Alternating normal form in the standard braid monoid

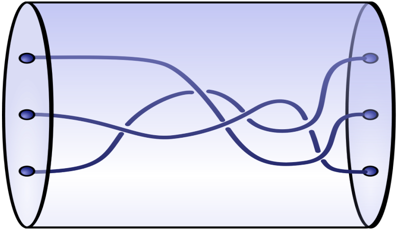
LIGM, Université Gustave Eiffel

June Roupin (she/they)
Joint work with Vincent Jugé

20 June 2024

Braids

Objects: Braids [6]



(Upper picture source: [1])
[6] E. Artin, 1925

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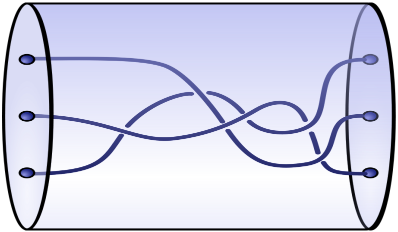

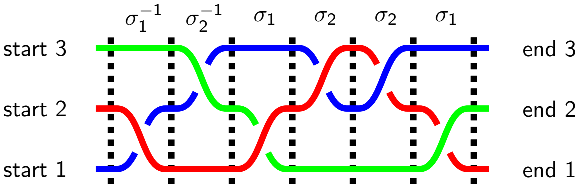

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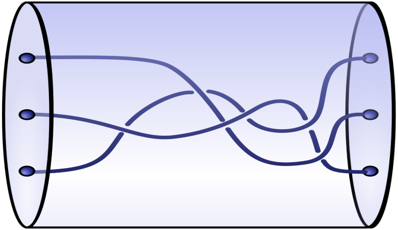

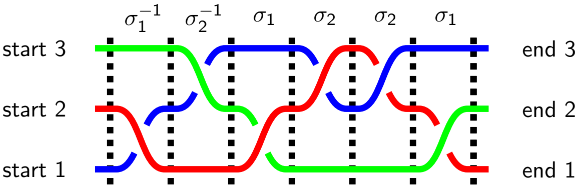

Objective: Study normal forms

(Upper picture source: [1])
[6] E. Artin, 1925

Braids

n strings

Generators/Letters: $\mathbb{A}_n = \{\sigma_1, \ldots, \sigma_{n-1}\}$

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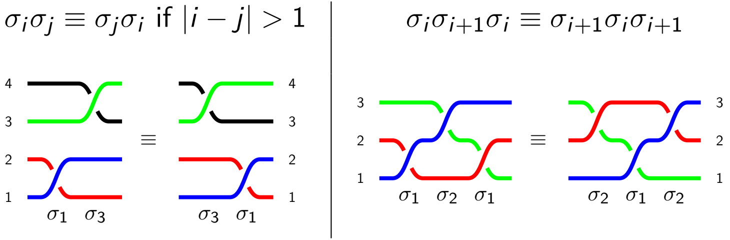

Figure: Commutation and braid relations

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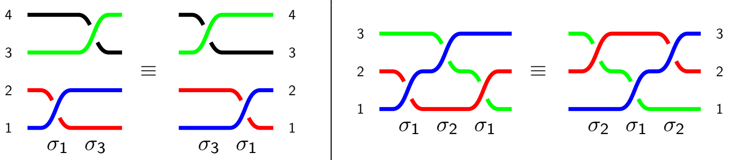

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Braids: Equivalence classes of words

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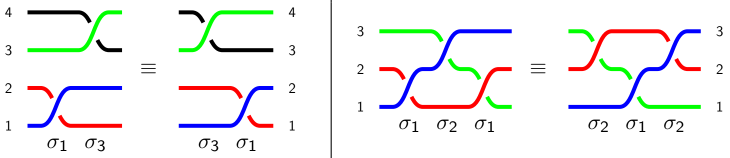

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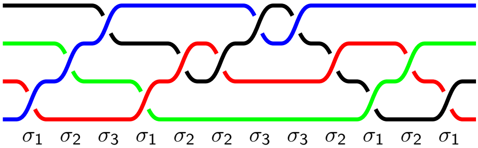

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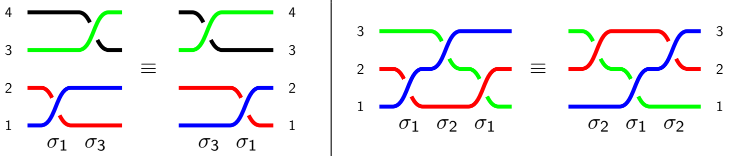

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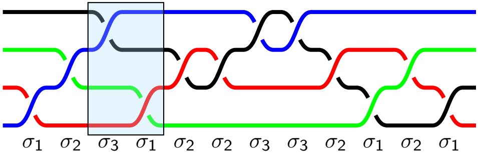

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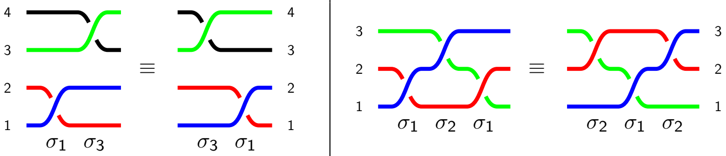

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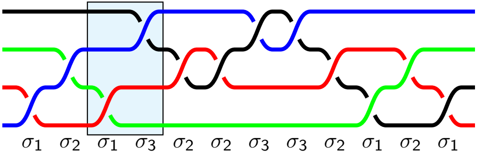

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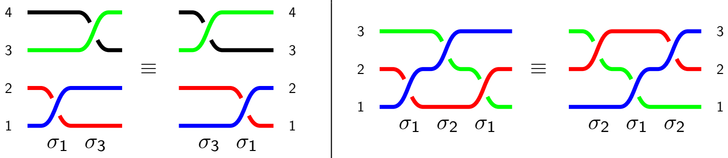

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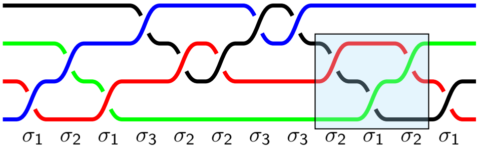

