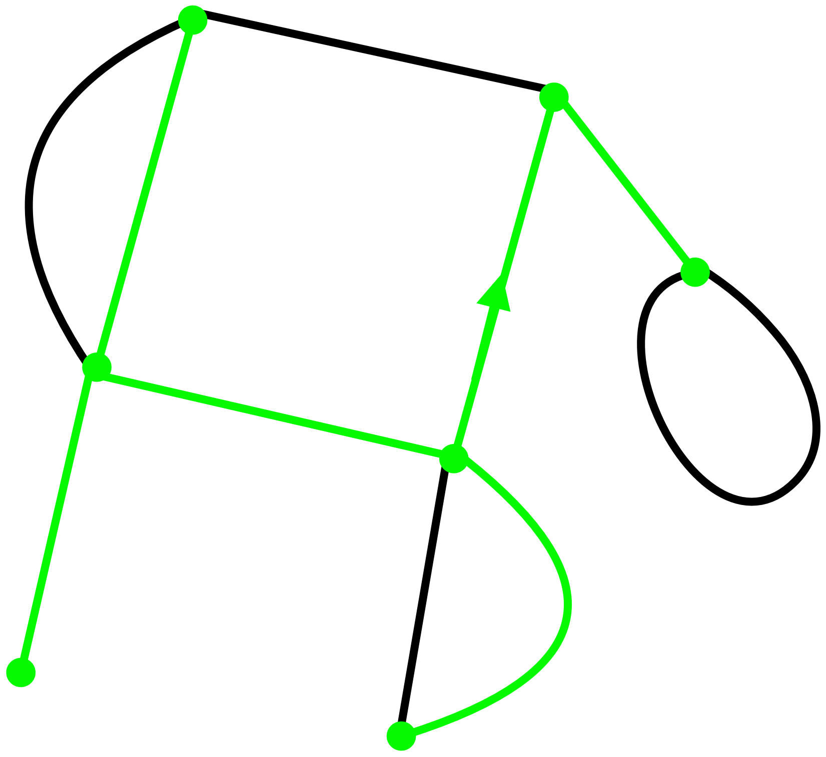
Decorated maps are interesting

Theoretical physics point of view:

- Undecorated maps: “pure gravity” case (nothing happens on the surface);
- Decorated maps: things happen! new asymptotic behaviours! new universality classes! excitement!



Tree-rooted maps



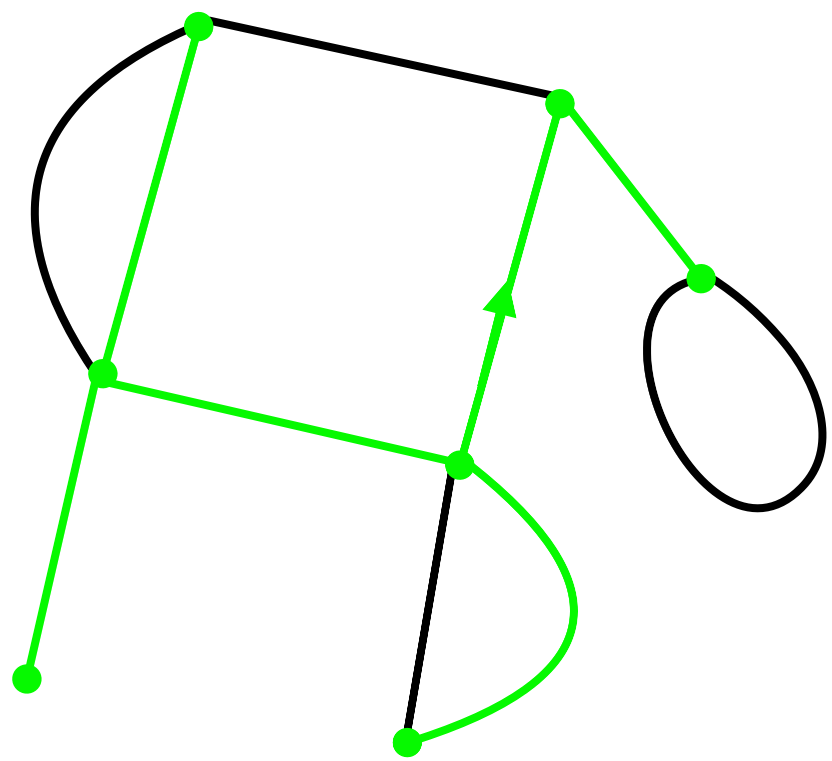
- Combinatorics well understood : Mullin's bijection;

$$[z^n]M(z) = \mathrm{Cat}_n \mathrm{Cat}_{n+1}$$

[Mullin 67]

- Geometry not so much.

Tree-rooted maps



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[Mullin 67]

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We want a phase transition in tree-rooted maps.

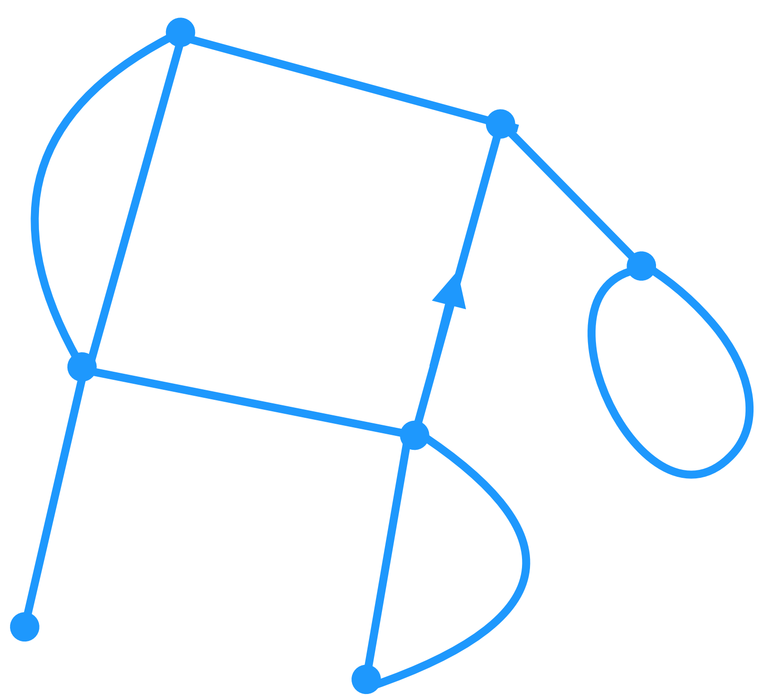
=> Block-weighted tree-rooted maps.

I. "Block-weighted maps"?

Joint work with William Fleurat

Universality results for planar maps

- Enumeration: $\kappa\rho^{-n}n^{-5/2}$ [Tutte 1963];
- Distance between vertices: $n^{1/4}$ [Chassaing, Schaeffer 2004];
- Scaling limit: Brownian sphere for quadrangulations [Le Gall 2013, Miermont 2013] and general maps [Bettinelli, Jacob, Miermont 2014];



Brownian Sphere $\mathcal{S}_e$

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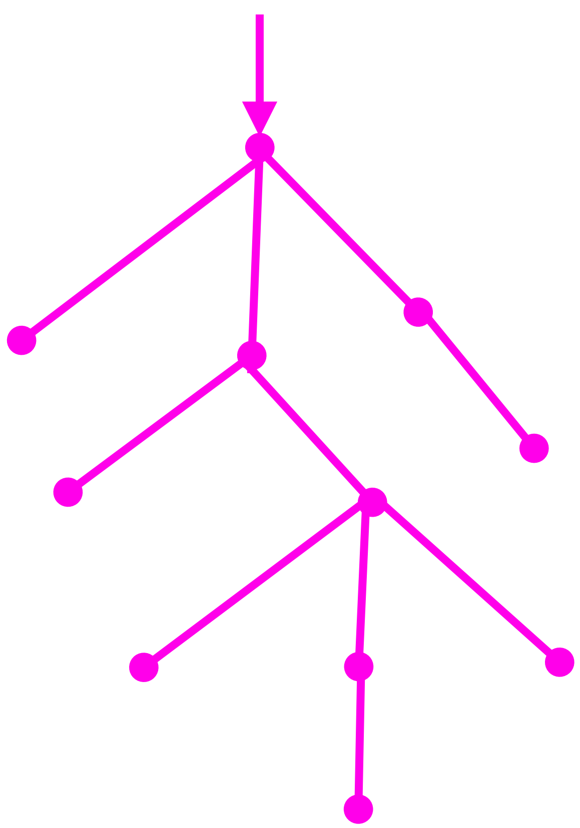
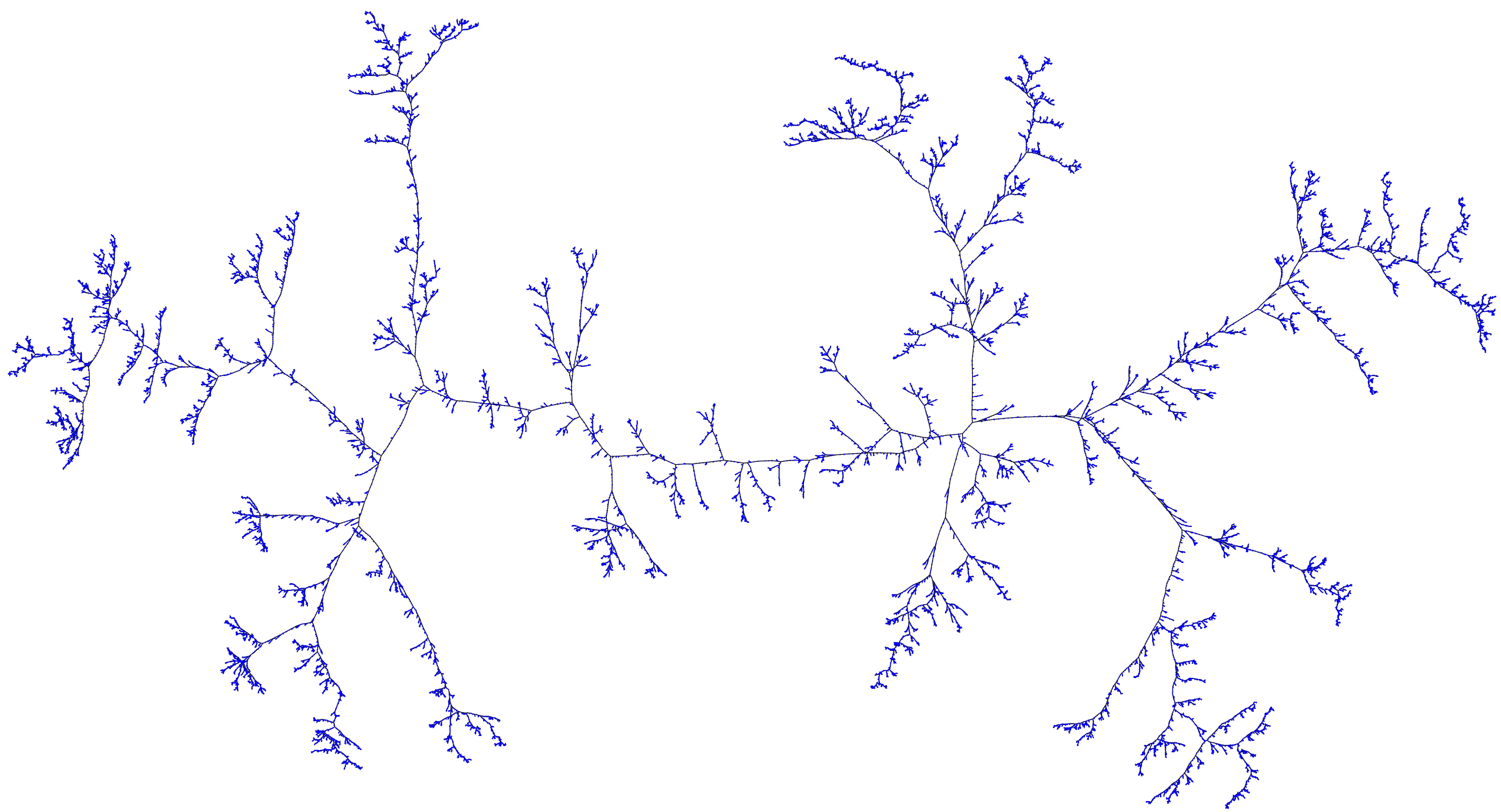
- Universality:
  - Same enumeration [Drmota, Noy, Yu 2020];
  - Same scaling limit, *e.g.* for triangulations & $2q$-angulations [Le Gall 2013], simple quadrangulations [Addario-Berry, Albenque 2017].

Universality results for plane trees

- Enumeration: $\kappa\rho^{-n}n^{-3/2}$;
- Distance between vertices: $n^{1/2}$ [Flajolet, Odlyzko 1982];
- Scaling limit: Brownian tree [Aldous 1993, Le Gall 2006];





Brownian Tree $\mathscr{T}_e$

Universality results for plane trees

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- Universality:
  - Same enumeration;
  - Same scaling limit;

Even for some classes of maps; e.g. outerplanar maps [Caraceni 2016], maps with a boundary of size >> $n^{1/2}$ [Bettinelli 2015].

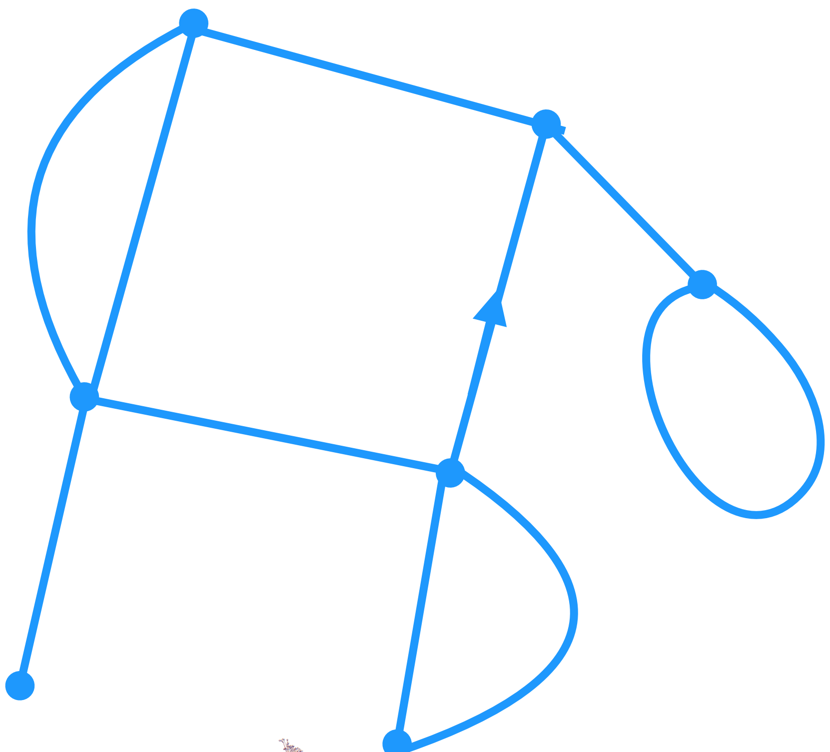
Models with (very) constrained boundaries

Motivation

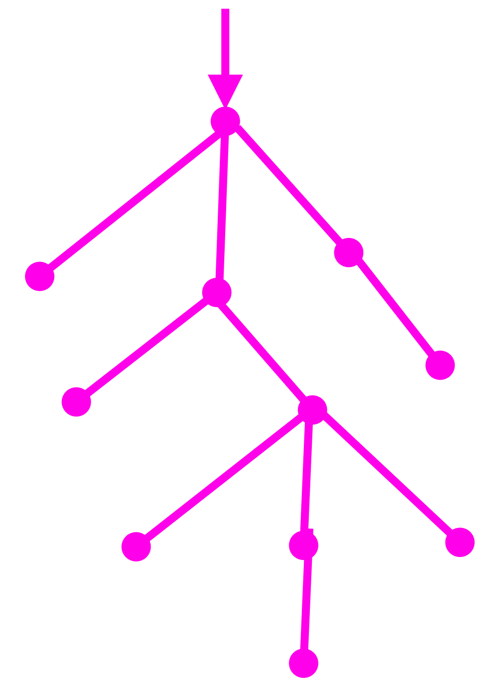
Inspired by [Bonzom 2016].

Two rich situations with universality results:

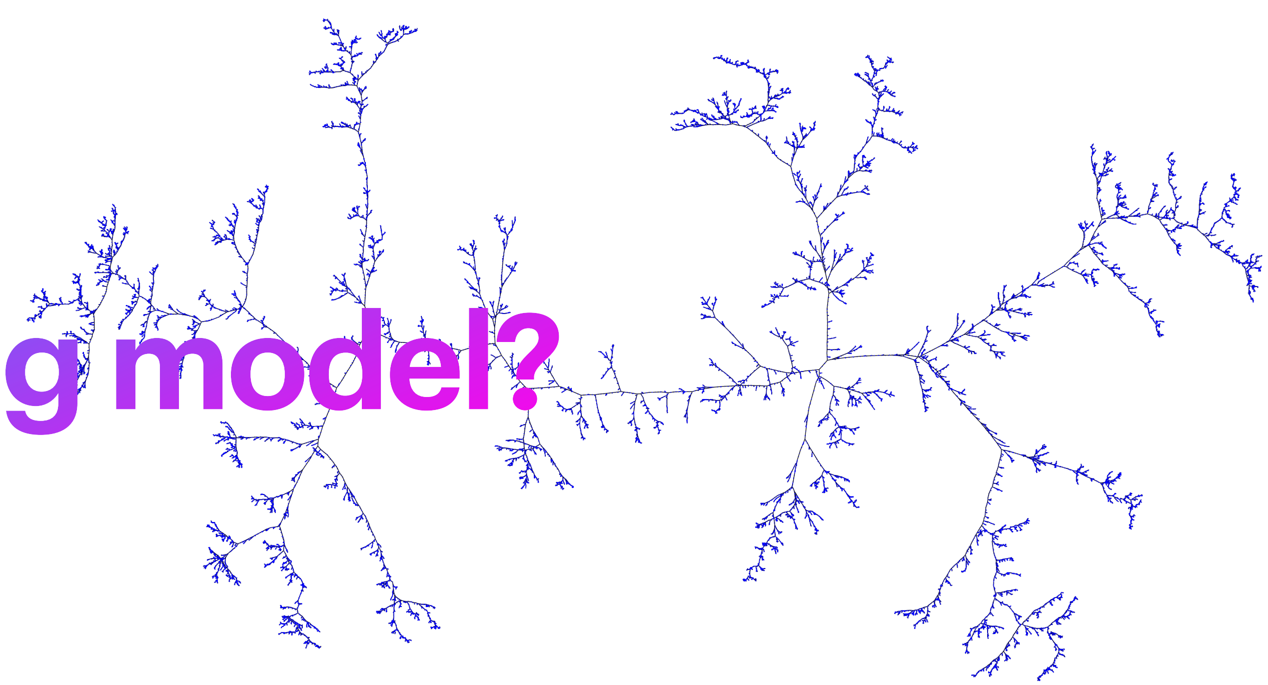
Planar maps



Plane trees



Interpolating model?



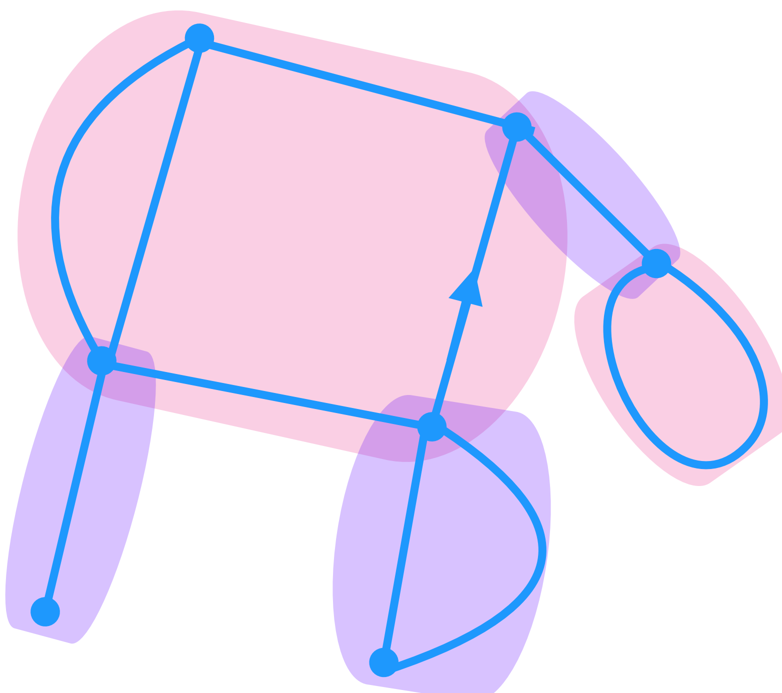
Brownian Sphere $\mathcal{S}_e$

Brownian Tree $\mathcal{T}_e$

Model definition

2-connected = two vertices must be removed to disconnect.

Block = maximal (for inclusion) 2-connected submap.



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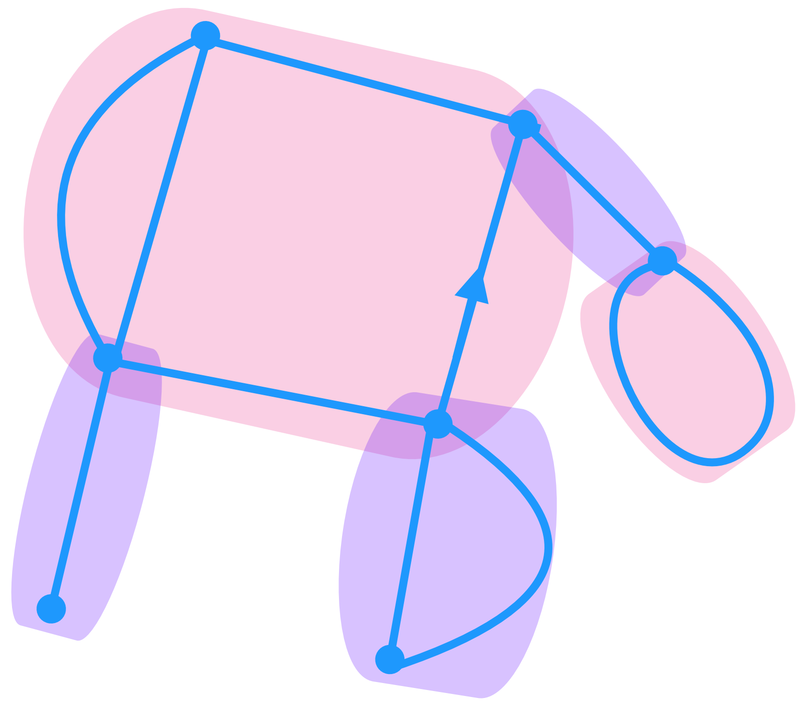

Condensation phenomenon: a large block concentrates a macroscopic part of the mass [Banderier, Flajolet, Schaeffer, Soria 2001; Jonsson, Stefánsson 2011].

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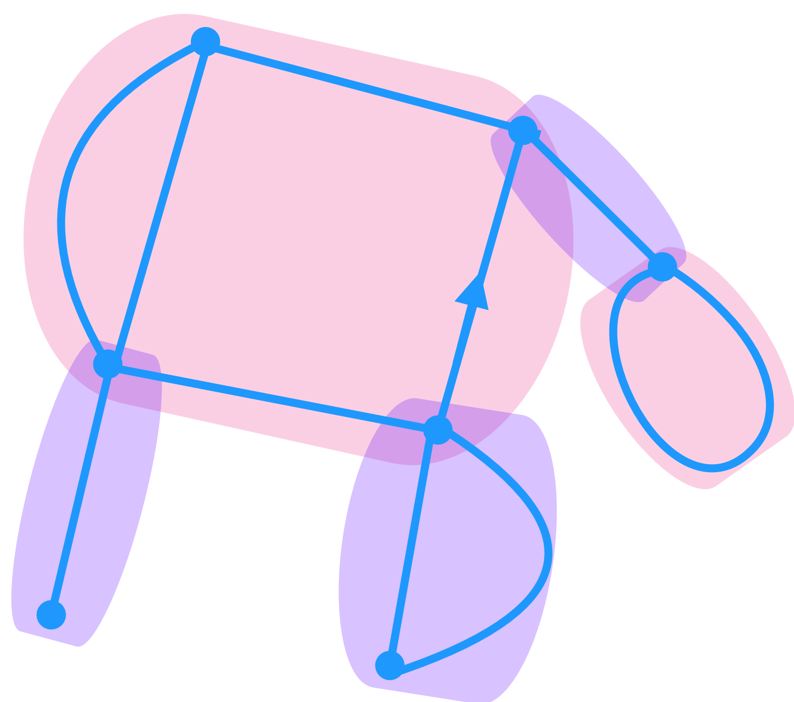

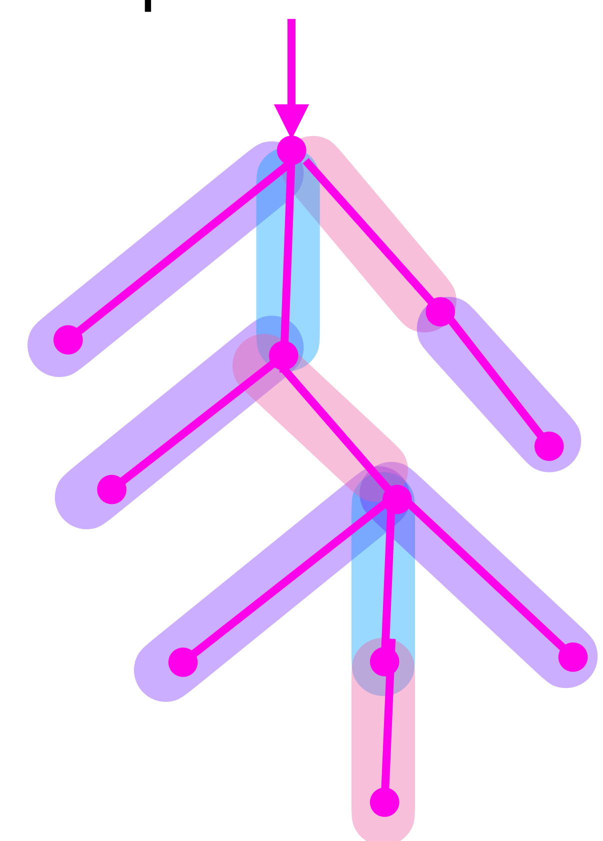

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Only small blocks.

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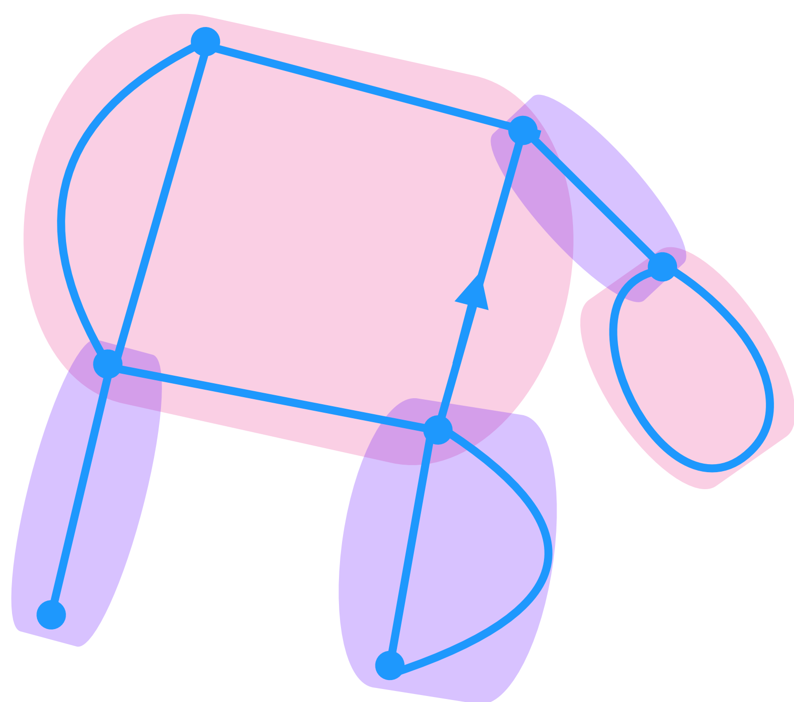

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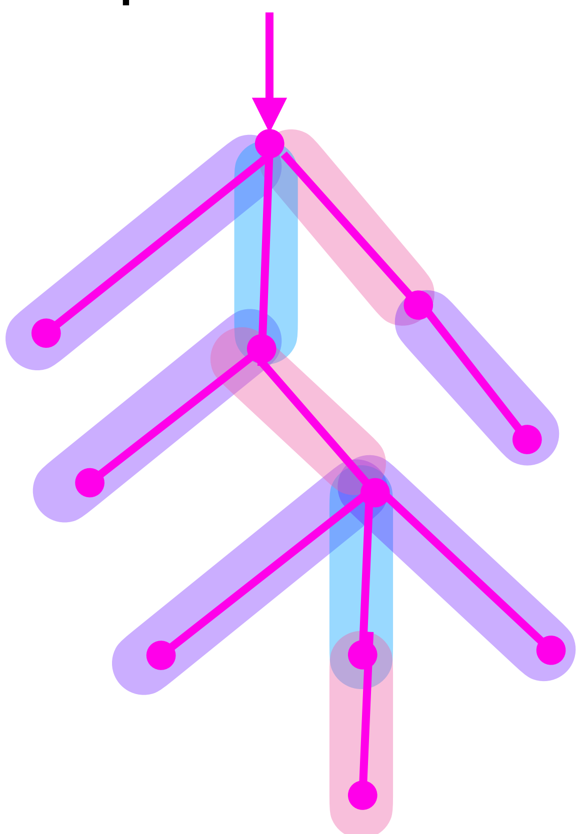


Only small blocks.

Interpolating model using blocks!

Model

Inspired by [Bonzom 2016];
General setting in [Stufler 2020].

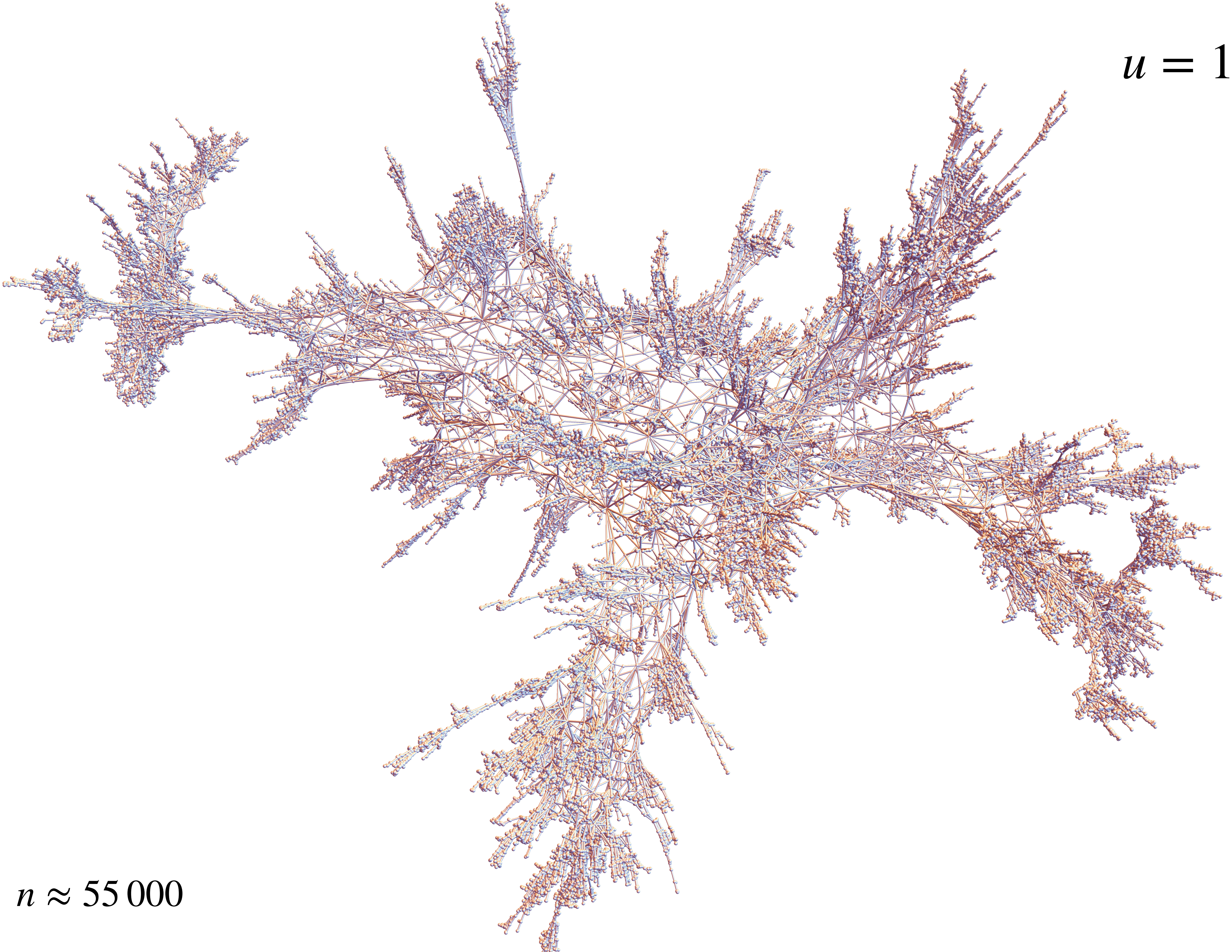
Goal: parameter that affects the typical number of blocks.

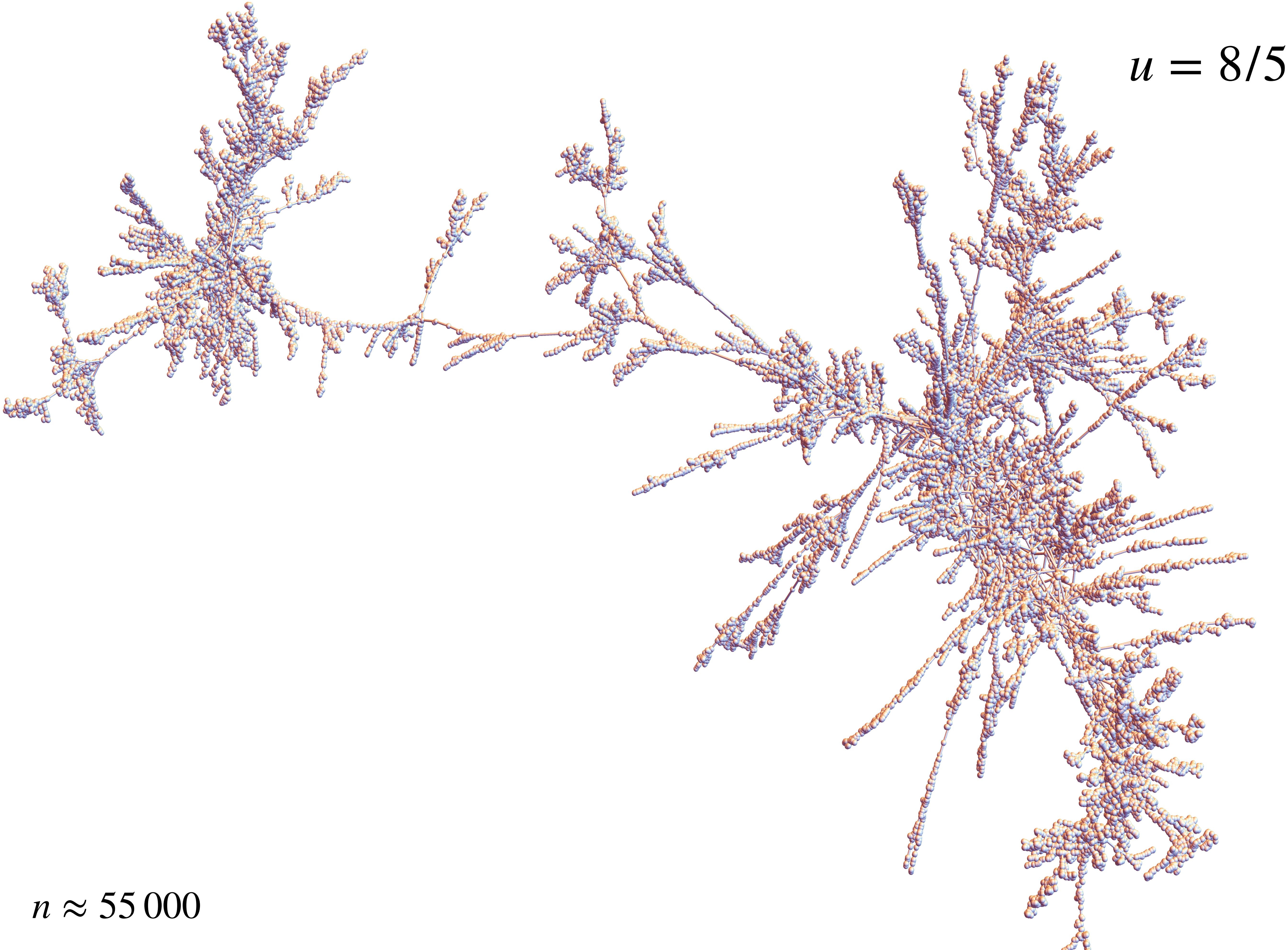
We choose: $\mathbb{P}_{n,u}(\mathfrak{m}) = \dfrac{u^{\#blocks(\mathfrak{m})}}{Z_{n,u}}$ where

$u > 0$,
$\mathcal{M}_n$ = {maps of size n},
$\mathfrak{m} \in \mathcal{M}_n$,
$Z_{n,u}$ = normalisation.

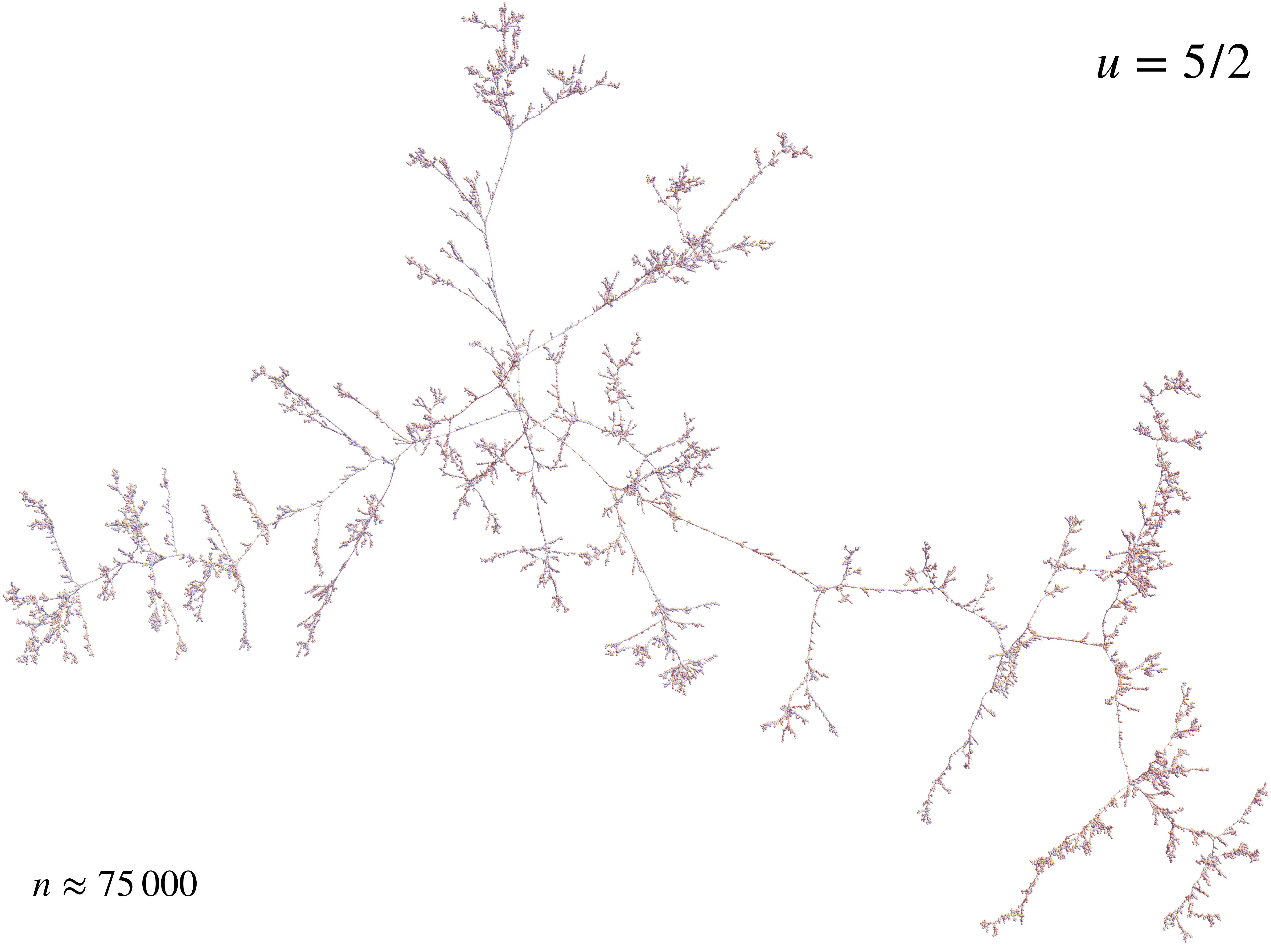
- $u = 1$: uniform distribution on maps of size n;
- $u \to 0$: minimising the number of blocks (=2-connected maps);
- $u \to \infty$: maximising the number of blocks (= trees!).

Given u, asymptotic behaviour when $n \to \infty$?