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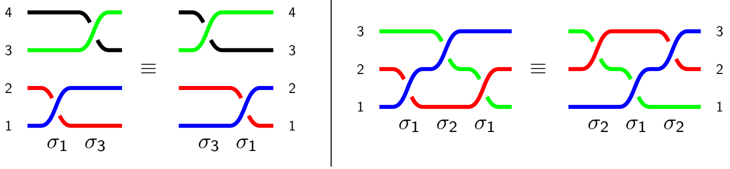

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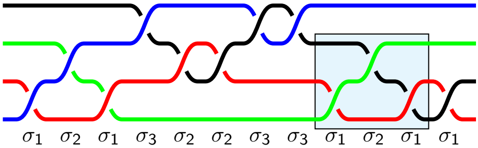

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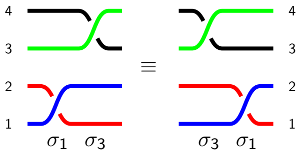

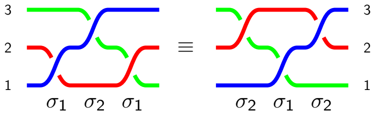

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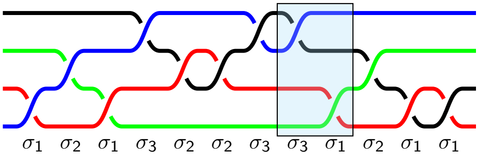

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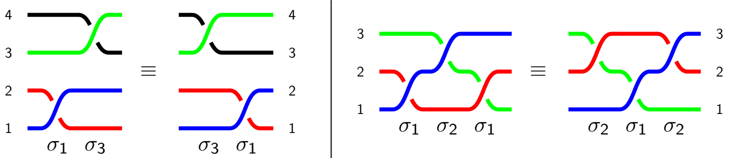

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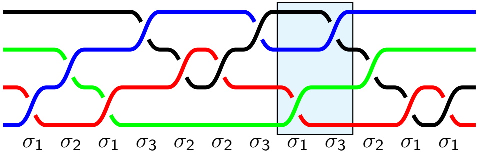

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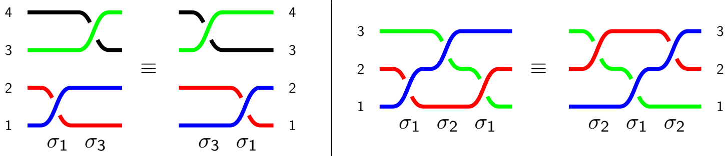

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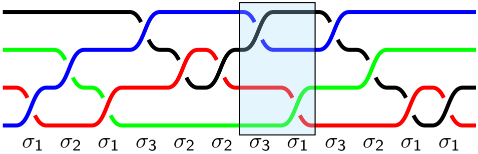

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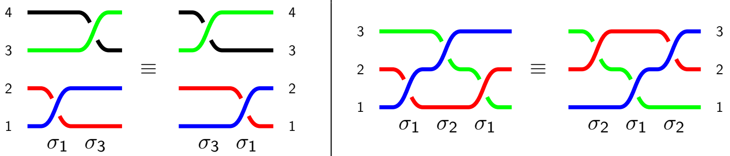

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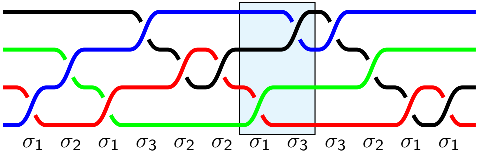

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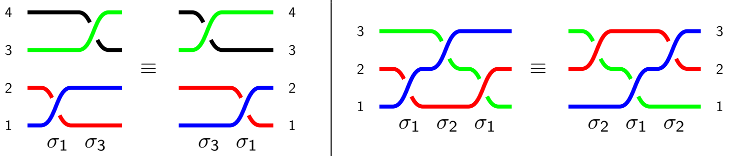

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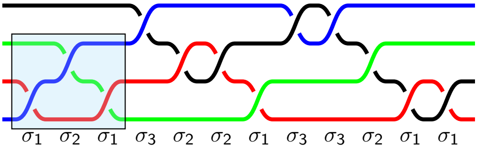

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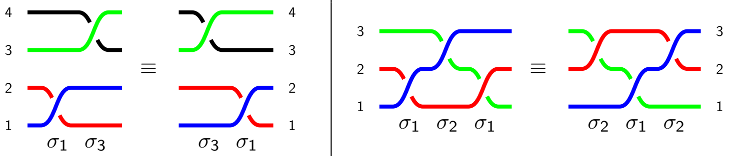

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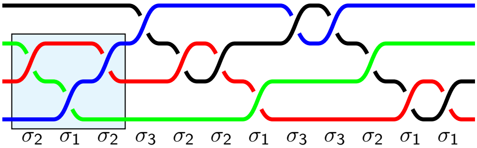

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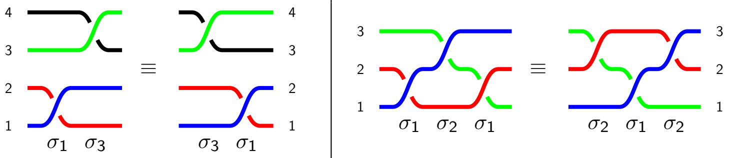

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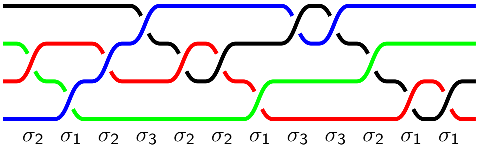