$u = 1$
$n \approx 55\,000$


$u = 8/5$
$n \approx 55\,000$


$u = 9/5$
$n \approx 80\,000$

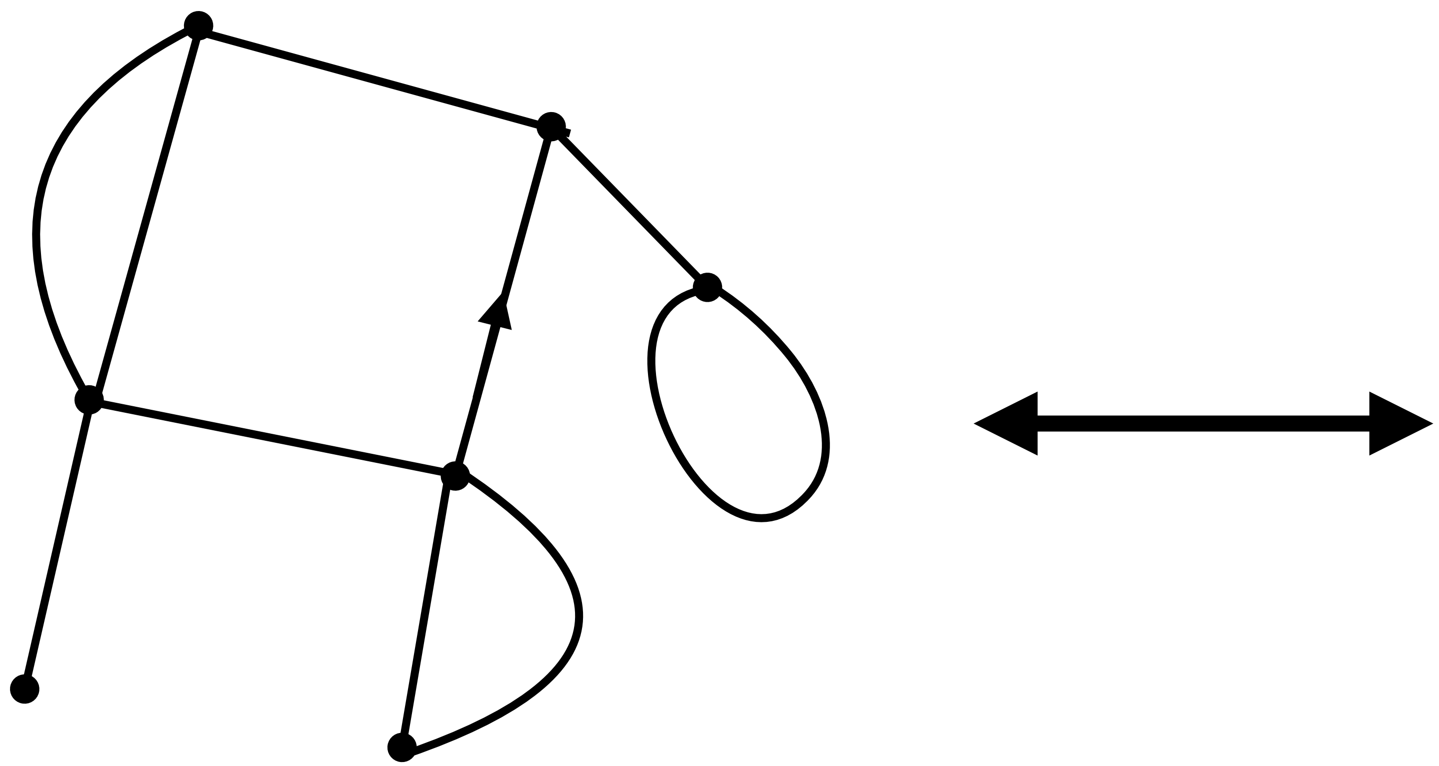

$u = 5/2$
$n \approx 75\,000$


$u = 5$
$n \approx 50\,000$

II. Block tree of a map

Decomposition of a map into blocks

Inspiration from [Tutte 1963]



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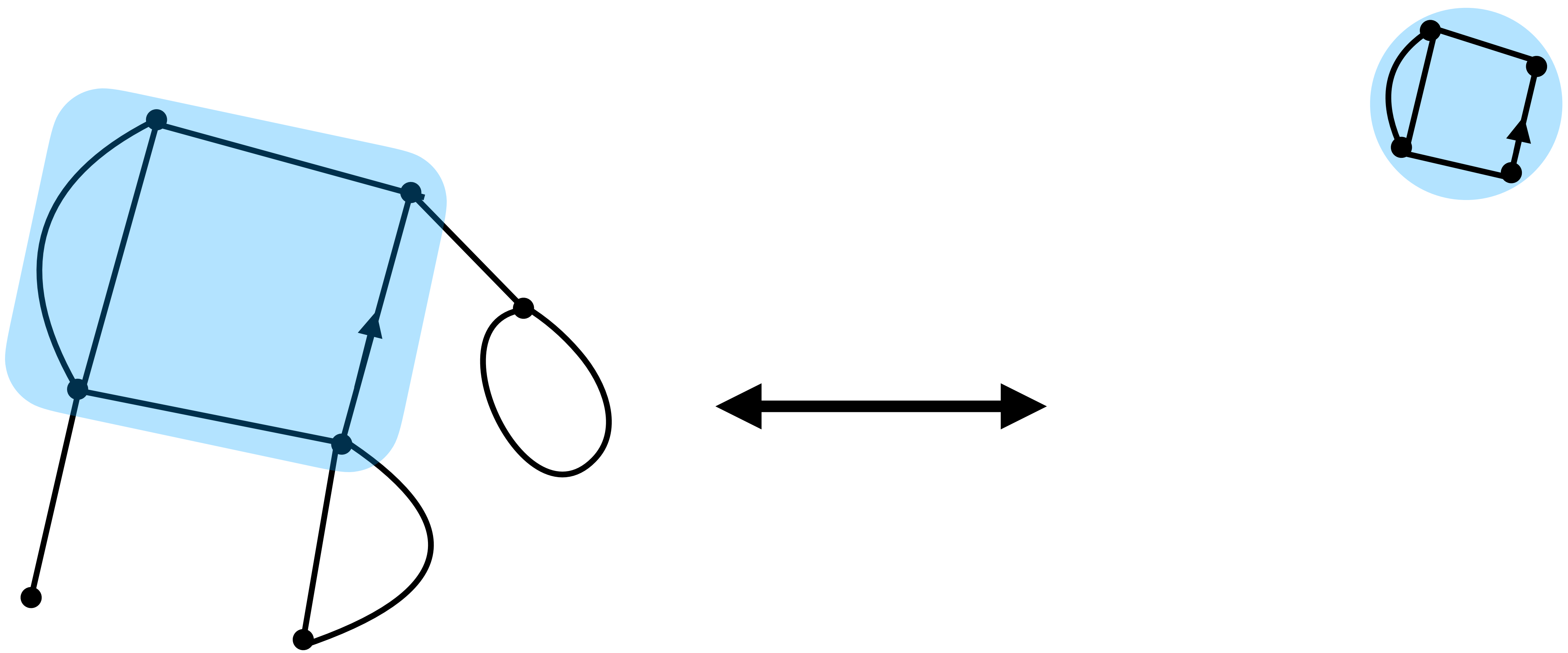

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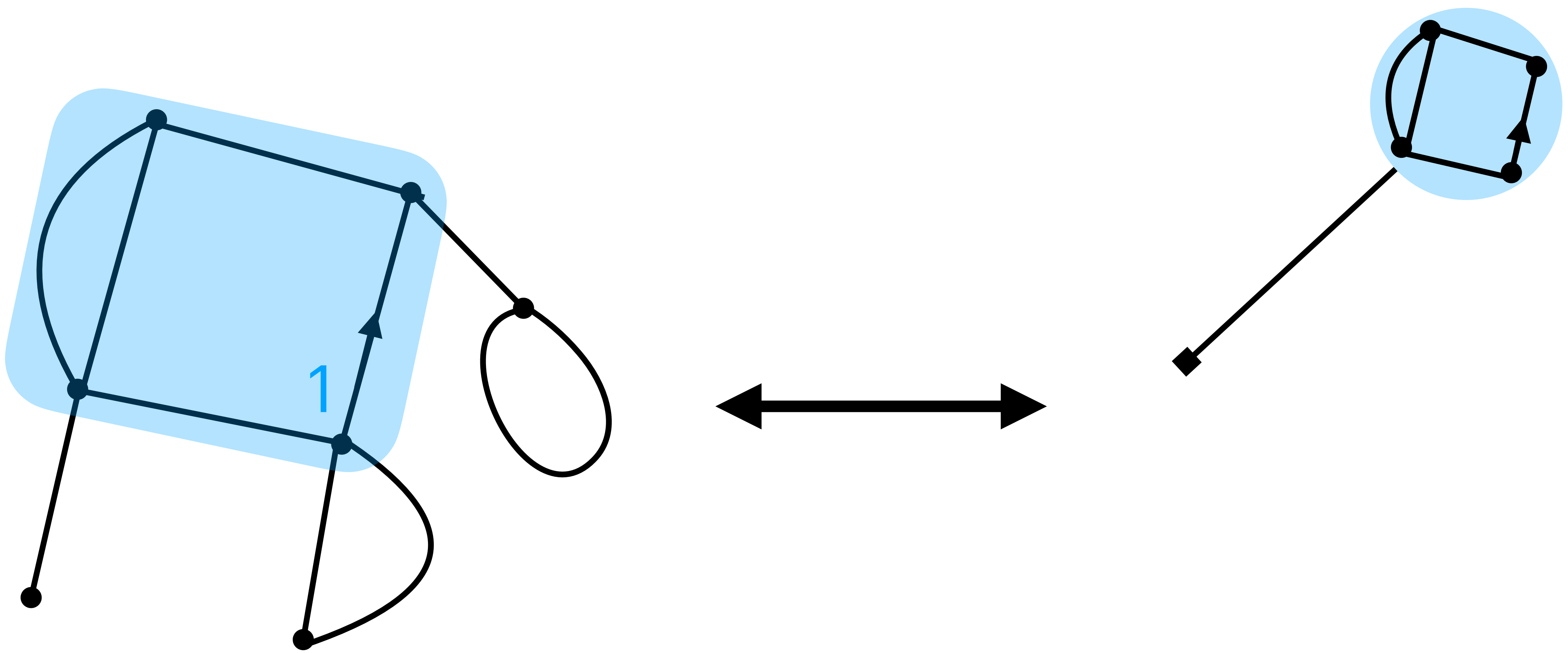

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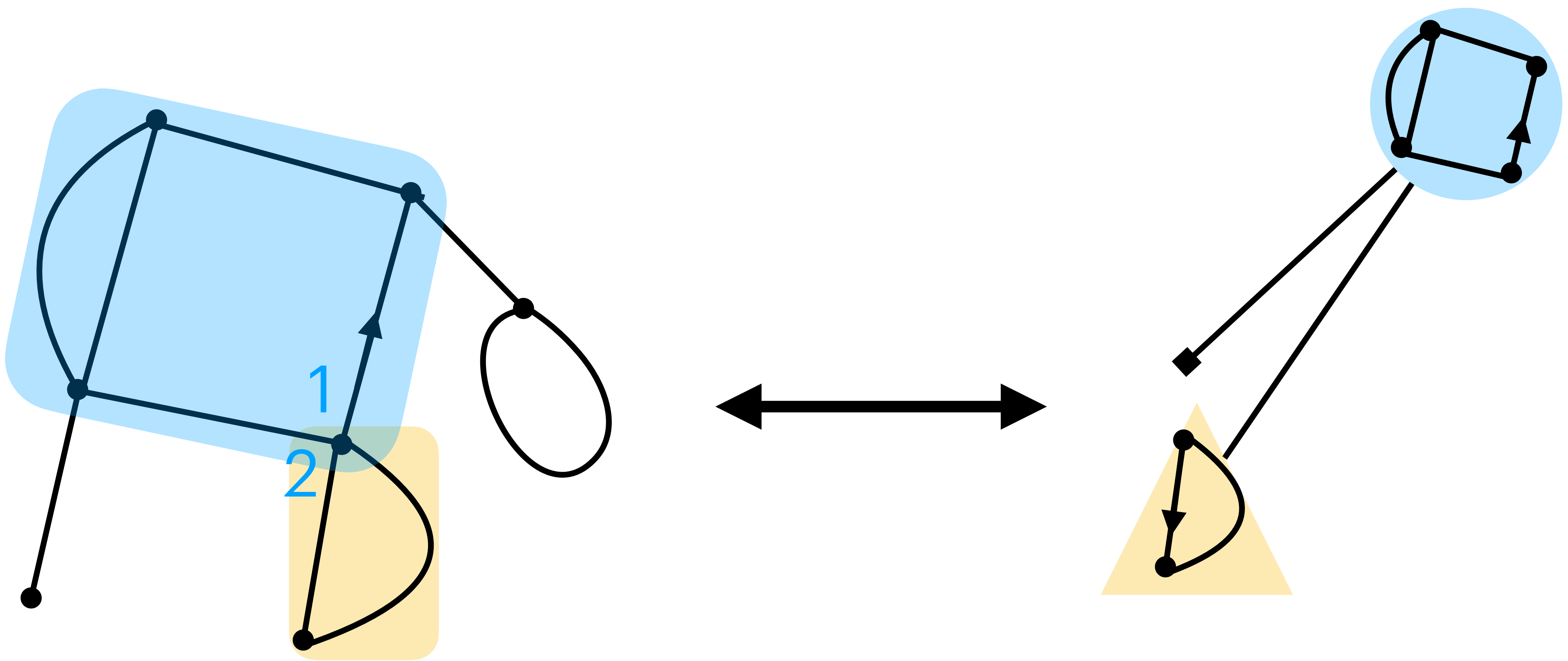

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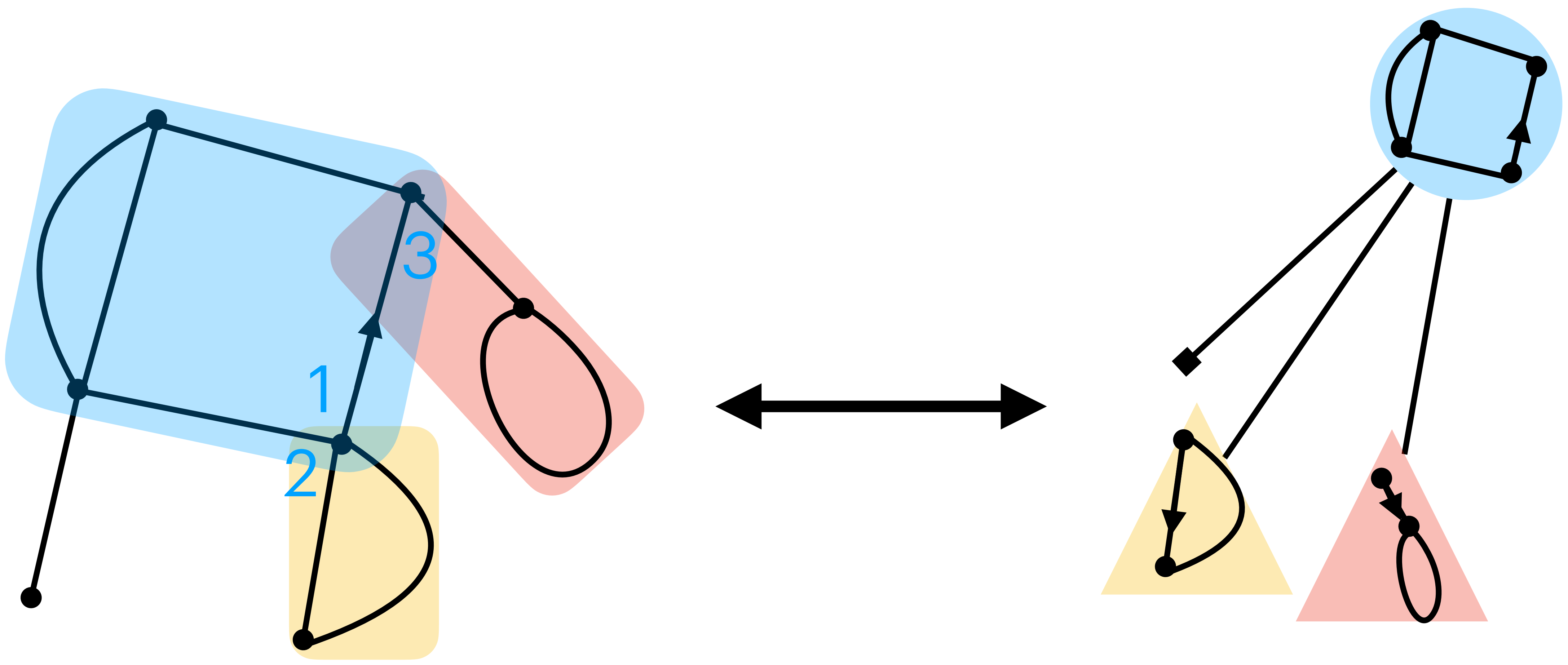

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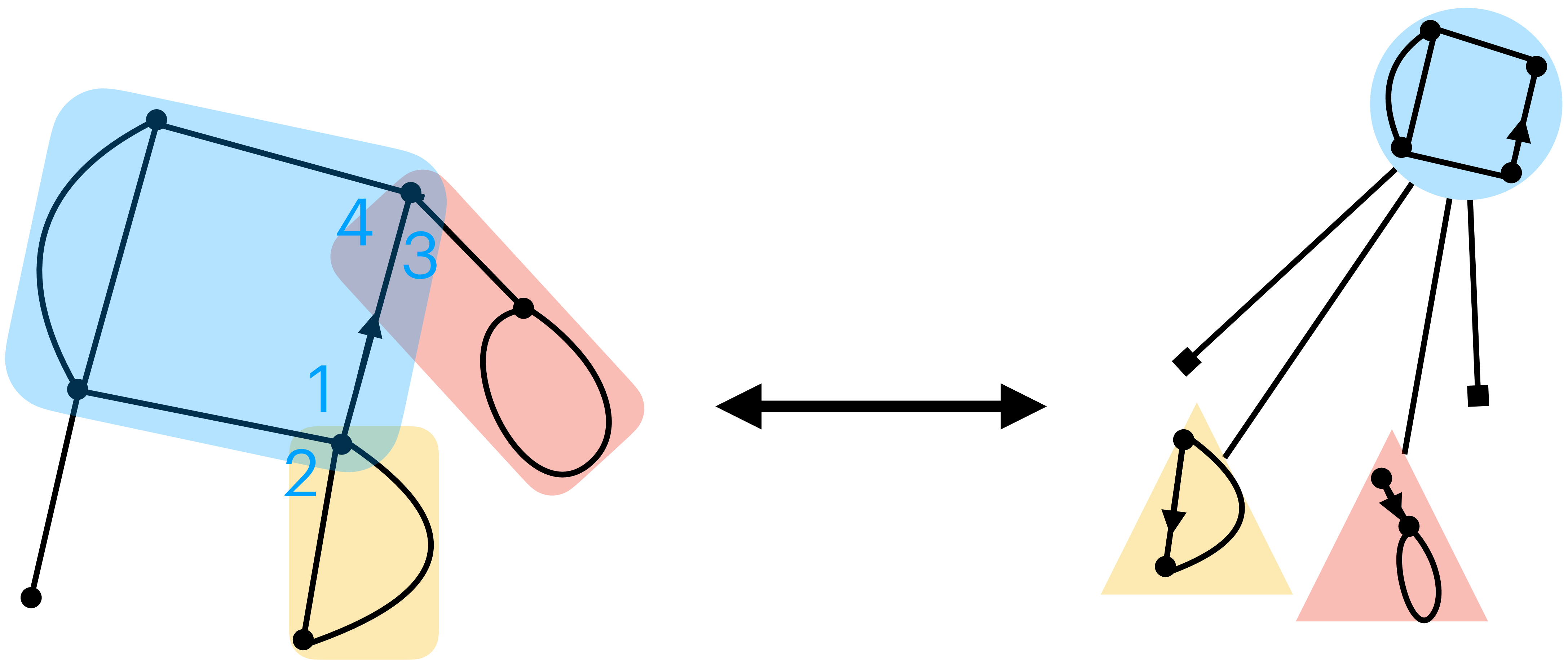

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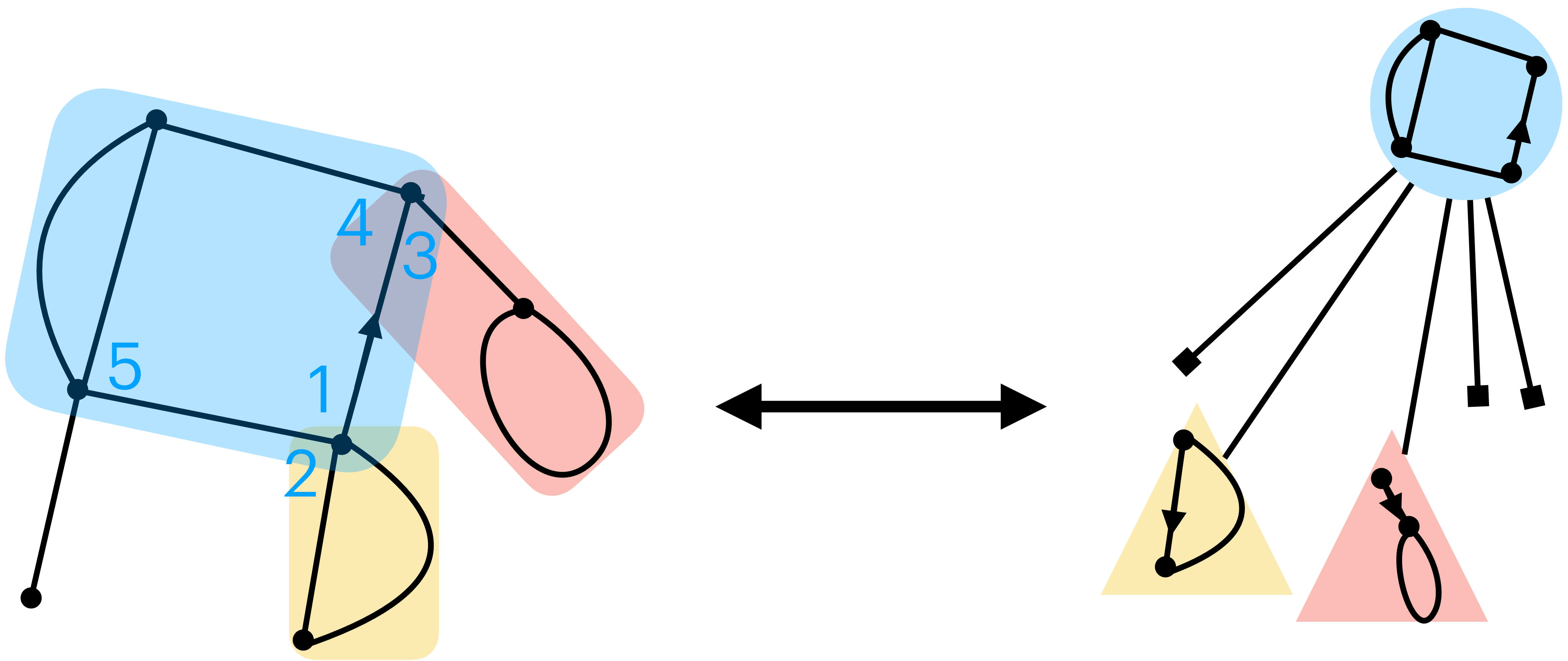

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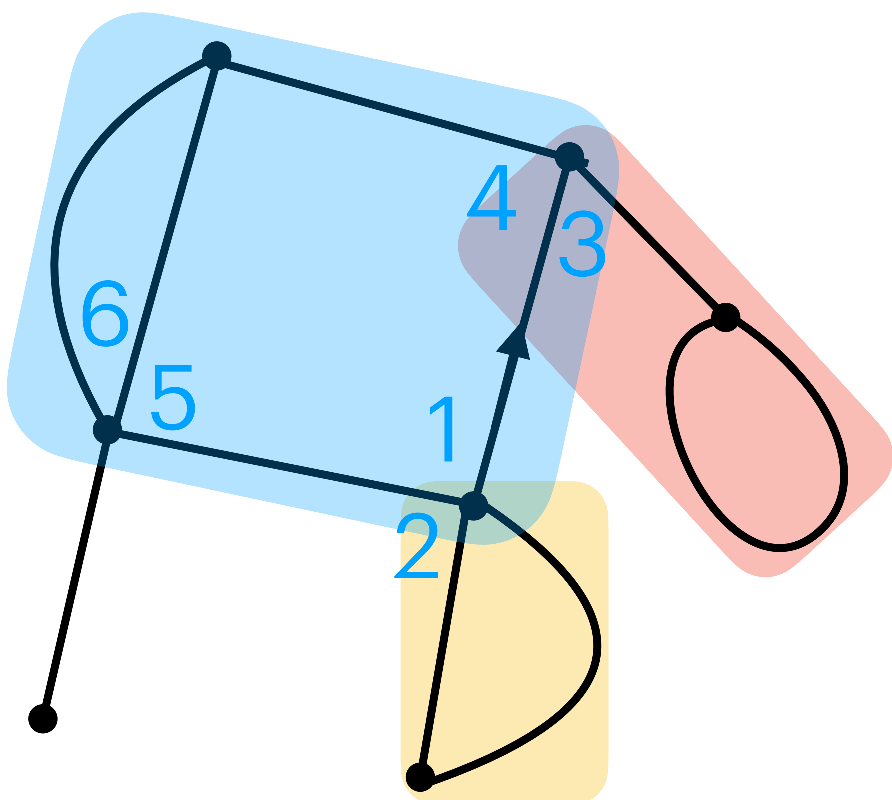

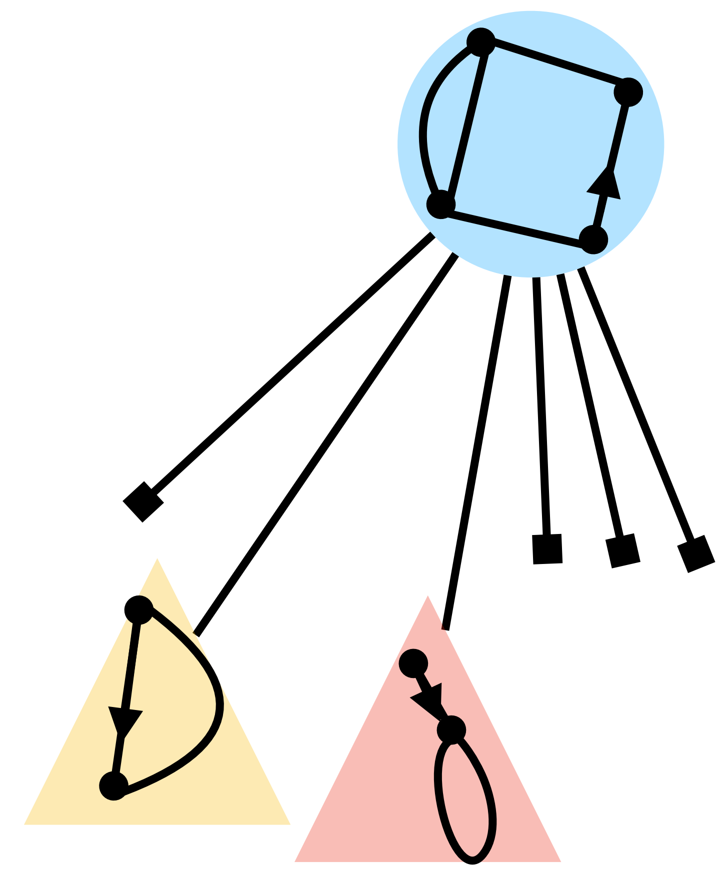

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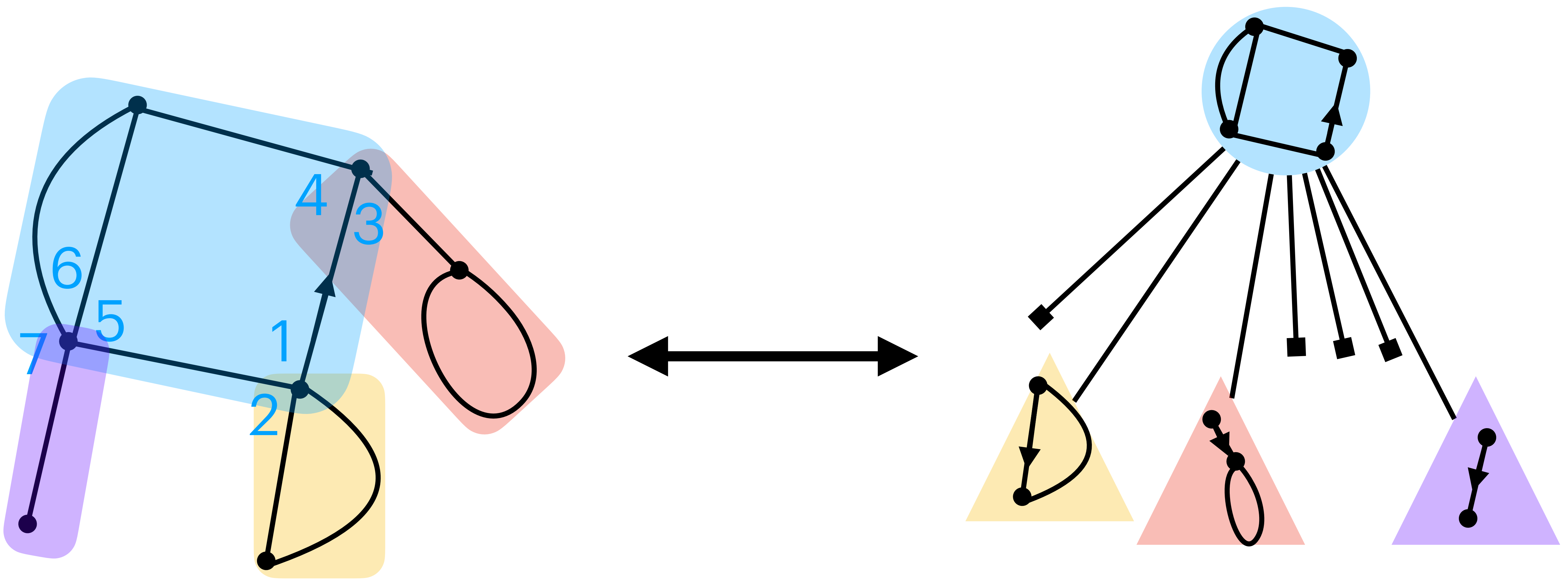

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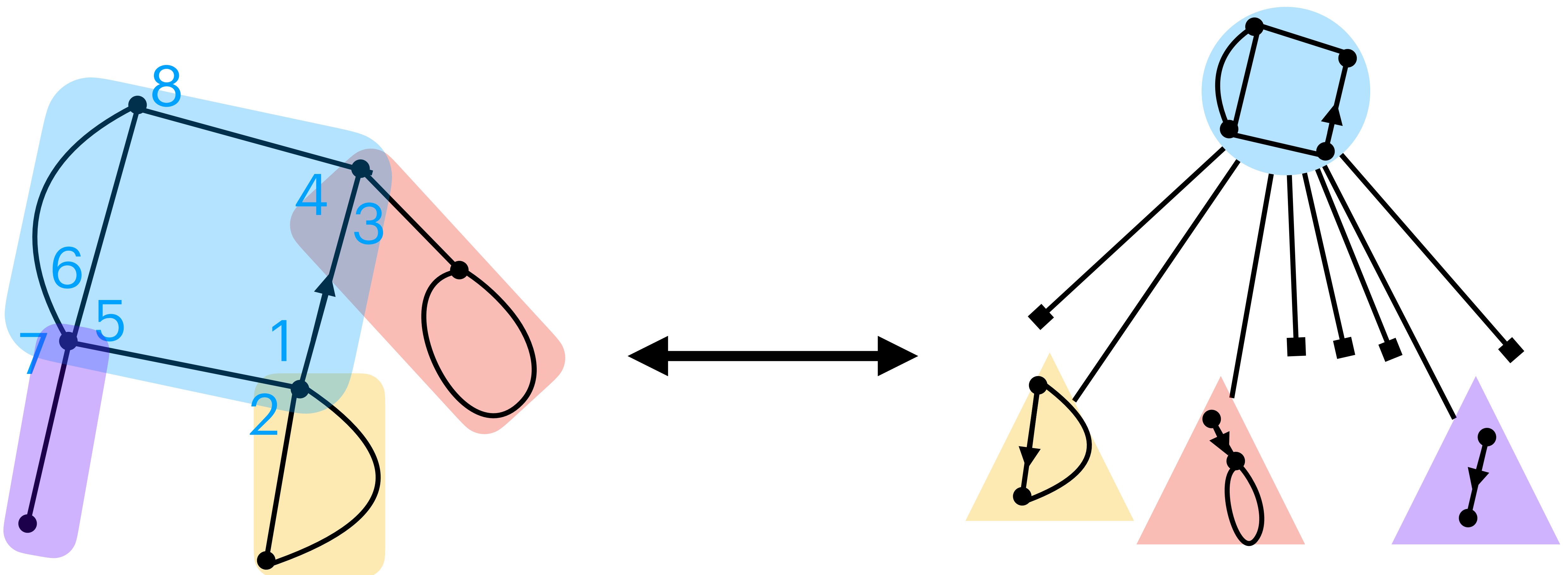

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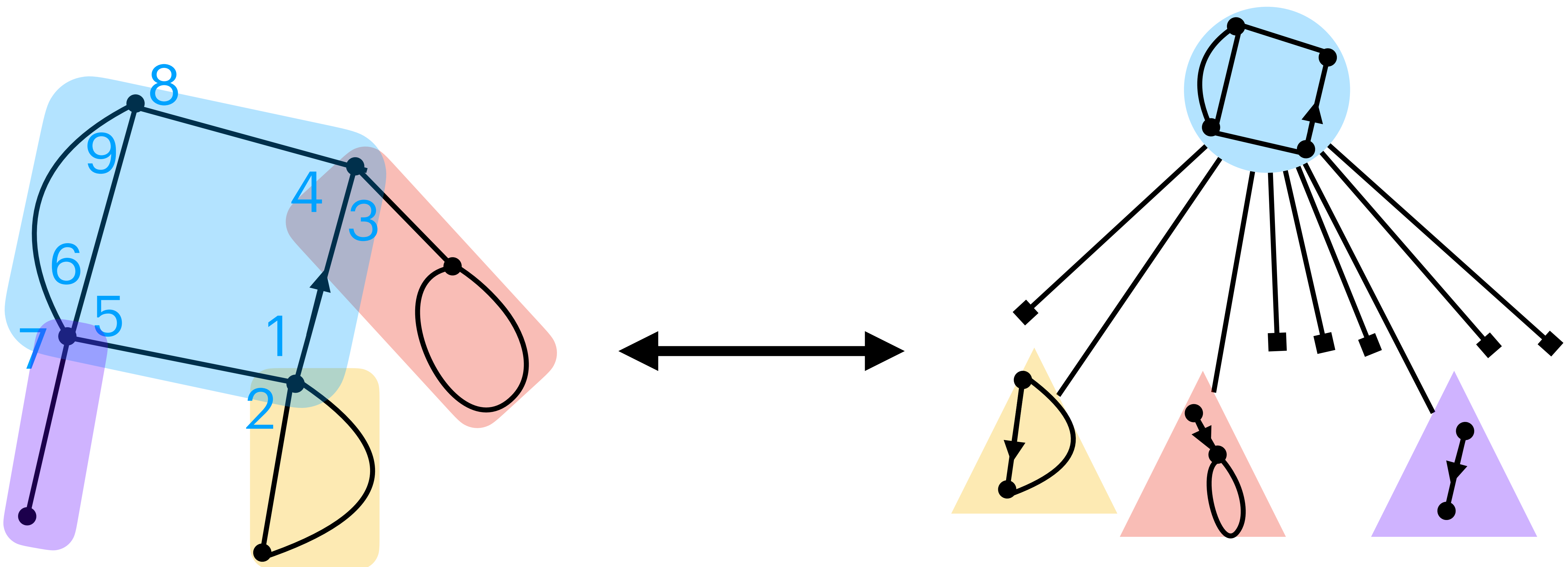

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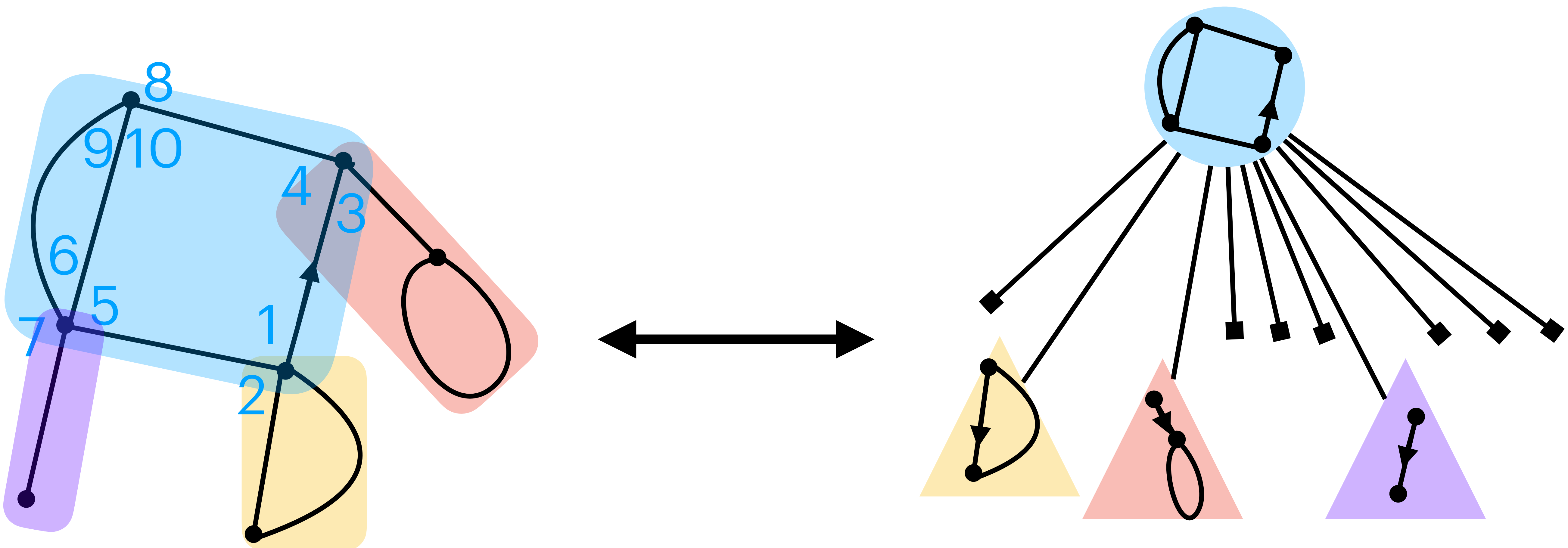

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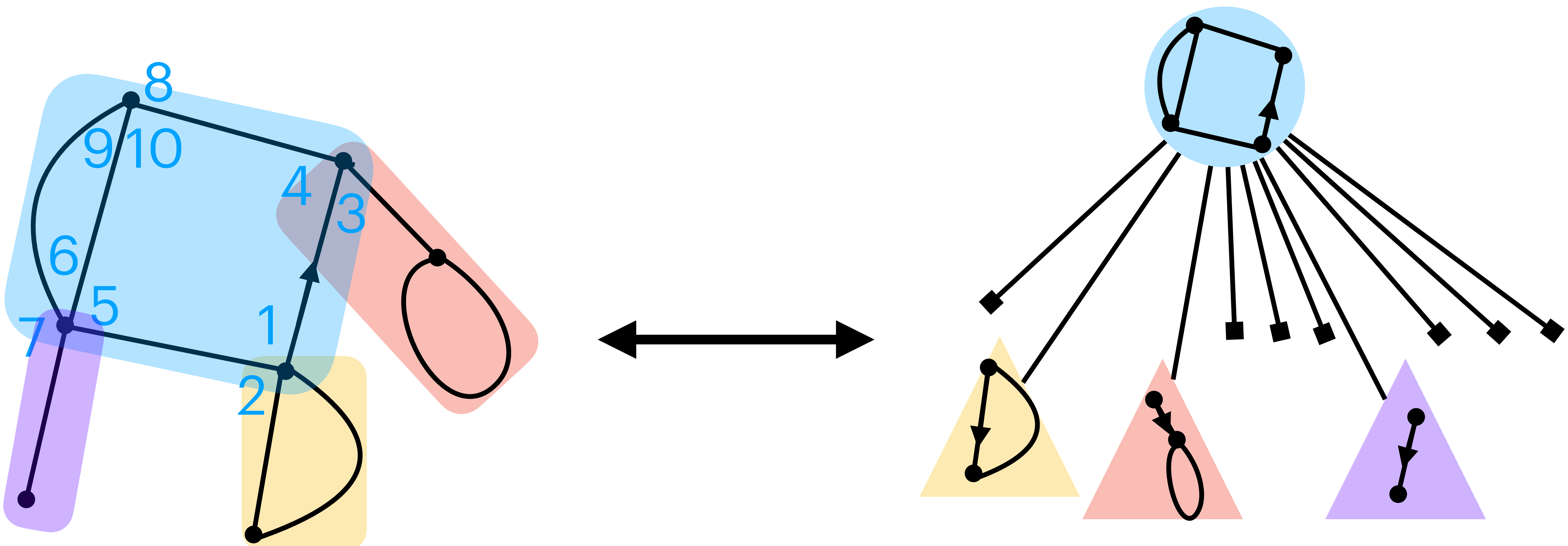

$$M(z) = B(zM^2(z))$$

GS of maps

GS of 2-connected maps

Decomposition of a map into blocks

$$M(z,u) = \sum_{\mathfrak{m}\in\mathscr{M}} z^{|\mathfrak{m}|} u^{\#blocks(\mathfrak{m})}$$

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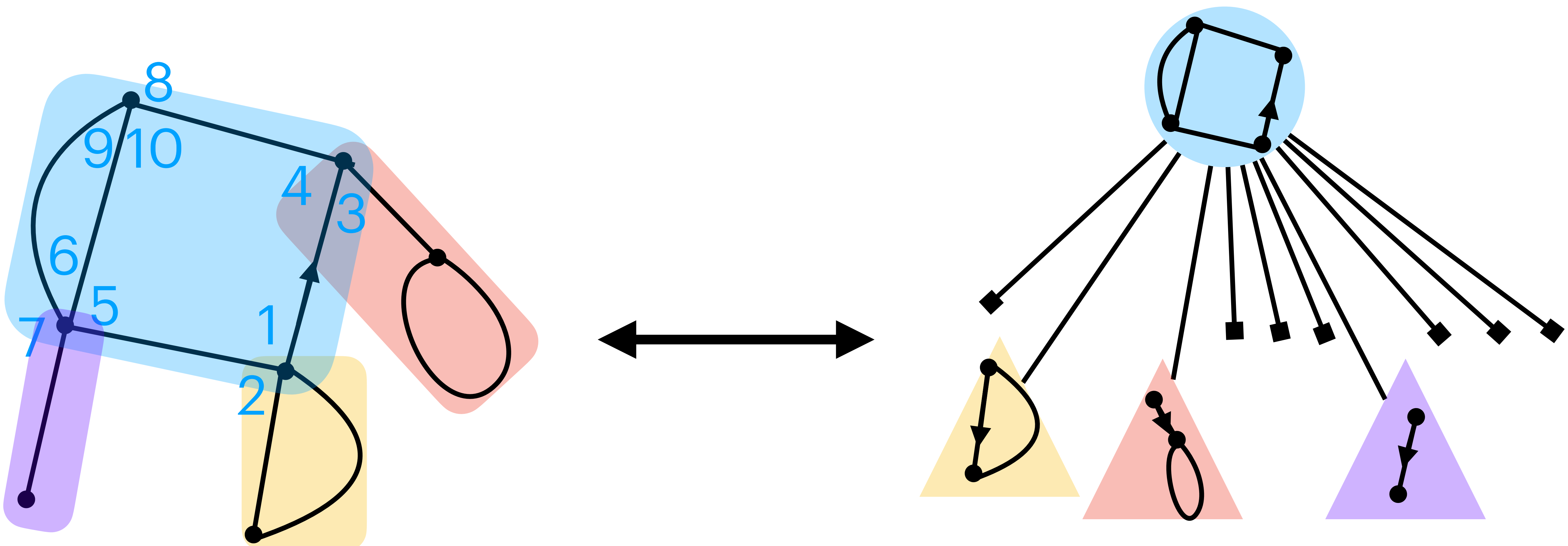

$$M(z) = B(zM^2(z))$$

GS of maps

GS of 2-connected maps

With a weight u on blocks: $M(z,u) = uB(zM^2(z,u)) + 1 - u$

Decomposition of a map into blocks

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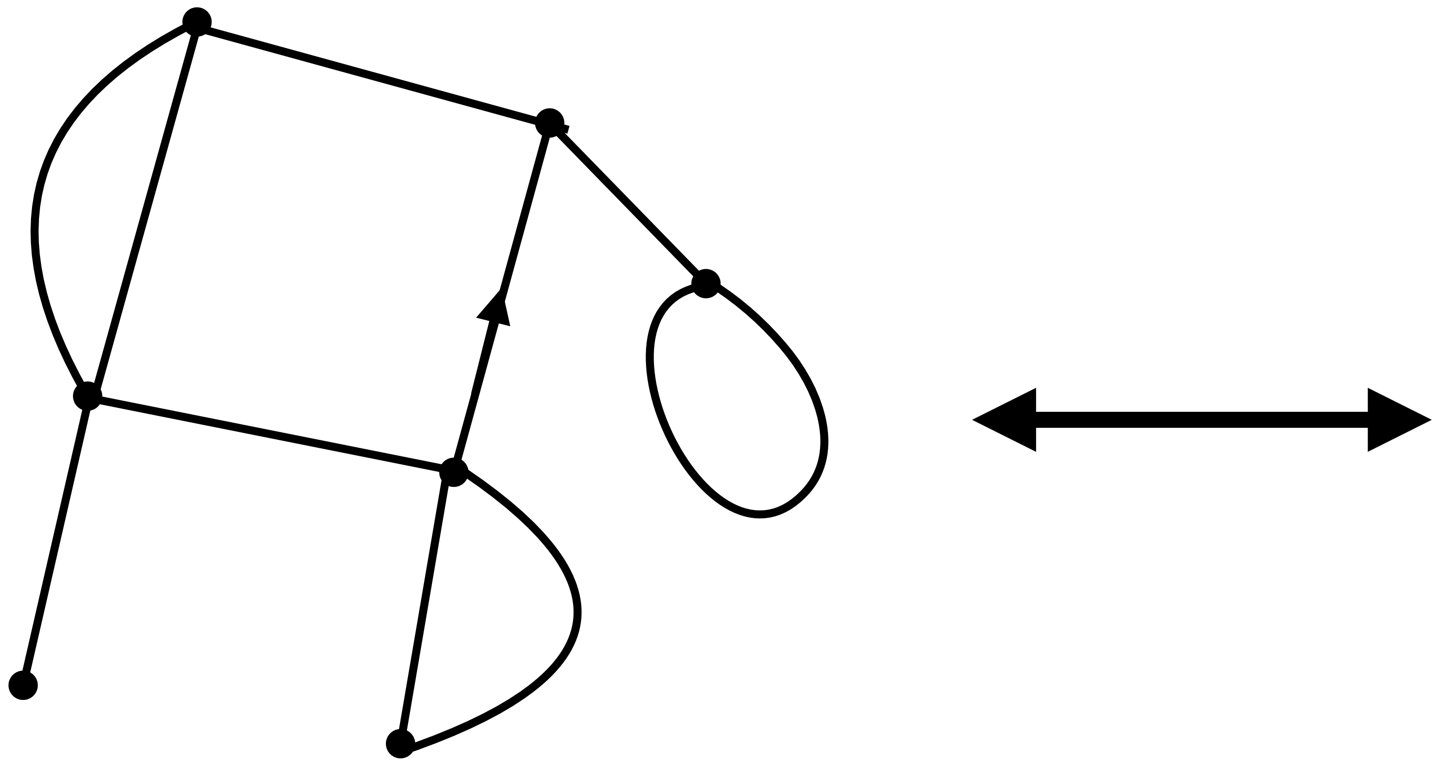

⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

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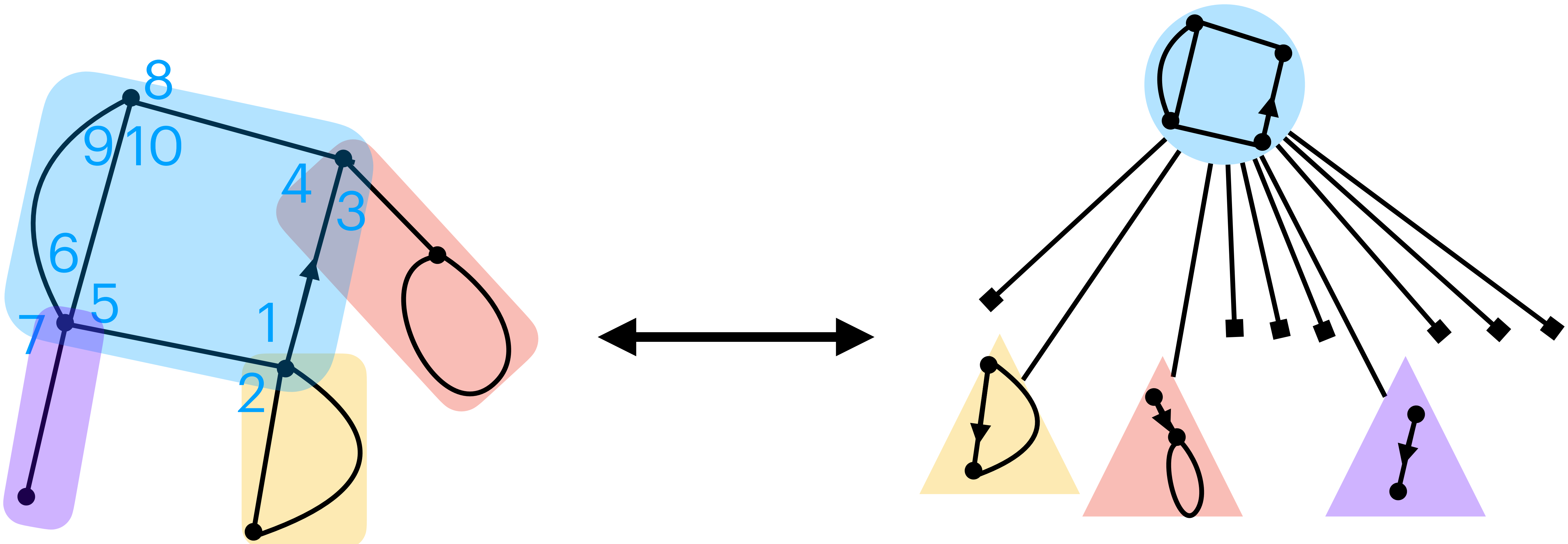

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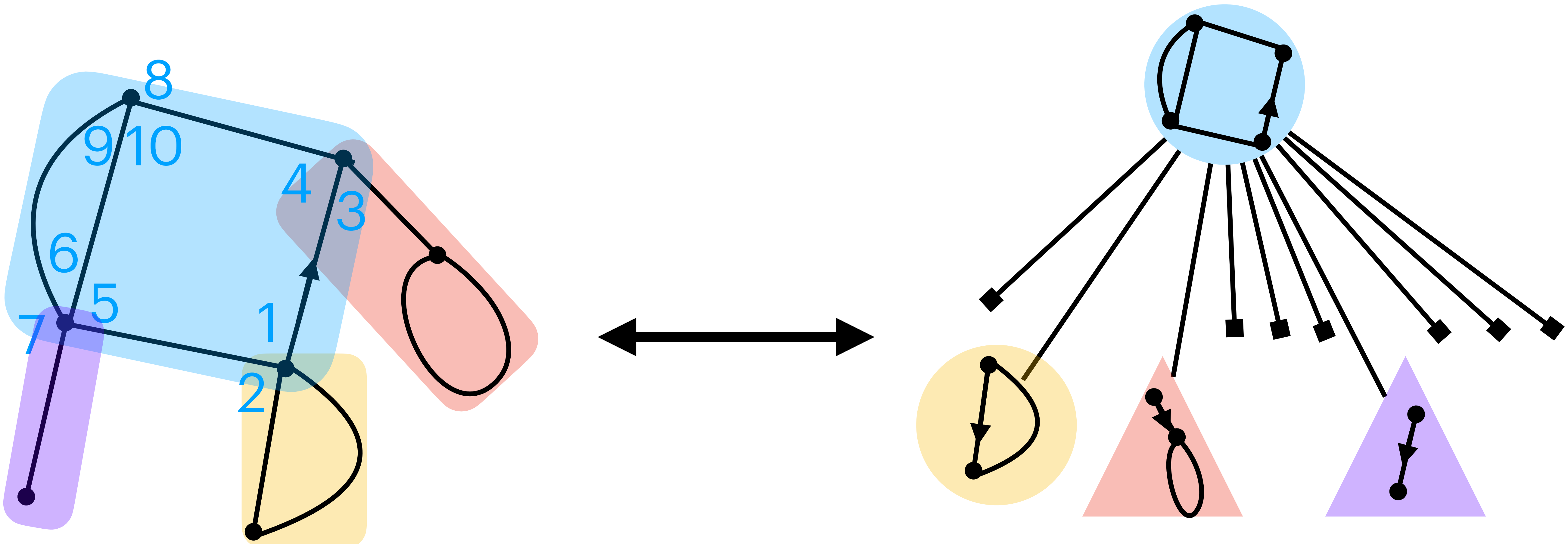

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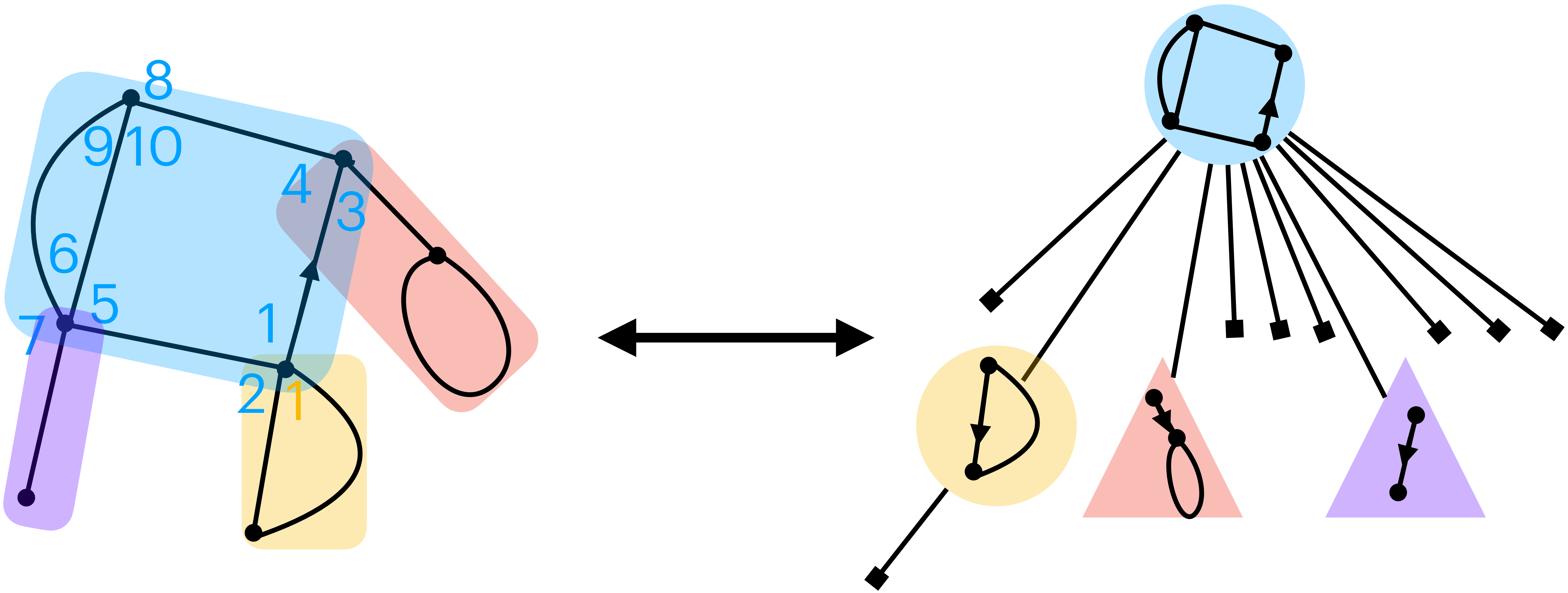

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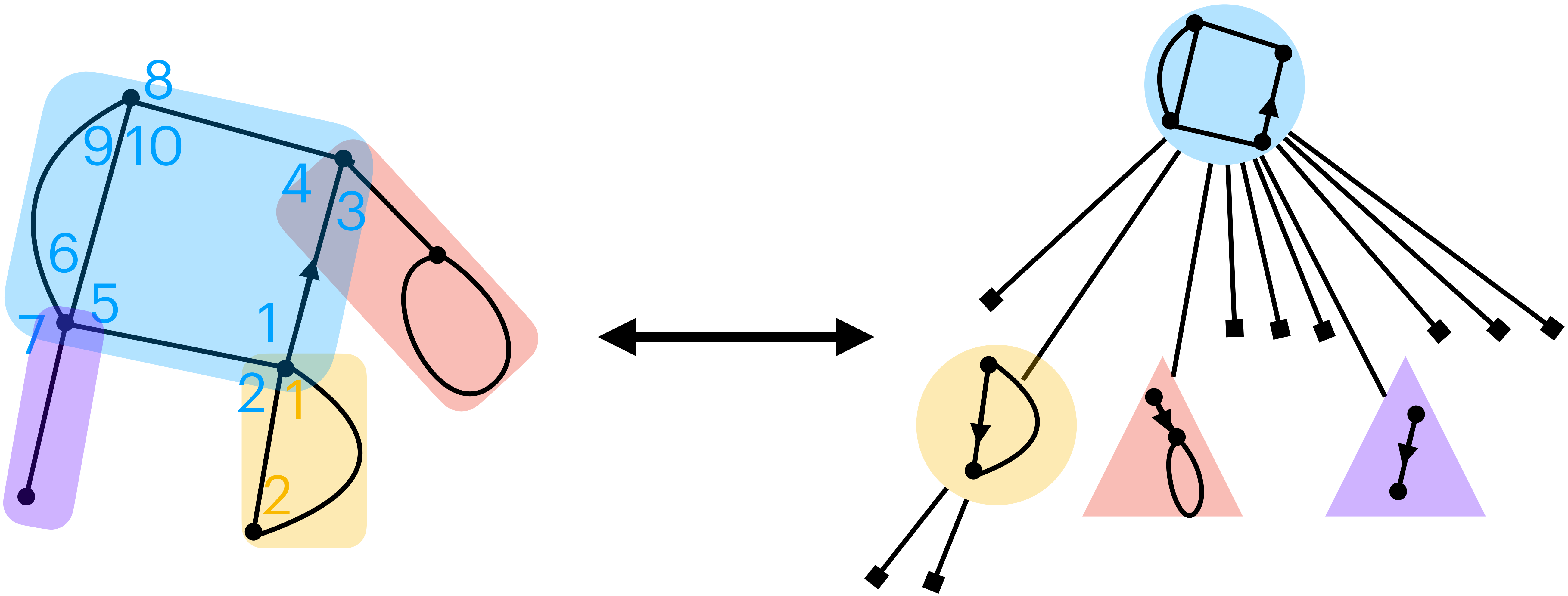

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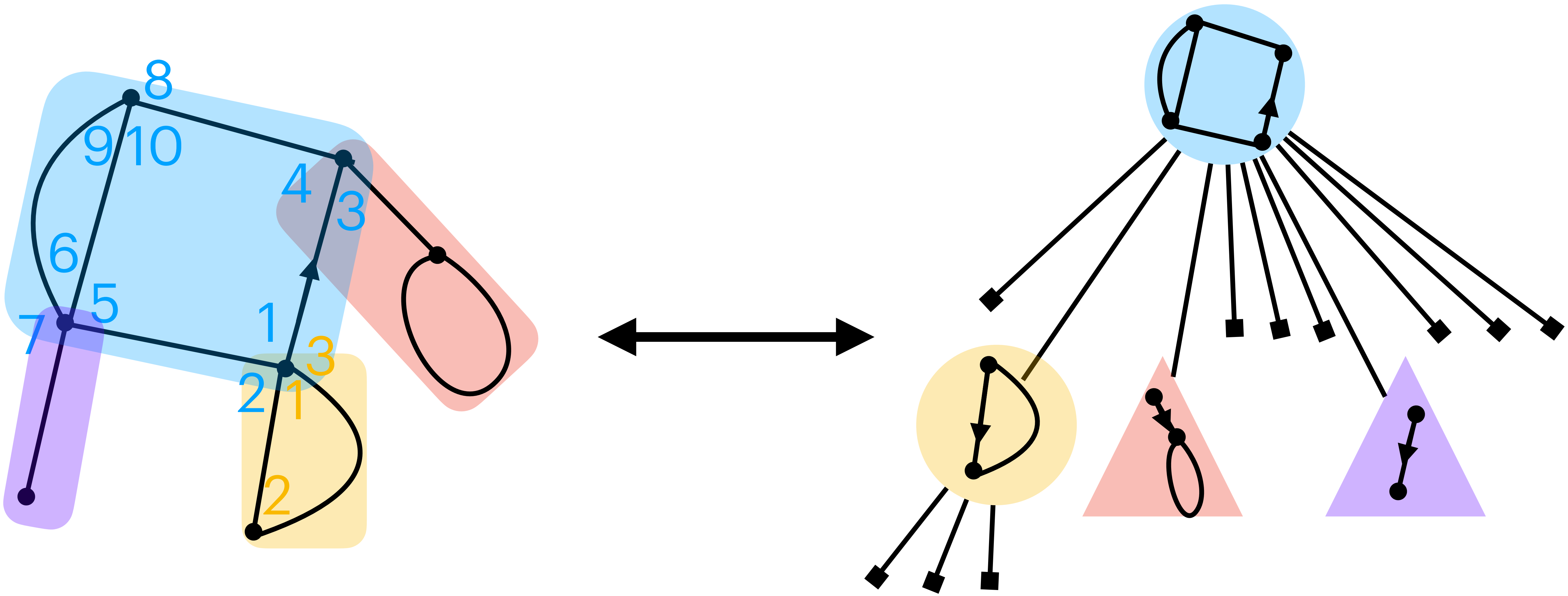

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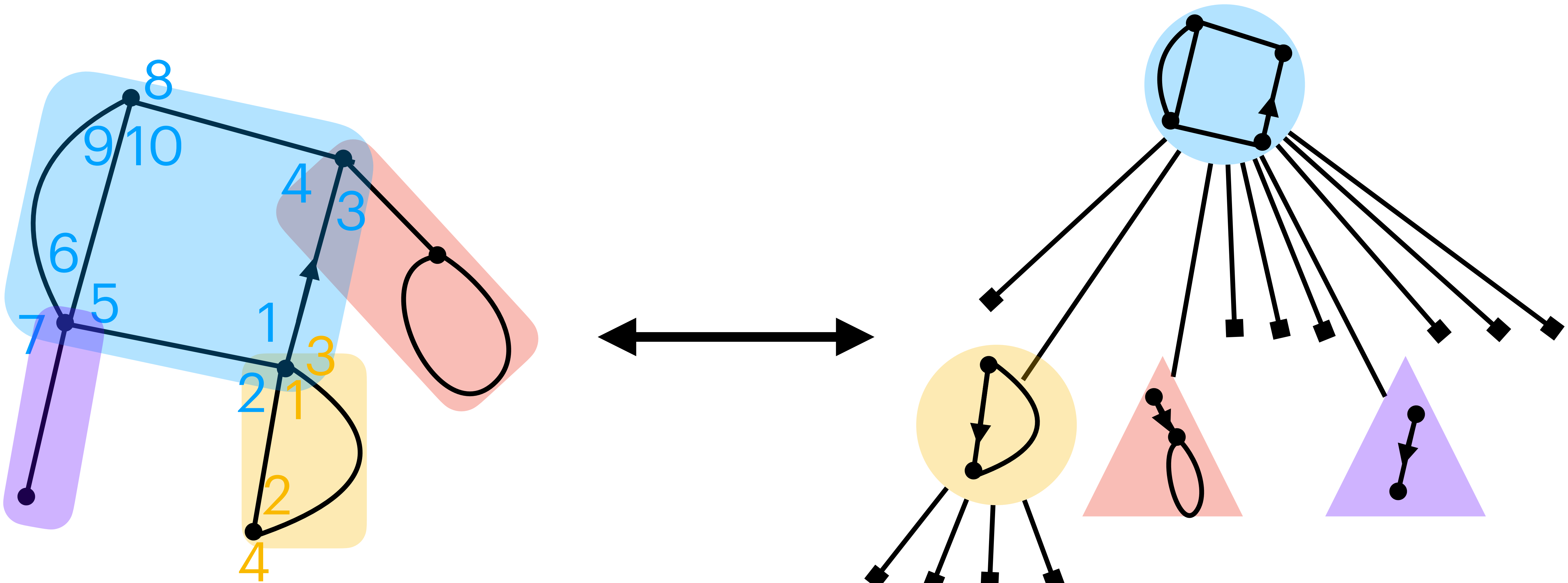

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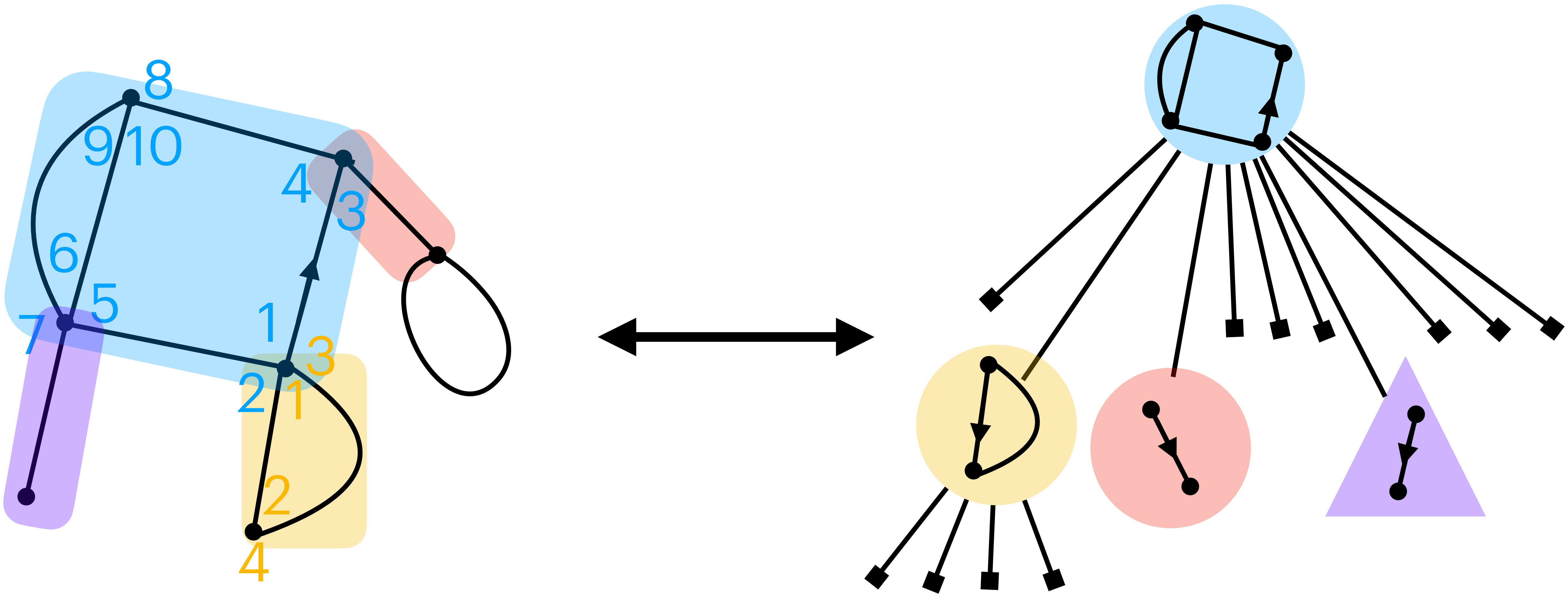

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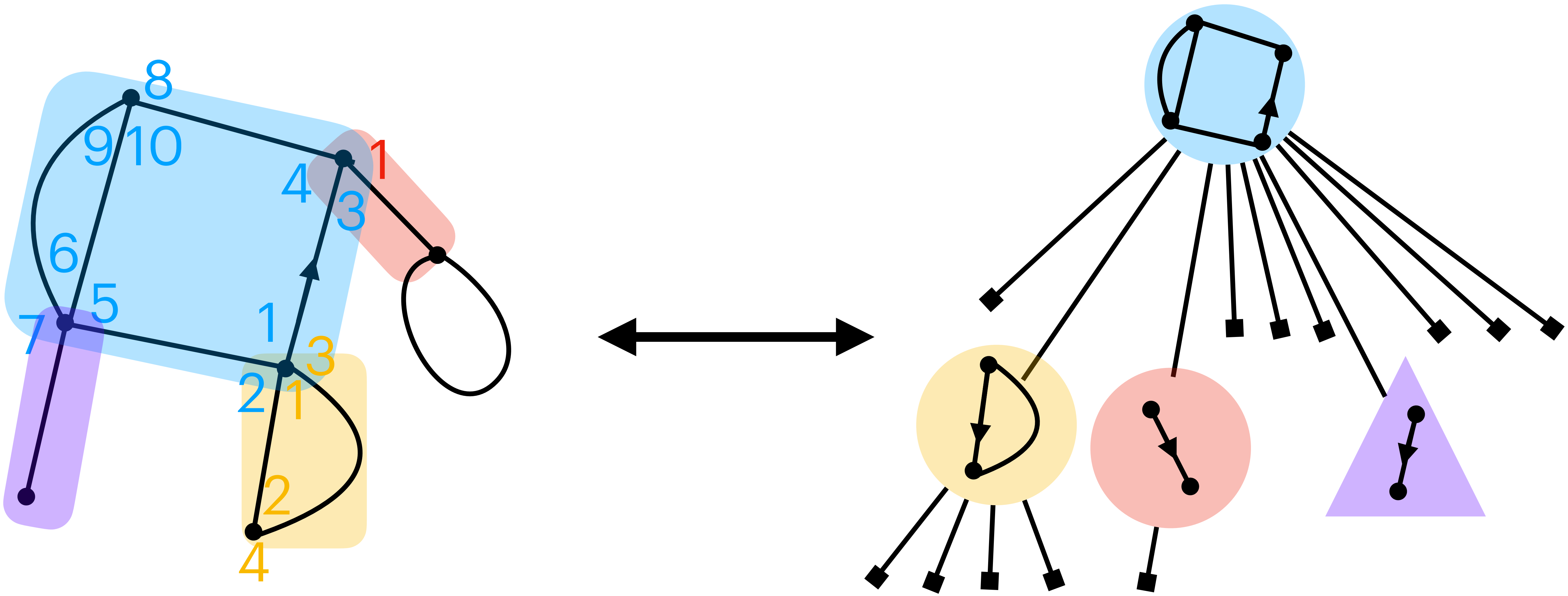