Braids

Representatives

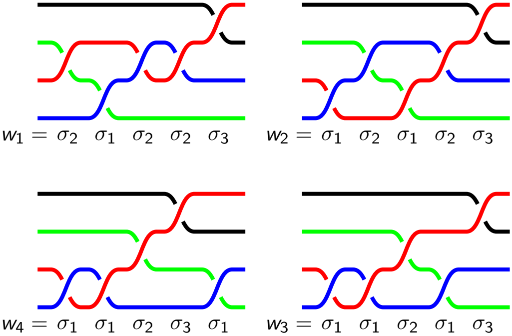

Figure: Representatives of the braid $\beta = \{w_1, w_2, w_3, w_4\}$

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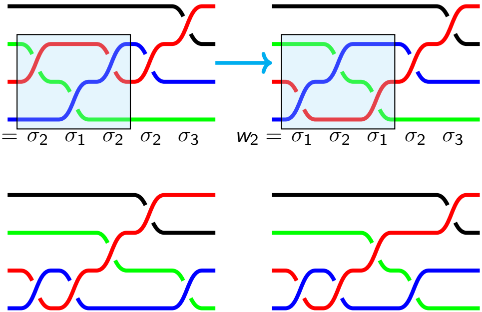

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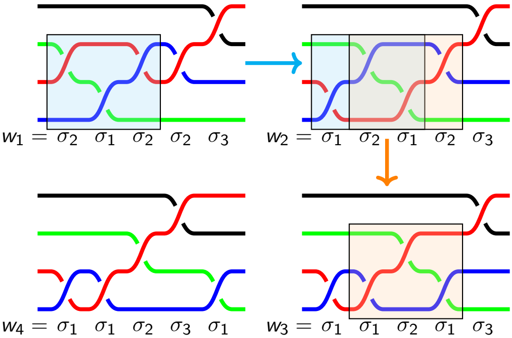

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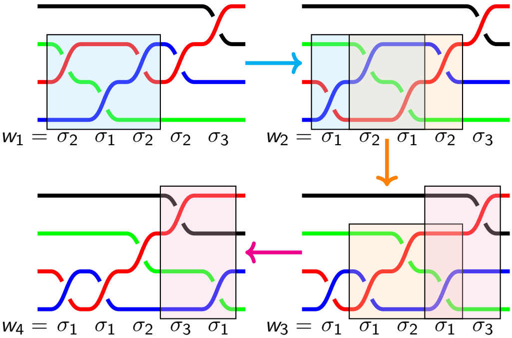

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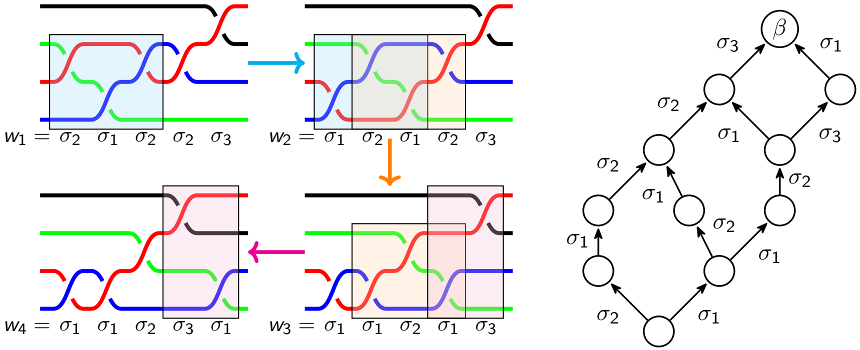

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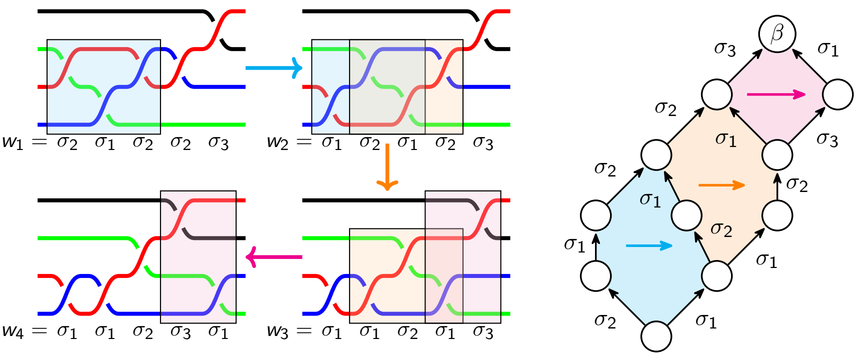

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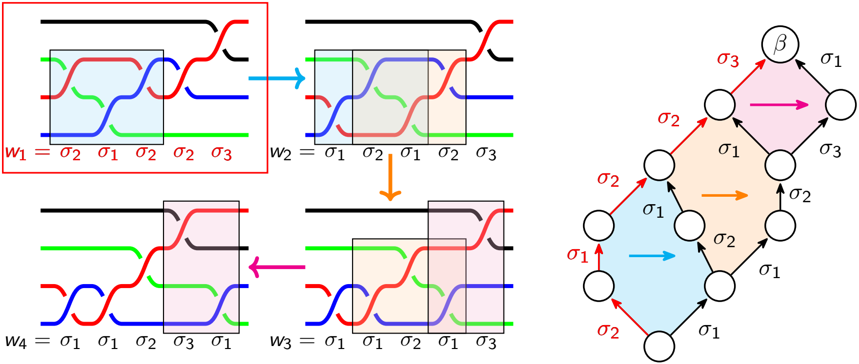

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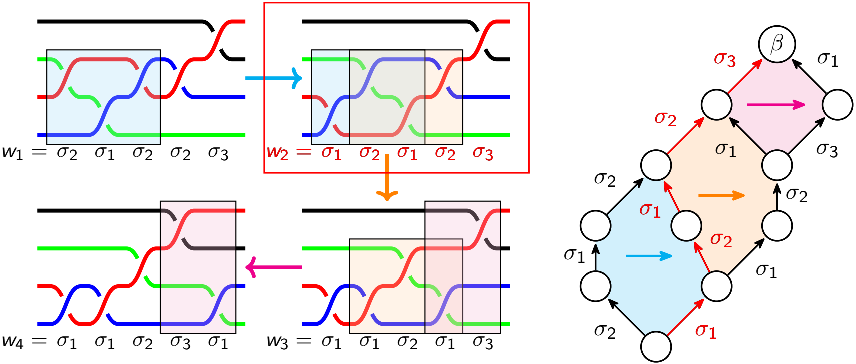