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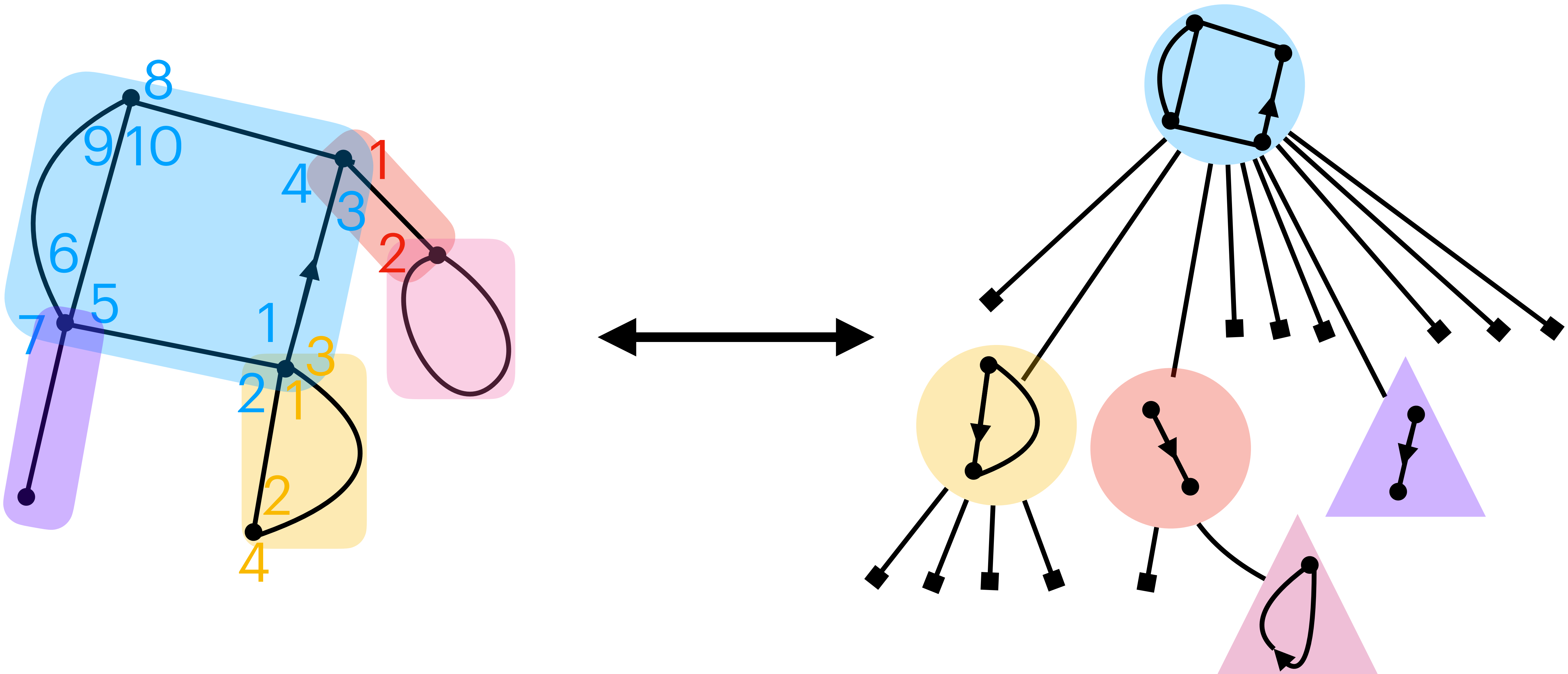

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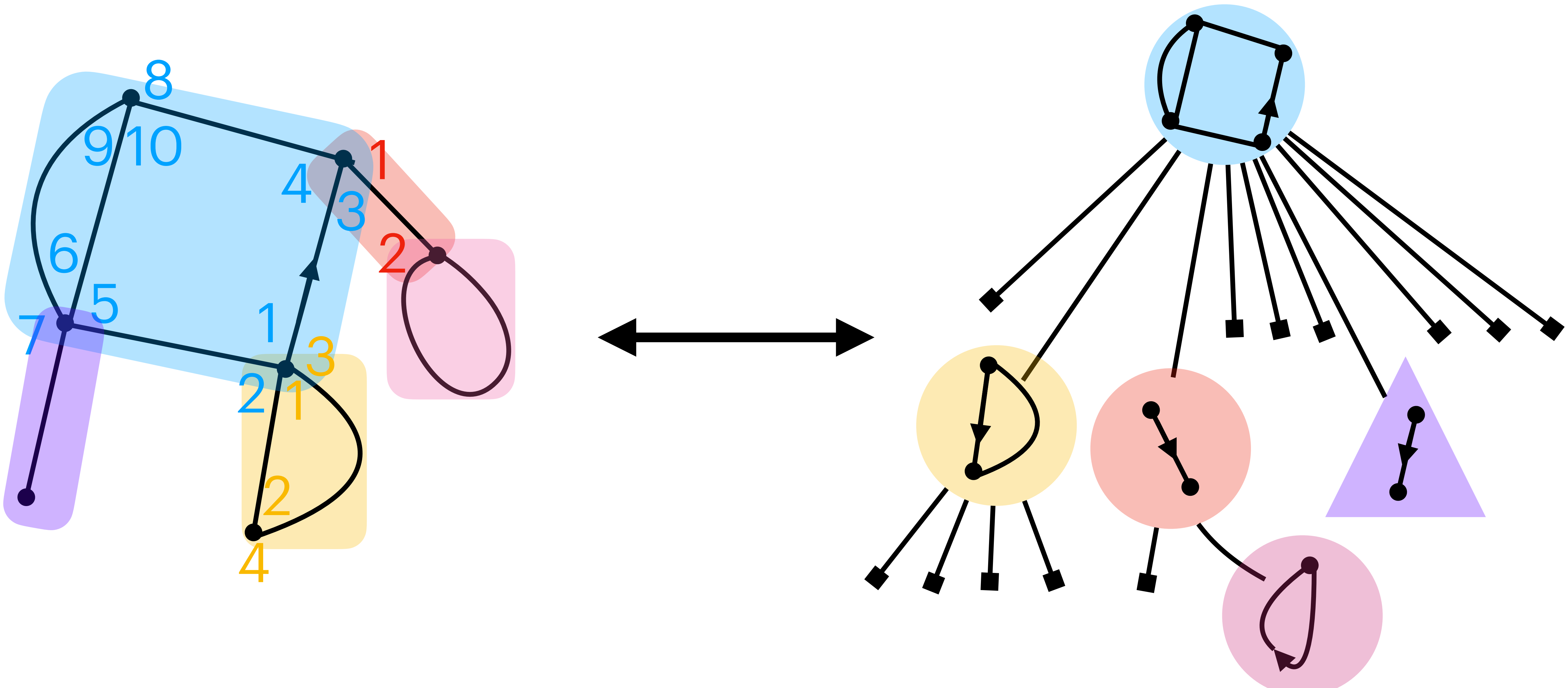

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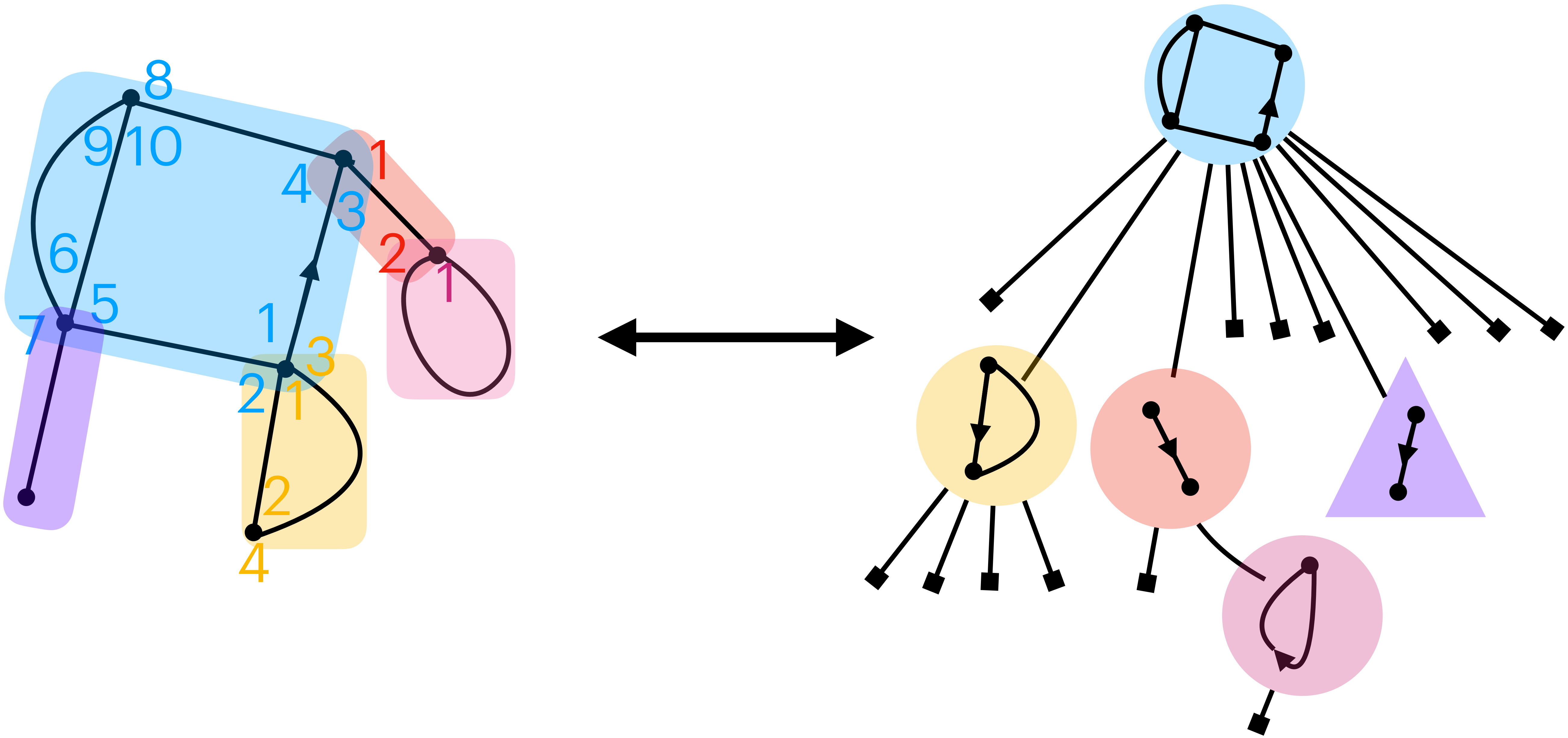

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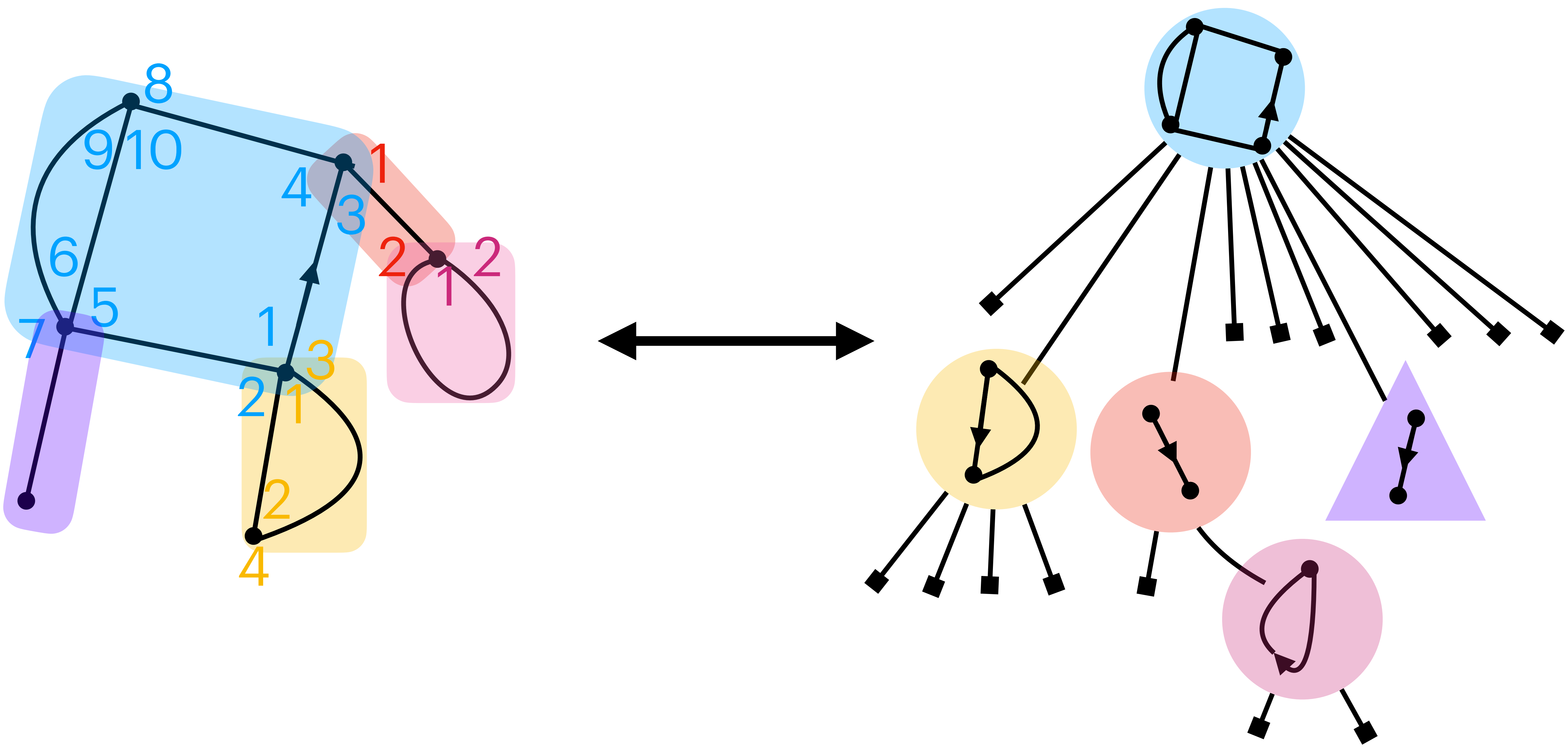

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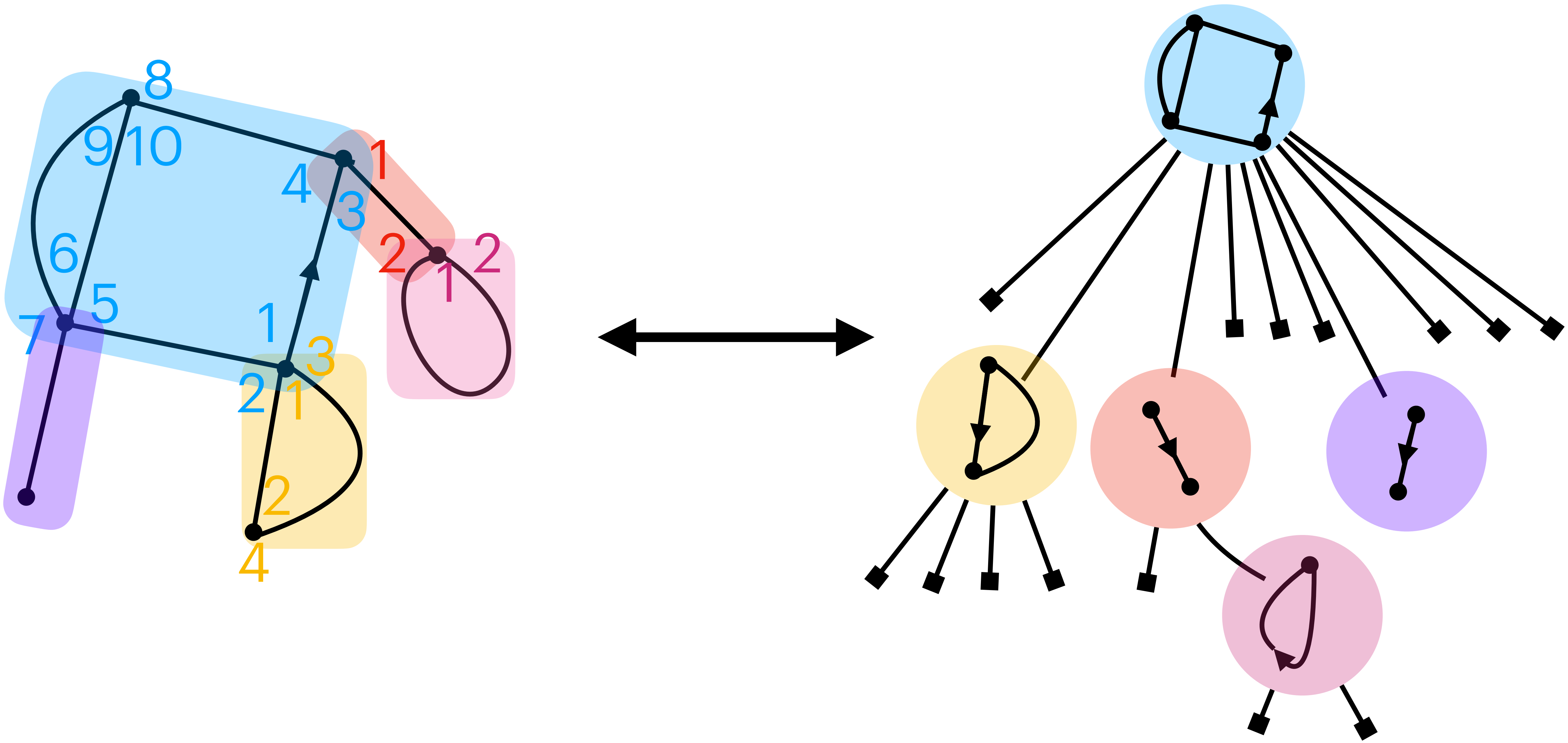

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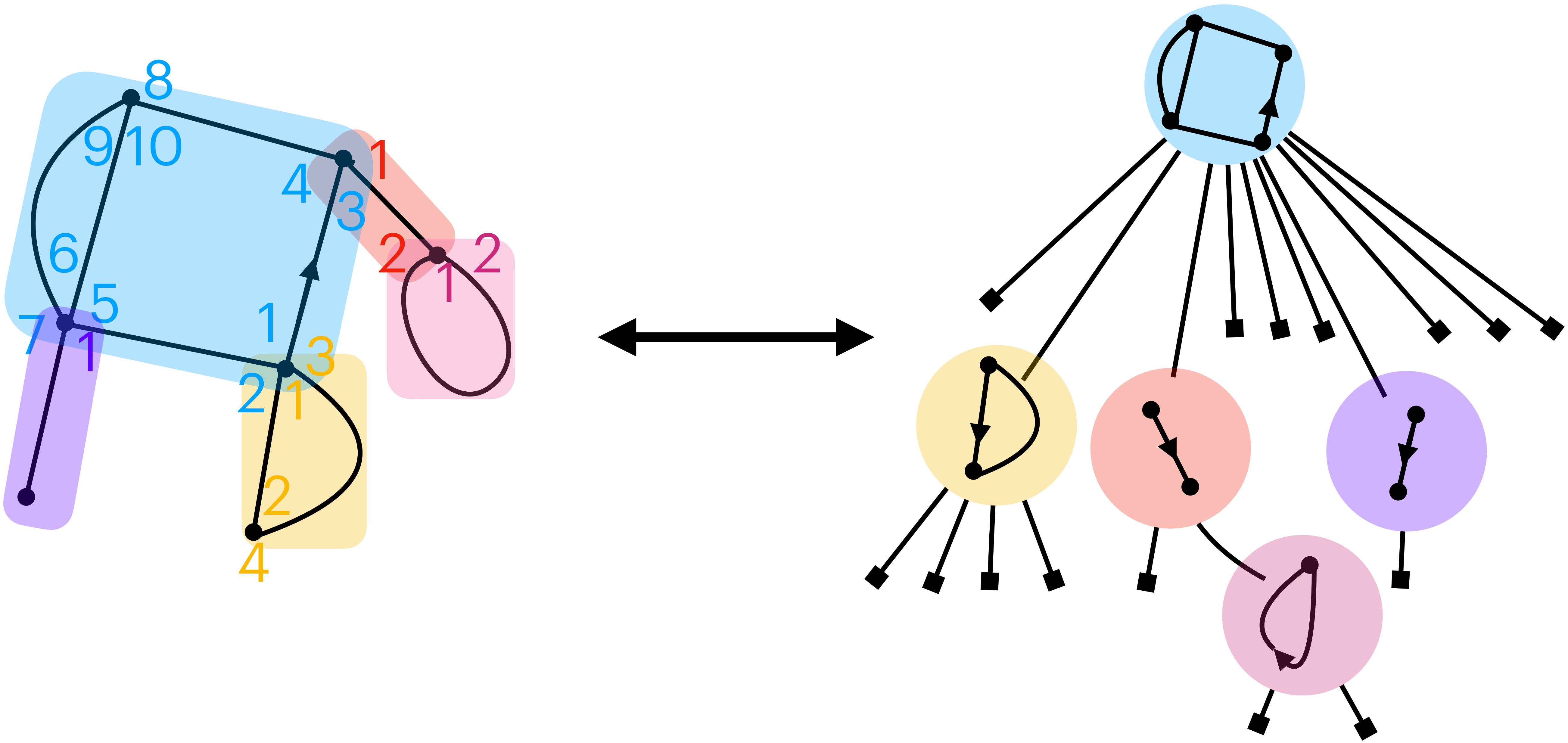