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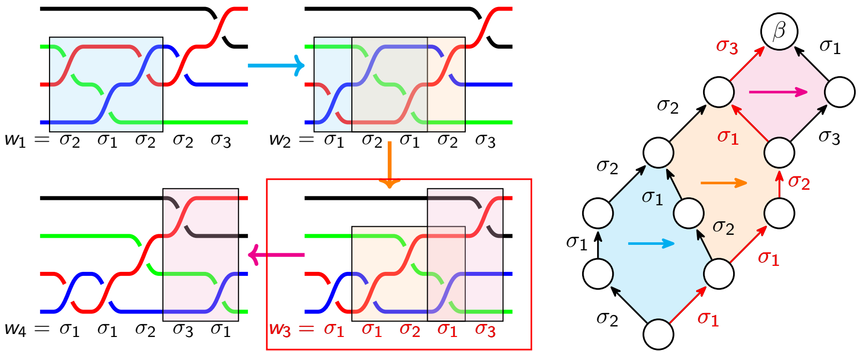

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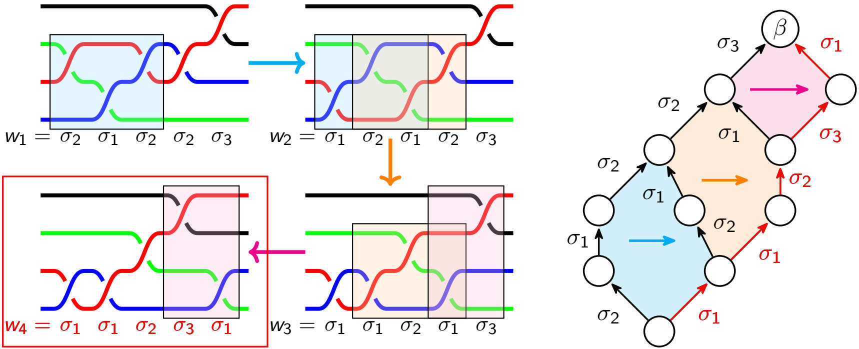

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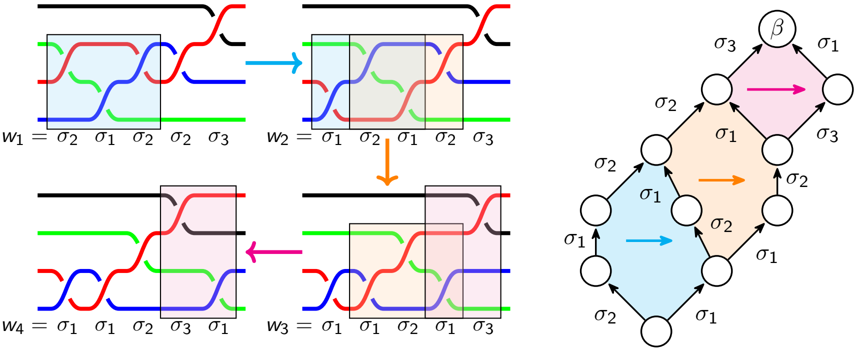
Left-divisors



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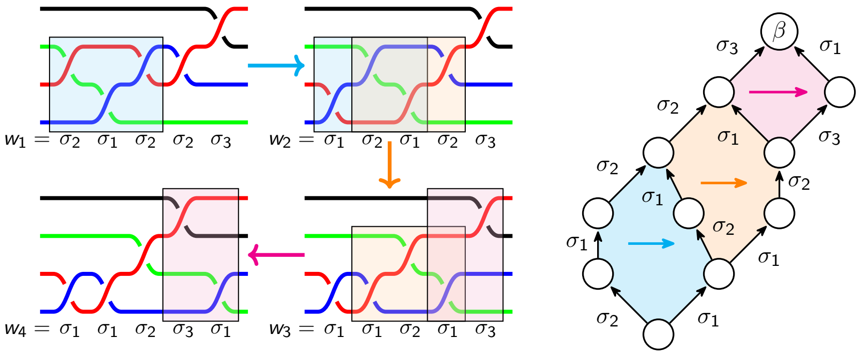
Braids

Left-divisors $\rightarrow$ Lattice structure



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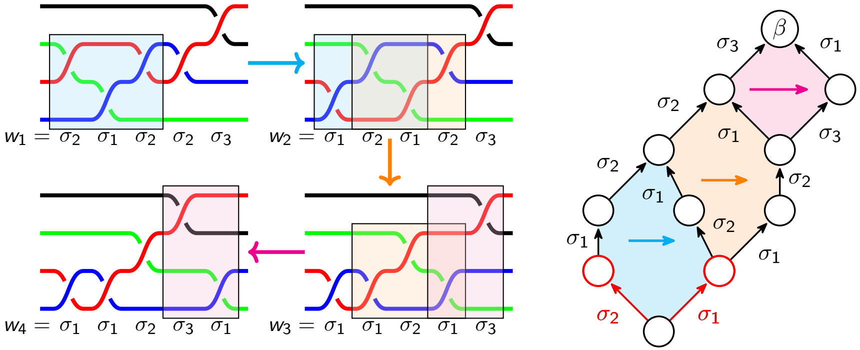

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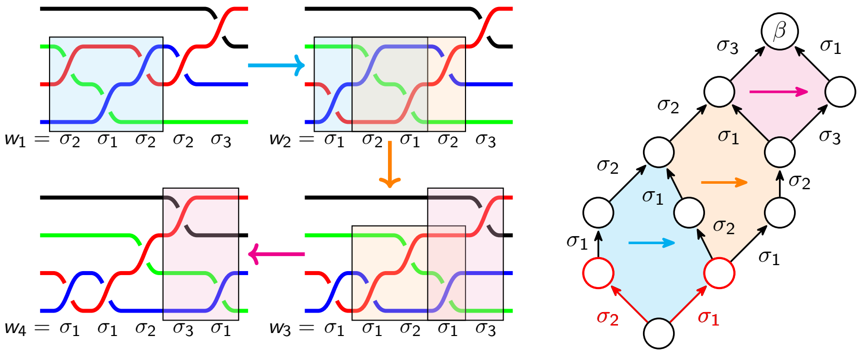

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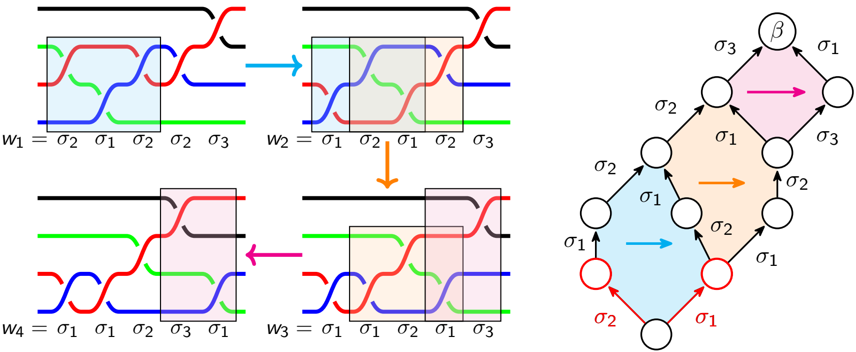

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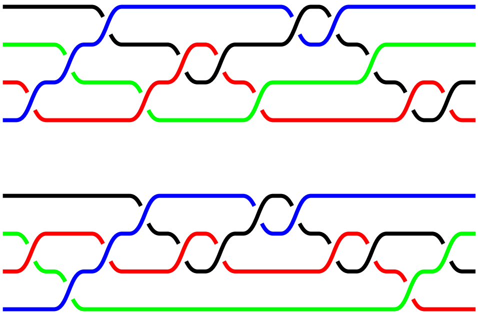
I. Normal forms

Normal forms

Normal form = choice of a unique representative per braid



Lexicographic normal form [4]

Choose the minimal representant in the lexicographic order

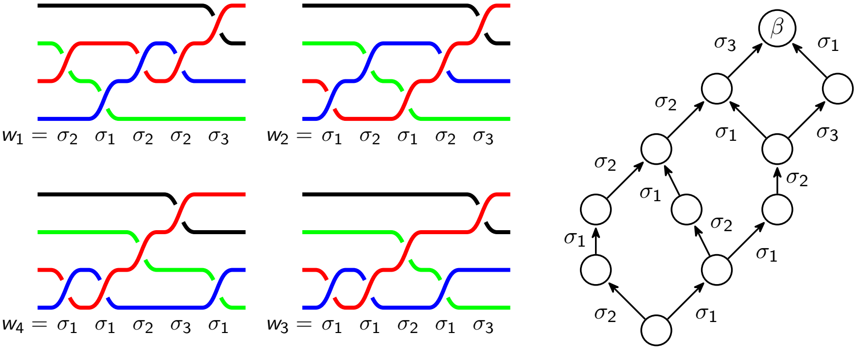


Figure: Lexicographic normal form ($\mathcal{LEX}_4$) of β

[4] V. Gebhardt and J. González-Meneses, 2012

Lexicographic normal form [4]

Choose the minimal representant in the lexicographic order

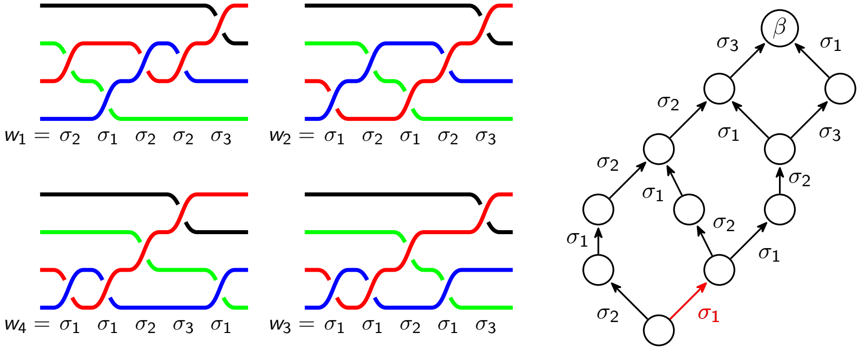
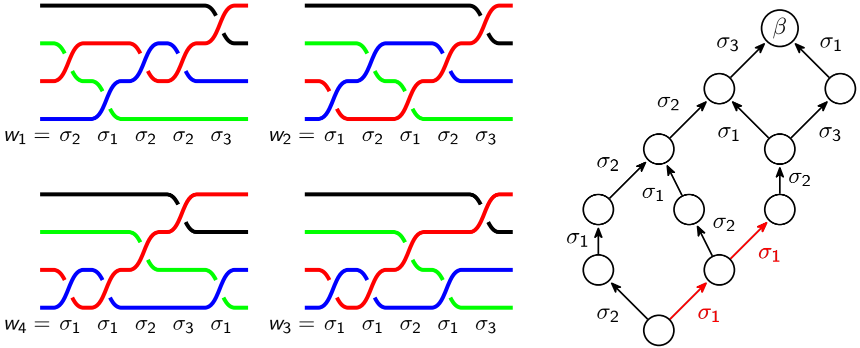


Figure: Lexicographic normal form ($\mathcal{LEX}_4$) of β

[4] V. Gebhardt and J. González-Meneses, 2012

Lexicographic normal form [4]

Choose the minimal representant in the lexicographic order



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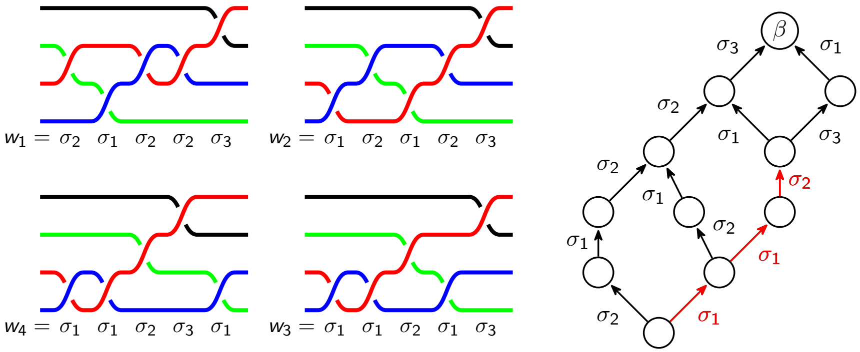