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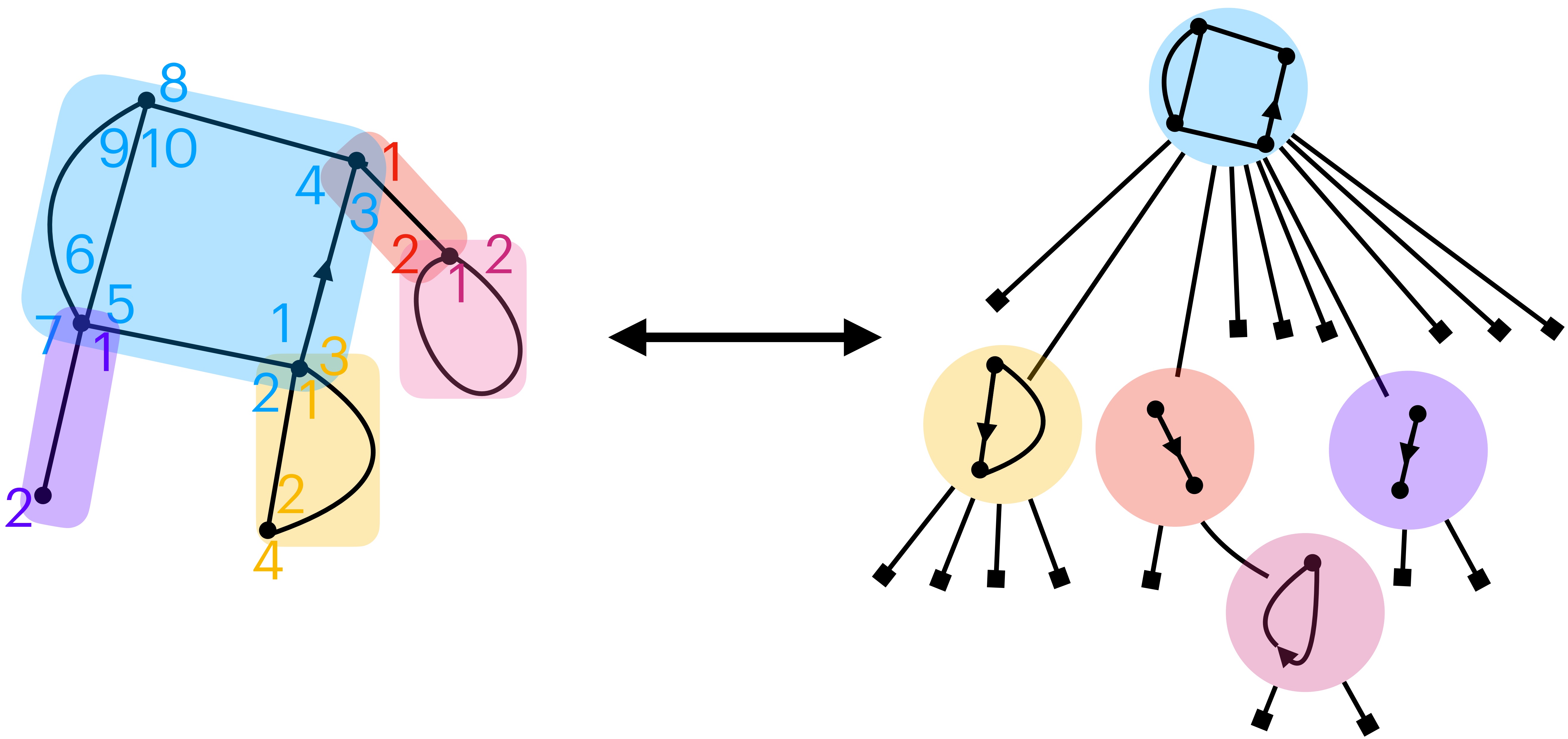

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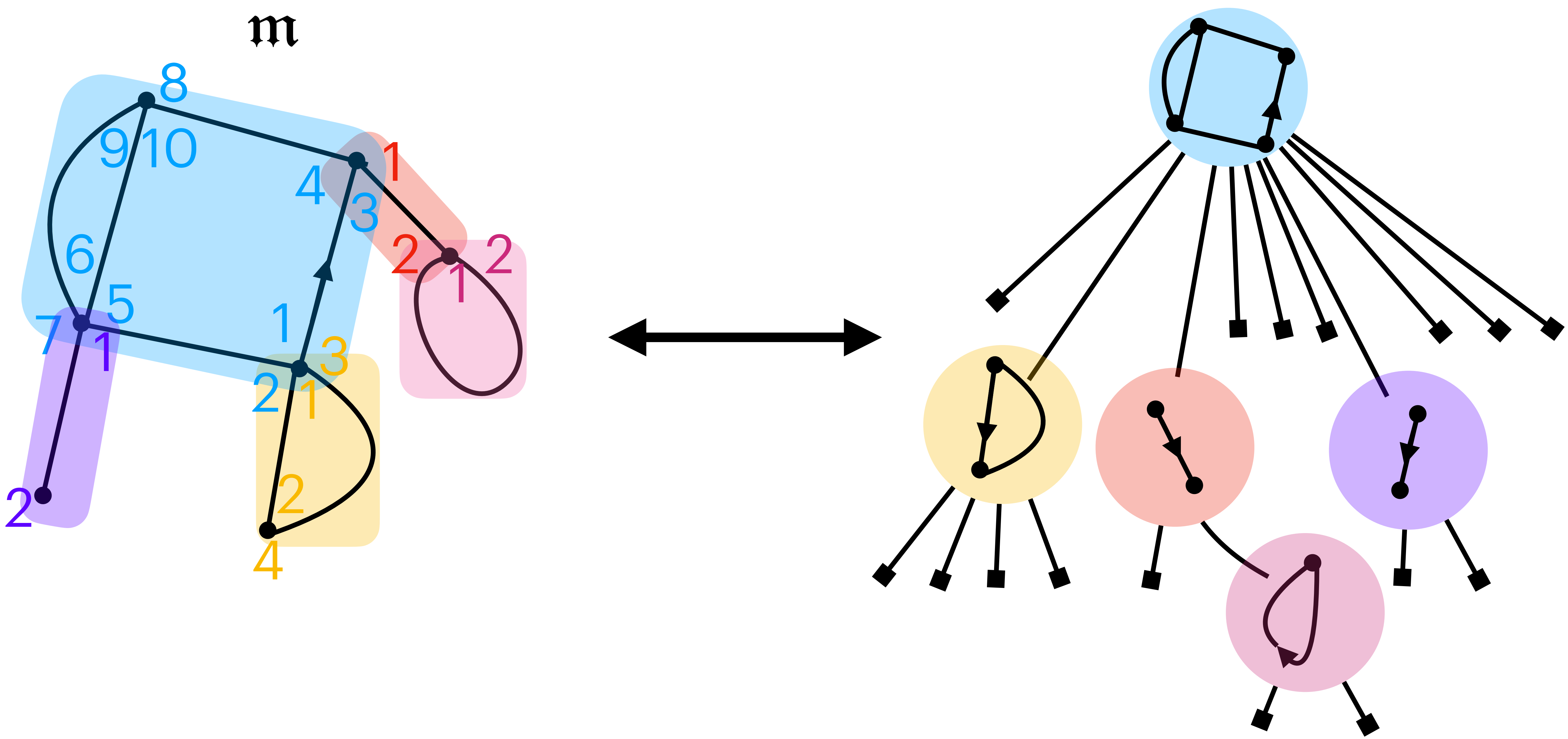
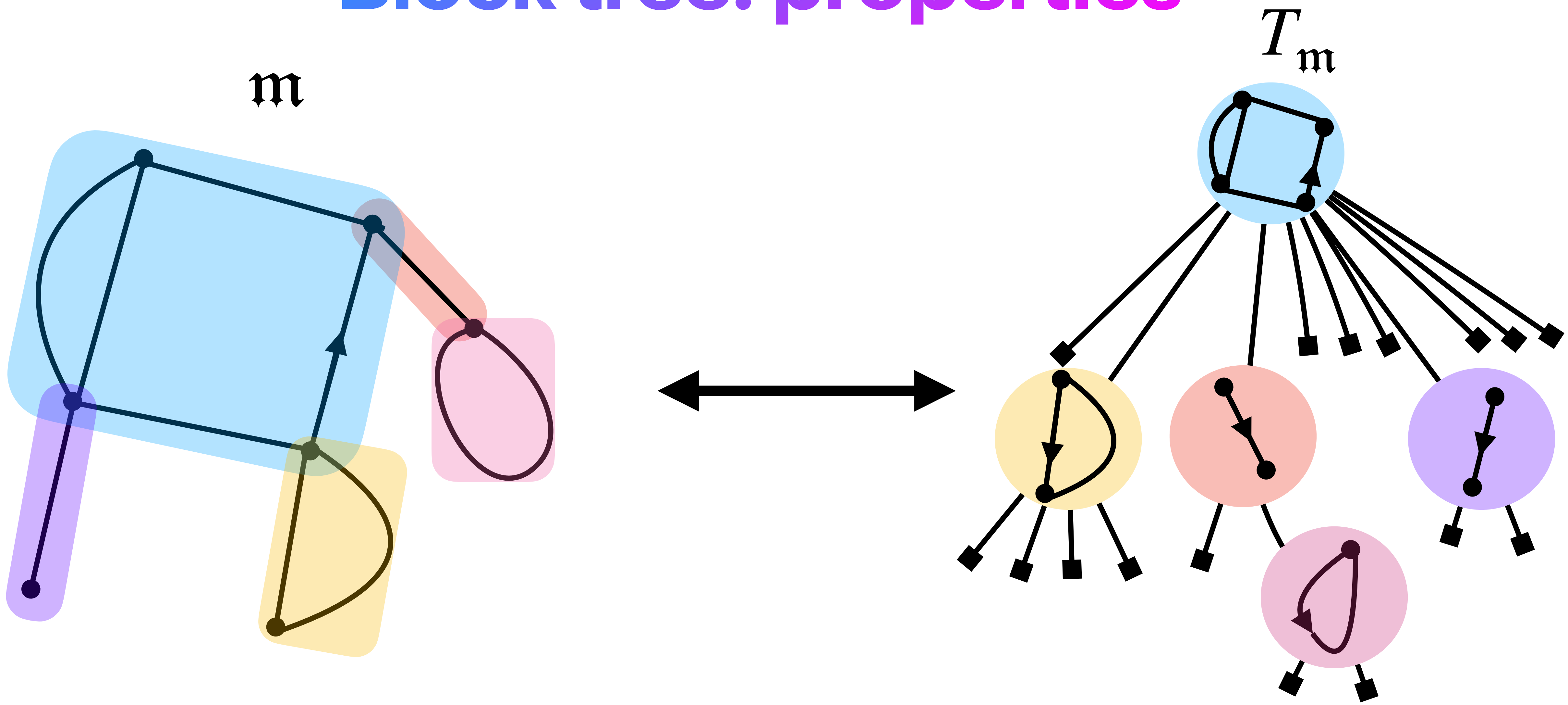
Decomposition of a map into blocks

Inspiration from [Tutte 1963]



⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

Block tree: properties



- $\mathfrak{m}$ is entirely determined by $T_{\mathfrak{m}}$ and $(\mathfrak{b}_v, v \in T_{\mathfrak{m}})$ where $\mathfrak{b}_v$ is the block of $\mathfrak{m}$ represented by v in $T_{\mathfrak{m}}$;
- Internal node (with k children) of $T_{\mathfrak{m}} \leftrightarrow$ block of $\mathfrak{m}$ of size $k/2$.

T_{M_n} gives the block sizes of a random map M_n.

Galton-Watson trees for map blocks

μ-Galton-Watson tree : random tree where the number of children of each node is given by μ independently, with μ = probability law on $\mathbb{N}$.

Galton-Watson trees for map blocks

μ-Galton-Watson tree : random tree where the number of children of each node is given by μ independently, with μ = probability law on $\mathbb{N}$.

<u>Theorem</u> [Fleurat, S. 24] $u > 0$

If $M_n \hookrightarrow \mathbb{P}_{n,u'}$, then T_{M_n} has the law of a Galton-Watson tree of explicit reproduction law μ^u conditioned to be of size $2n$.

Results for non tree-rooted maps

<u>Theorem</u> [Fleurat, S. 24] Model exhibits a phase transition at $u = 9/5$. When $n \to \infty$:

- Subcritical phase $u < 9/5$: "general map phase" one huge block;
- Critical phase $u = 9/5$: a few large blocks;
- Supercritical phase $u > 9/5$: "tree phase" only small blocks.

We obtain explicit results on enumeration, size of blocks and scaling limits in each case.

→ *A phase transition in block-weighted random maps*
W. Fleurat & Z. S., Electronic Journal of Probability, 2024

How can we do the same for tree-rooted maps?

III. Tree-rooted maps

Joint work with Marie Albenque and Éric Fusy

Model

Goal: parameter that affects the typical number of blocks.

We choose: $\mathbb{P}_{n,u}(\mathfrak{m}) = \dfrac{u^{\#blocks(\mathfrak{m})}}{Z_{n,u}}$ where

$u > 0$,

$\mathcal{M}_n$ = {tree-rooted maps of size n},

$\mathfrak{m} \in \mathcal{M}_n$,

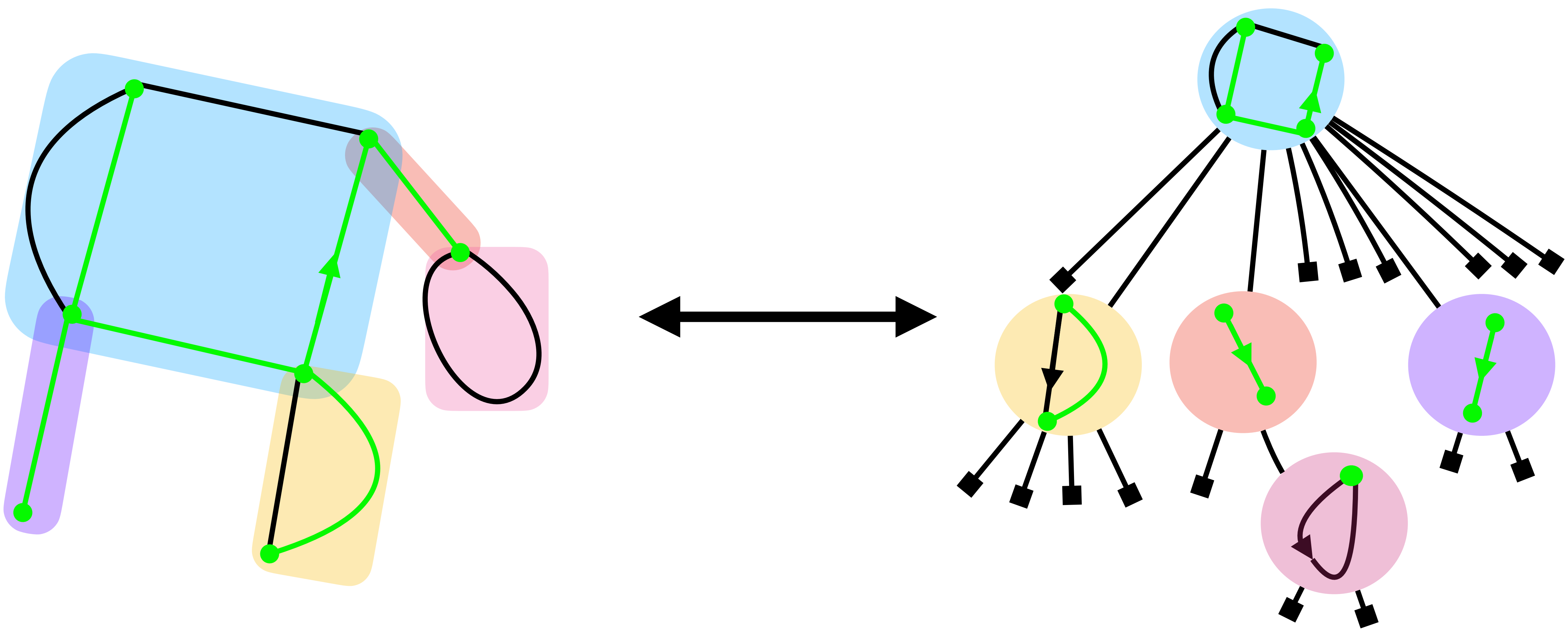
$Z_{n,u}$ = normalisation.

- $u = 1$: uniform distribution on tree-rooted maps of size n;
- $u \to 0$: minimising the number of blocks (=2-connected tree-rooted maps);
- $u \to \infty$: maximising the number of blocks (= tree-rooted trees!).

Given u, asymptotic behaviour when $n \to \infty$?

Block decomposition of tree-rooted maps

The decomposition of maps into blocks extends into a decomposition of tree-rooted maps into tree-rooted blocks.



$$M(z) = B(zM^2(z))$$

GS of 2-connected tree-rooted maps

GS of tree-rooted maps

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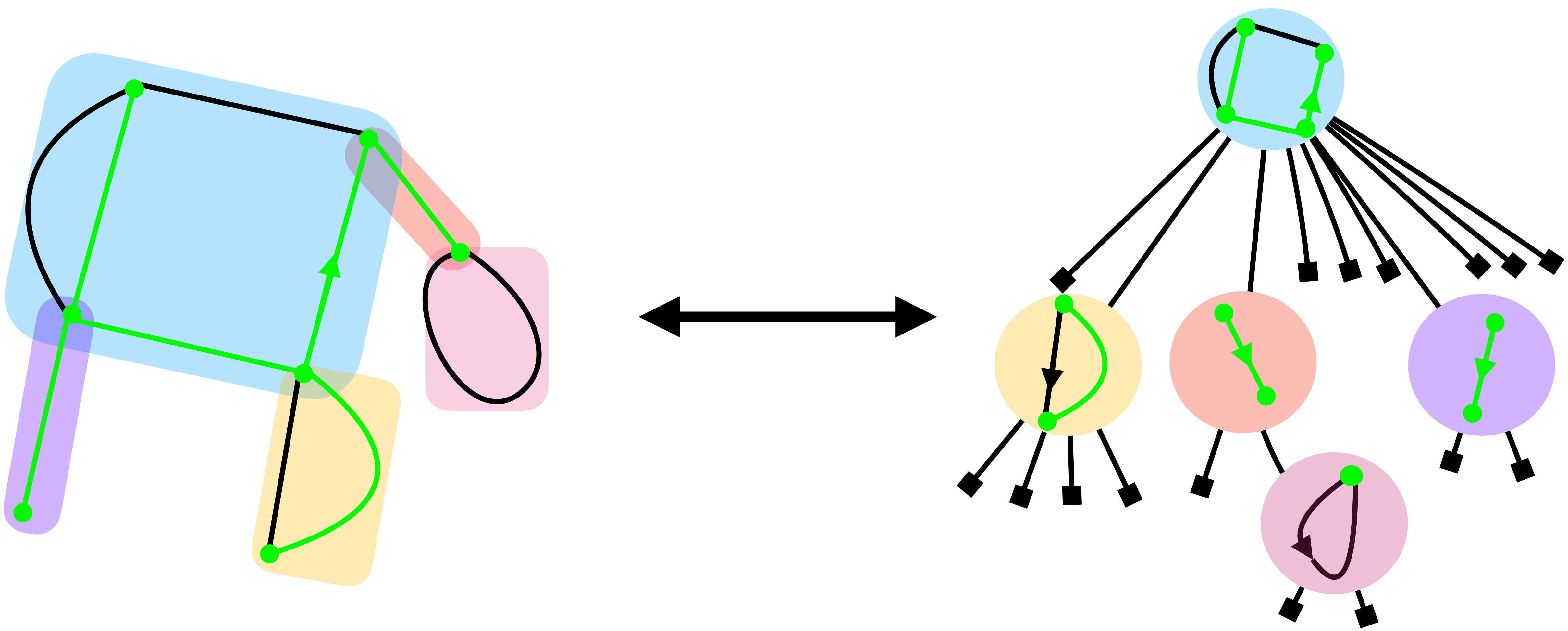
The decomposition of maps into blocks extends into a decomposition of tree-rooted maps into tree-rooted blocks.



$$M(z,u) = uB(zM^2(z,u)) + 1 - u$$

GS of 2-connected tree-rooted maps

GS of tree-rooted maps

So everything should be easy, right?