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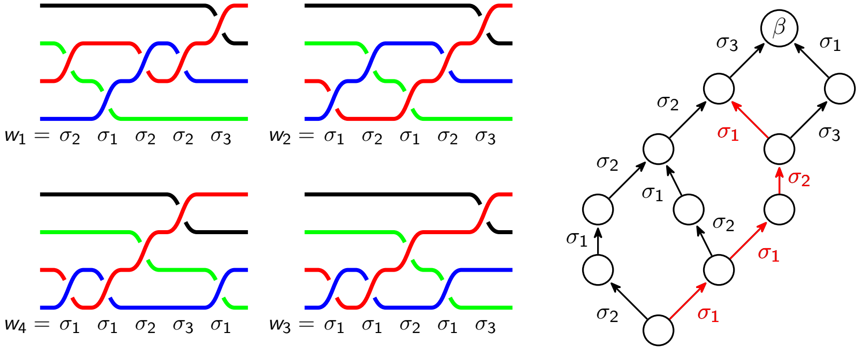

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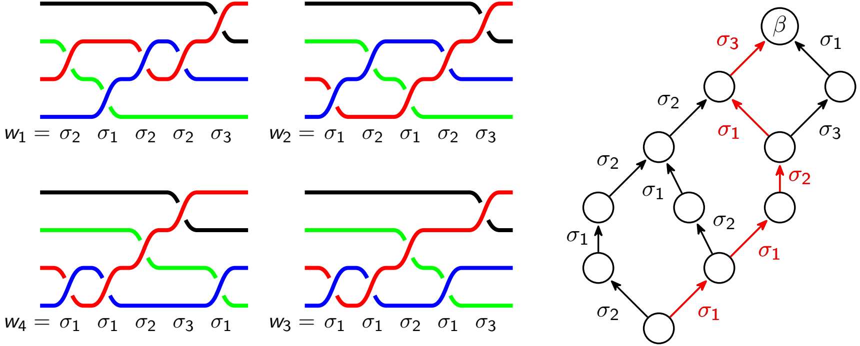

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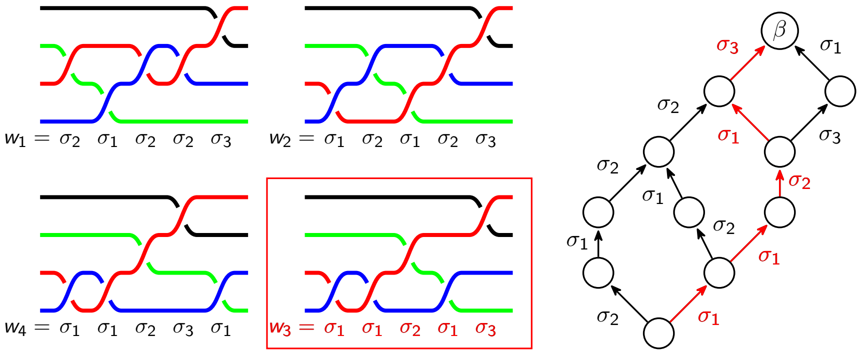


Figure: Lexicographic normal form $(\mathcal{LEX}_4)$ of β

[4] V. Gebhardt and J. González-Meneses, 2012

Alternating normal form [2,3]

Forbid a letter, alternatively and recursively

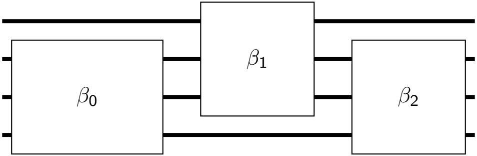
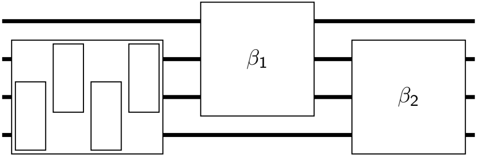


Figure: Alternating normal form $\mathcal{ANF}_n$

[2] P. Dehornoy, 2008
[3] S. Burkel, 1997

Alternating normal form [2,3]

Forbid a letter, alternatively and recursively



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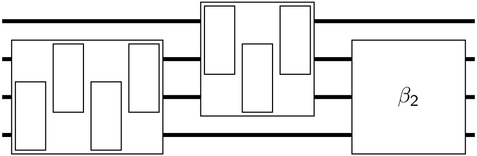

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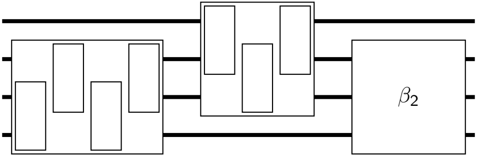

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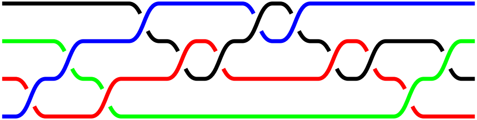


Figure: β in alternating normal form ($\mathcal{ANF}_4$)

[2] P. Dehornoy, 2008
[3] S. Burkel, 1997

Alternating normal form [2,3]

Forbid a letter, alternatively and recursively

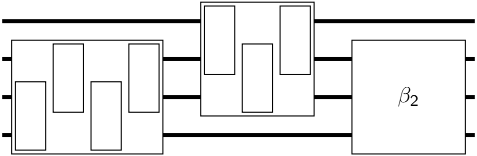
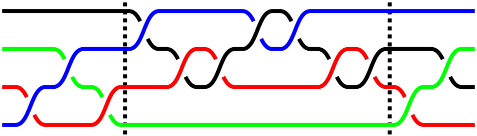


Figure: Alternating normal form $\mathcal{ANF}_n$



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[3] S. Burkel, 1997

Alternating normal form [2,3]

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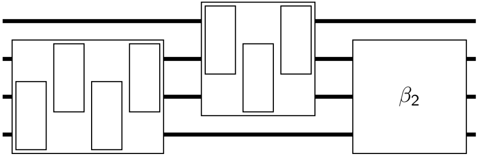
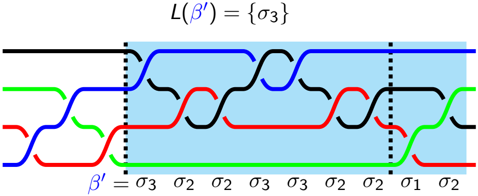


Figure: Alternating normal form $\mathcal{ANF}_n$



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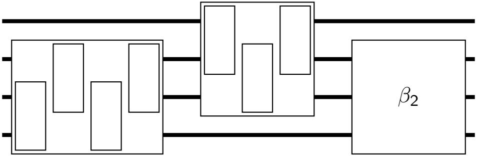

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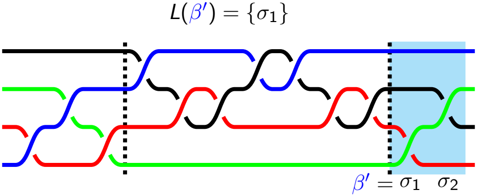


Figure: β in alternating normal form ($\mathcal{ANF}_4$)

[2] P. Dehornoy, 2008
[3] S. Burkel, 1997

What is a good normal form?

- Checking that a word is in normal form is easy

- Computing the normal form of a word is easy

What is a good normal form?

- Checking that a word is in normal form is easy

$\rightarrow$ Regularity

- Computing the normal form of a word is easy

What is a good normal form?

- Checking that a word is in normal form is easy

$\rightarrow$ Regularity

$\rightarrow$ Small minimal automaton

- Computing the normal form of a word is easy

What is a good normal form?

- Checking that a word is in normal form is easy

$\rightarrow$ Regularity

$\rightarrow$ Small minimal automaton

- Computing the normal form of a word is easy

$\rightarrow$ Automaticity

What is a good normal form?

- Checking that a word is in normal form is easy

→ Regularity

→ Small minimal automaton

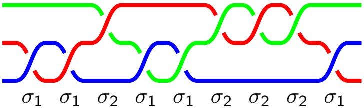
- Computing the normal form of a braid is easy

→ Automaticity

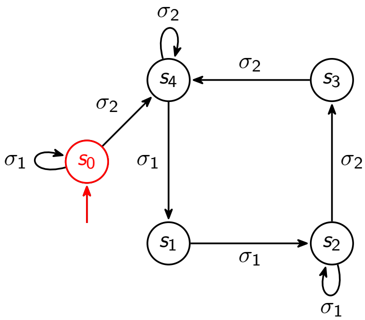
II. Regularity

of normal forms

Regularity

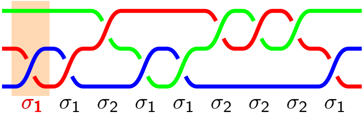


σ_1 σ_1 σ_2 σ_1 σ_1 σ_2 σ_2 σ_2 σ_1

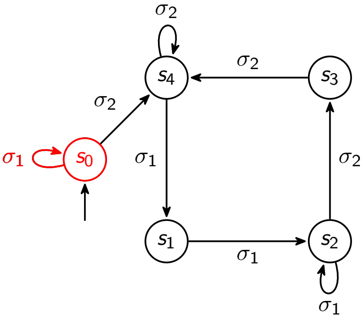


σ_2
s_4
σ_2
s_3
σ_2
σ_1
s_0
σ_1
σ_2
s_1
σ_1
s_2
σ_1

Regularity

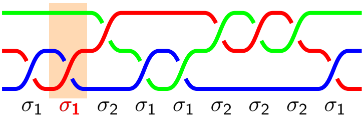


σ_1 σ_1 σ_2 σ_1 σ_1 σ_2 σ_2 σ_2 σ_1

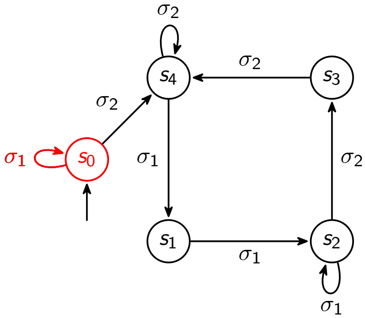


σ_2
s_4
σ_2
s_3
σ_2
σ_1
s_0
σ_1
σ_2
s_1
σ_1
s_2
σ_1

Regularity

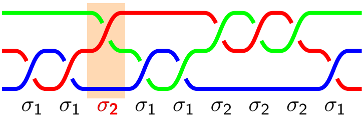


σ_1 σ_1 σ_2 σ_1 σ_1 σ_2 σ_2 σ_2 σ_1

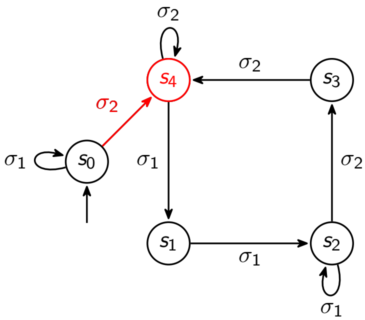


s_0
s_1
s_2
s_3
s_4
σ_1
σ_2
σ_2
σ_1
σ_1
σ_1
σ_2
σ_2

Regularity

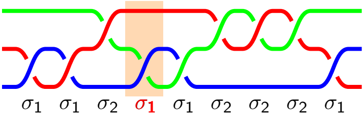


σ_1 σ_1 σ_2 σ_1 σ_1 σ_2 σ_2 σ_2 σ_1

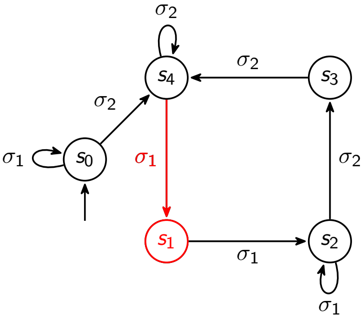


σ_2
s_4
σ_2
s_3
σ_2
σ_1
s_0
σ_1
σ_2
s_1
σ_1
s_2
σ_1

Regularity



σ_1 σ_1 σ_2 σ_1 σ_1 σ_2 σ_2 σ_2 σ_1



σ_2
σ_2
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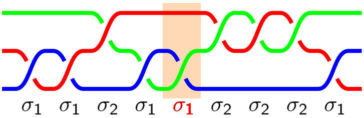
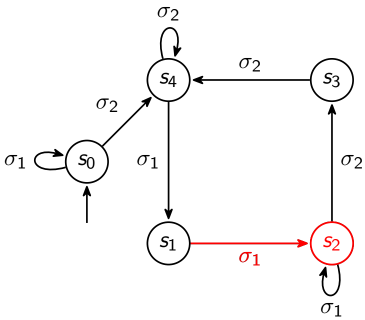
σ_1