Tree-rooted maps are not so nice

$$M(z) = \sum_{n\geq 0} \mathsf{Cat}_n \mathsf{Cat}_{n+1} z^n \text{ so}$$

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- $[z^n]M(z) \sim \dfrac{4}{\pi} \times 16^n \times n^{-3};$

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- $M(\rho_M) = 8 - \dfrac{64}{3\pi} \simeq 1.2$ so M is not algebraic...

$$\cancel{P\left(z, M(z)\right) = 0}$$

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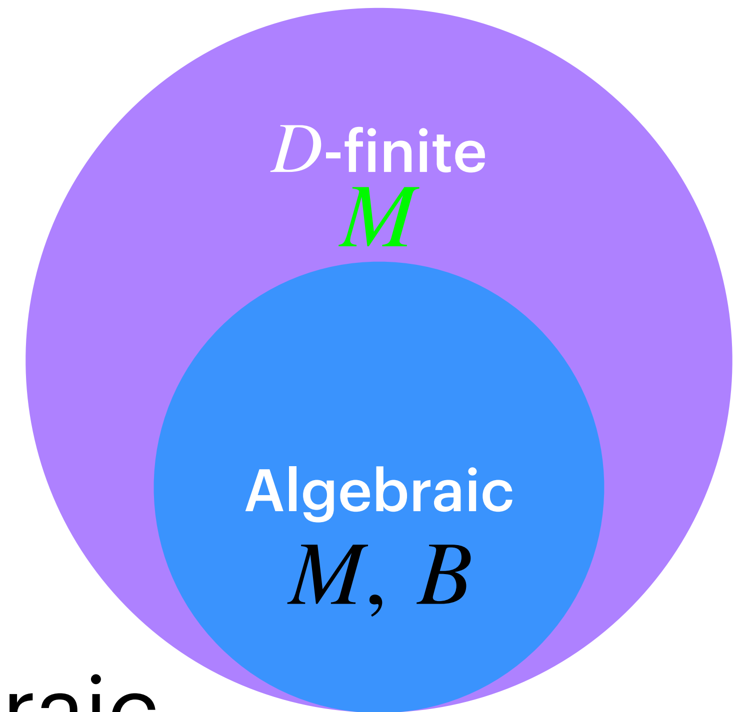

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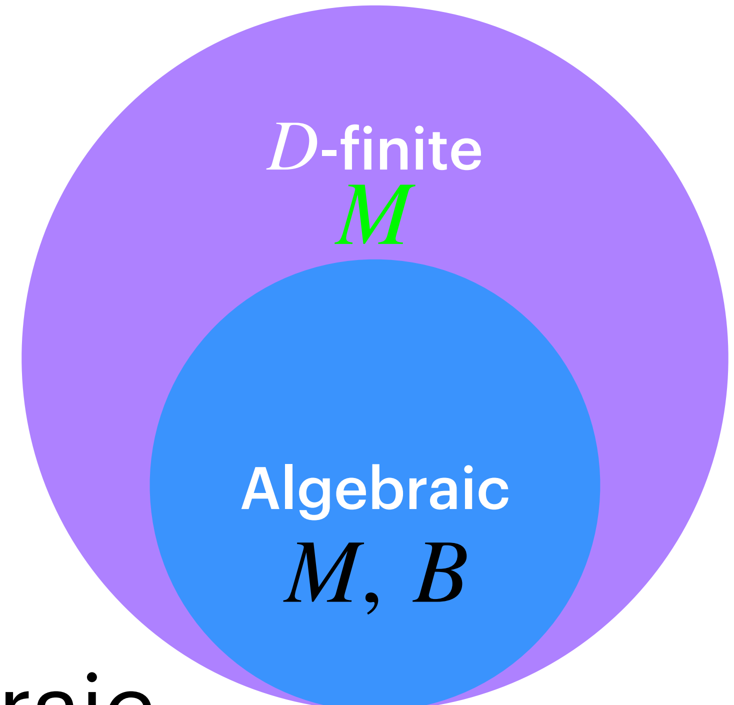

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$$P\left(z, \cancel{M(z)}\right) = 0$$

- Fortunately, it is still D-finite

$$P_0(z)\frac{\partial^2 M}{\partial z^2}(z) + P_1(z)\frac{\partial M}{\partial z}(z) + P_2(z)M(z) + P_3(z) = 0.$$

2-connected tree-rooted maps are naughty

Using $M(z) = B(zM^2(z))$ and the properties of M, we show

2-connected tree-rooted maps are naughty

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- $\rho_B = \rho_M M^2(\rho_M) = \dfrac{4(3\pi - 8)^2}{9\pi^2} \approx 0.091$

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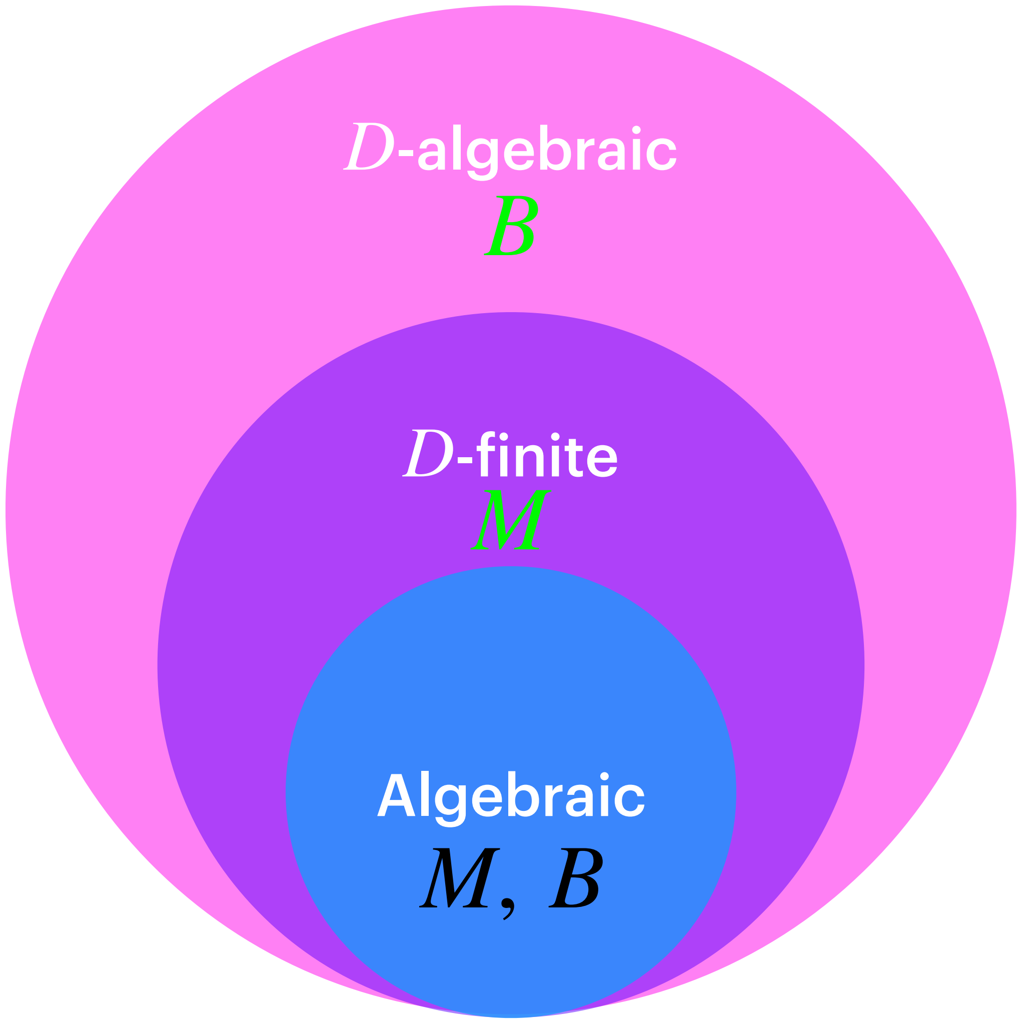

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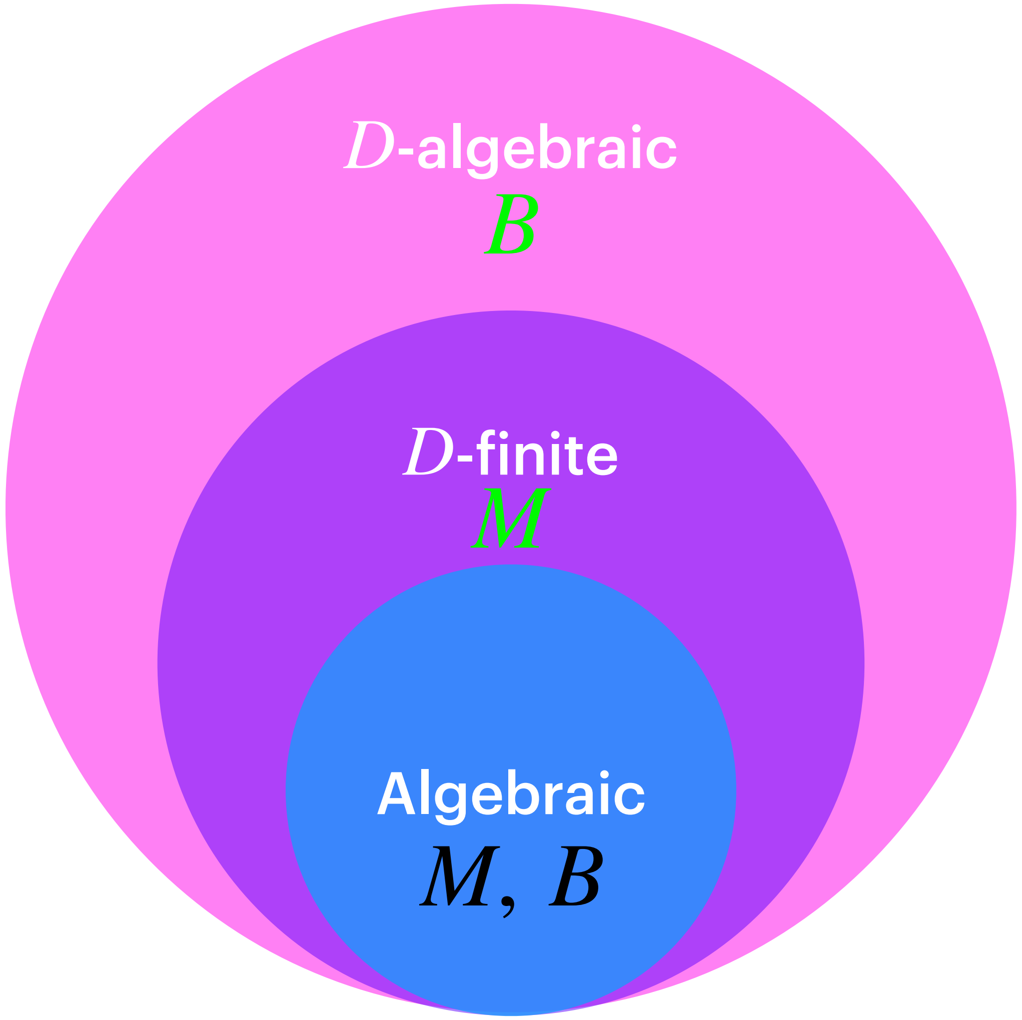

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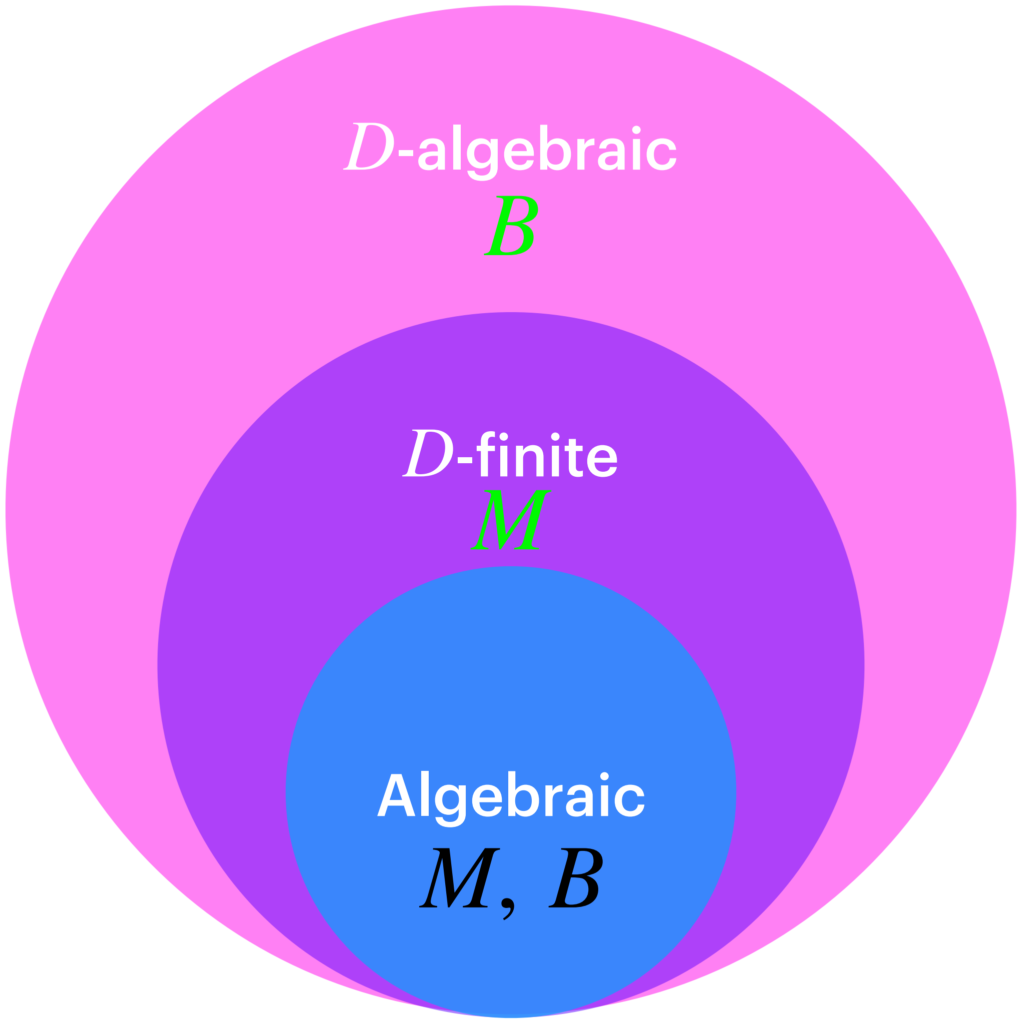

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$$P\left(\frac{\partial^2 B}{\partial y^2}(y), \frac{\partial B}{\partial y}(y), B(y), y\right) = 0.$$

Enumeration of 2-connected tree-rooted maps

Using $M(z) = B(zM^2(z))$ and the properties of M, we show

Theorem [Albenque, Fusy, S. 24]

$$[y^n]B(y) \sim \frac{4(3\pi - 8)^3}{27\pi(4-\pi)^3} \times \rho_B^{-n} \times n^{-3}.$$

Phase transition

Theorem [Albenque, Fusy, S. 24] Model exhibits a phase transition at $u_C = \dfrac{9\pi(4-\pi)}{420\pi - 81\pi^2 - 512} \simeq 3.02.$

When $n \to \infty$:

- Subcritical phase $u < u_C$: "general tree-rooted map phase" one huge block;
- Critical phase $u = u_C$: a few large blocks;
- Supercritical phase $u > u_C$: "tree phase" only small blocks.

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration			
Size of - the largest block - the second one			
Scaling limit of M_n			

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration	$\rho(u)^{-n} n^{-3}$	$\rho(u)^{-n} n^{-3/2} \ln(n)^{-1/2}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one			
Scaling limit of M_n			

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
Enumeration	$\rho(u)^{-n} n^{-3}$	$\rho(u)^{-n} n^{-3/2} \ln(n)^{-1/2}$	$\rho(u)^{-n} n^{-3/2}$
Size of - the largest block - the second one	$\sim (1 - \mathbb{E}(\mu^u))n$ $\Theta(n^{1/2})$	$\Theta(n^{1/2})$	$\frac{\ln(n)}{\ln\left(\frac{\rho_B}{y(u)}\right)} - \frac{3\ln(\ln(n))}{\ln\left(\frac{\rho_B}{y(u)}\right)} + O(1)$
Scaling limit of M_n			

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
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	Ordered atoms of a Poisson Point Process		
Scaling limit of M_n			

Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
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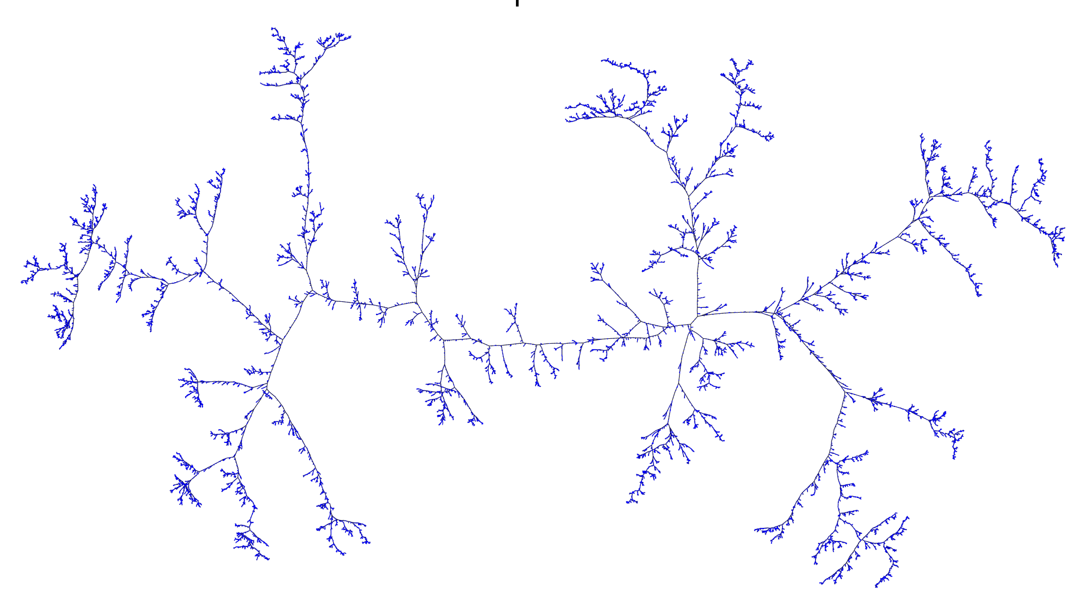
Scaling limit of M_n	?	$\frac{C_2 \ln(n)^{1/2}}{n^{1/2}} M_n \to \mathcal{T}_e$	[Stufler 2020] $\frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e$



Results

For $M_n \hookrightarrow \mathbb{P}_{n,u}$	$u < u_C$	$u = u_C$	$u > u_C$
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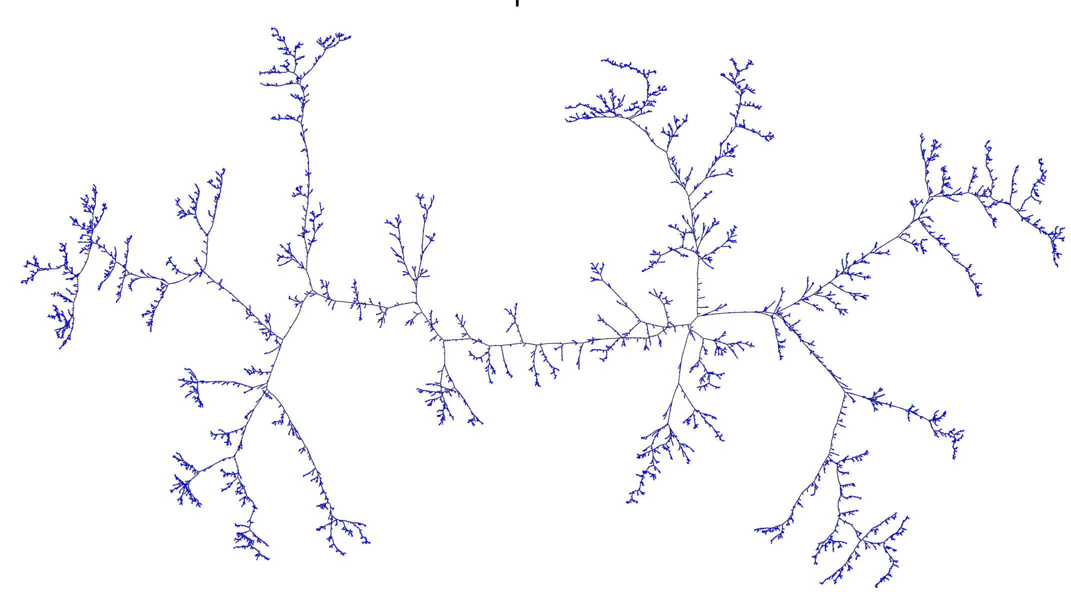

IV. Perspectives

Extensions to more involved decompositions

Block-weighted

- Tree-rooted quadrangulations;
- Forested maps;
- Maps endowed with a Potts model / Ising model;
- 2-oriented quadrangulations (resp. 3-oriented triangulations) decomposed into irreducible blocks...

Thank you!

▶ **Theorem 15.** *The random tree-rooted map* $\mathrm{M}_n^{(u)}$*, drawn according to* $\mathbb{P}_n^{(u)}$*, exhibits the following behaviours when* n *tends to infinity.*

Subcritical case. *For* $u < u_c$*, the largest bloc is macroscopic, and more precisely one has:*

$$\frac{\mathrm{LB}_1(\mathrm{M}_n^{(u)}) - (1 - E(u))n}{\sqrt{c(u) n \ln(n)}} \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1). \tag{23}$$

Furthermore, for any fixed $j \geq 2$*, it holds that* $\mathrm{LB}_j(\mathrm{M}_n^{(u)}) = \Theta_{\mathbb{P}}(n^{1/2})$ *and for* $x > 0$*:*

$$\mathbb{P}\left(\mathrm{LB}_j(\mathrm{M}_n^{(u)}) \leq x\sqrt{n}\right) \xrightarrow[n\to\infty]{} e^{-\lambda(x)} \sum_{p=0}^{j-2} \frac{\lambda(x)^p}{p!}, \qquad \text{where } \lambda(x) := \frac{c(u)}{2x^2}. \tag{24}$$

Critical case. *For* $u = u_C$*, for any fixed* $j \geq 1$*, it holds that* $\mathrm{LB}_j(\mathrm{M}_n^{(u)}) = \Theta_{\mathbb{P}}(n^{1/2})$*. More precisely, up to a shift of indices, the sizes of the blocks exhibit a similar behavior as the sizes of non-macroscopic blocks in the subcritical regime, namely, for* $x > 0$*:*

$$\mathbb{P}\left(\mathrm{LB}_j(\mathrm{M}_n^{(u)}) \leq x\sqrt{n}\right) \xrightarrow[n\to\infty]{} e^{-\lambda(x)} \sum_{p=0}^{j-1} \frac{\lambda(x)^p}{p!}, \qquad \text{where } \lambda(x) := \frac{c(u_C)}{2x^2}. \tag{25}$$

Supercritical case. *For* $u > u_C$*, for all fixed* $j \geq 1$*, it holds as* $n \to \infty$ *that*

$$\mathrm{LB}_j(\mathrm{M}_n^{(u)}) = \frac{\ln(n)}{\ln\left(\frac{\rho_B}{y(u)}\right)} - \frac{3\ln(\ln(n))}{\ln\left(\frac{\rho_B}{y(u)}\right)} + O_{\mathbb{P}}(1).$$

▶ **Remark 16.** One can get a local limit theorem for $\mathrm{LB}_1(\mathrm{M}_n^{(u)})$ in the subcritical case as in [24] (up to the technicality that nodes of the block tree have only even numbers of children). Furthermore, one can state a joint limit law for the sizes $\mathrm{LB}_j(\mathrm{M}_n^{(u)})$. For any fixed $r \geq 1$,

$$\left(\frac{c(u)}{2}\left(\frac{\mathrm{LB}_j(\mathrm{M}_n^{(u)})}{\sqrt{n}}\right)^{-2},\ 2 \leq j \leq r+1\right) \xrightarrow[n\to\infty]{(d)} (A_1, \ldots, A_r),$$

where the A_i are the decreasingly ordered atoms of a Poisson Point Process of rate 1 on $\mathbb{R}_+$. The same joint limit law holds at u_C (with j from 1 to r).