Regularity



σ_1 σ_1 σ_2 σ_1 σ_1 σ_2 σ_2 σ_2 σ_1



σ_2
σ_2
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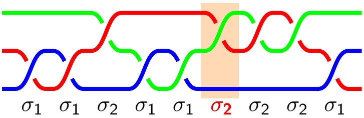
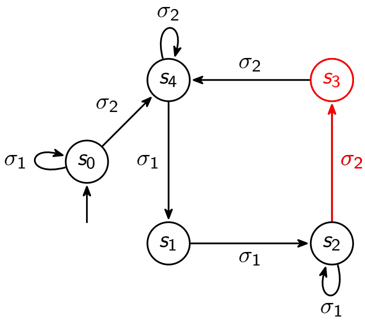
s_2
σ_1
σ_1

Regularity



σ_1 σ_1 σ_2 σ_1 σ_1 σ_2 σ_2 σ_2 σ_1



σ_2
s_4
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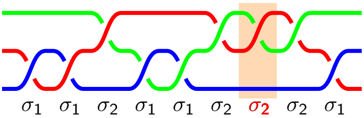
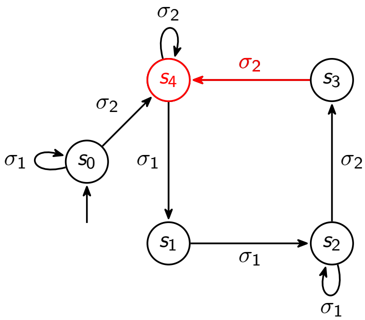
σ_1
s_2
σ_1

Regularity


σ_1 σ_1 σ_2 σ_1 σ_1 σ_2 σ_2 σ_2 σ_1


σ_2
s_4
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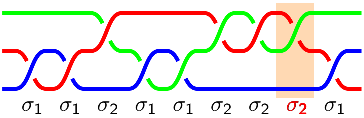
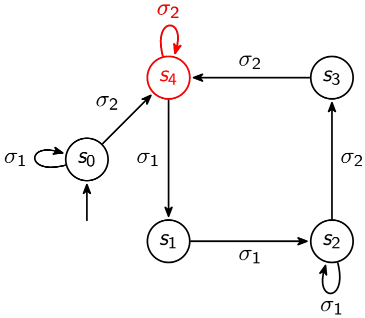
Regularity


σ_1 σ_1 σ_2 σ_1 σ_1 σ_2 σ_2 σ_2 σ_1


σ_2
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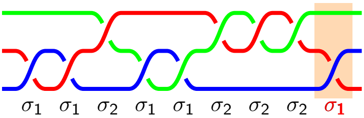
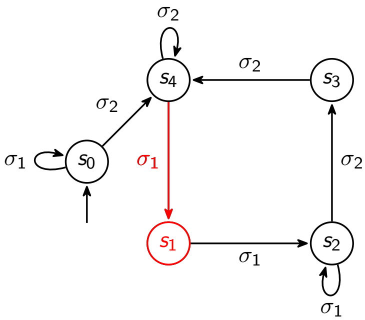
Regularity



σ_1 σ_1 σ_2 σ_1 σ_1 σ_2 σ_2 σ_2 σ_1



σ_2
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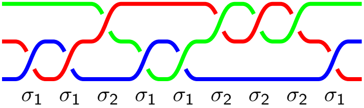
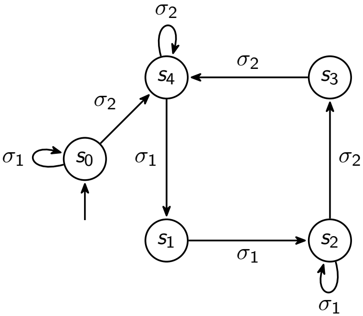
Regularity





$$L = \mathcal{ANF}_3 = \mathcal{LEX}_3$$

Regularity

Theorem ([3,4])

The lexicographic normal form and the alternating normal form are regular for all $n \geqslant 2$.

[3] S. Burkel, 1997
[4] V. Gebhardt and J. González-Meneses, 2012

Regularity

Theorem ([3,4])

The lexicographic normal form and the alternating normal form are regular for all $n \geqslant 2$.

$\rightarrow$ Gives no explicit small automaton for the alternating normal form!

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Regularity

Theorem ([3,4])

The lexicographic normal form and the alternating normal form are regular for all $n \geqslant 2$.

$\rightarrow$ Gives no explicit small automaton for the alternating normal form!

$\Rightarrow$ Need another characterization

[3] S. Burkel, 1997
[4] V. Gebhardt and J. González-Meneses, 2012

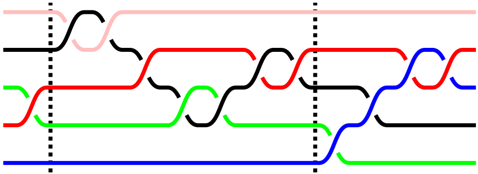
III. Local characterization

of the alternating normal form

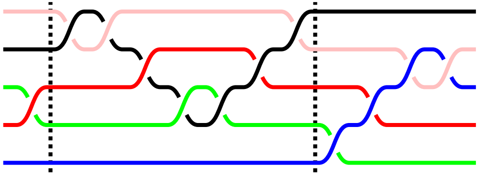
Who is in normal form?

Local characterization

$v =$



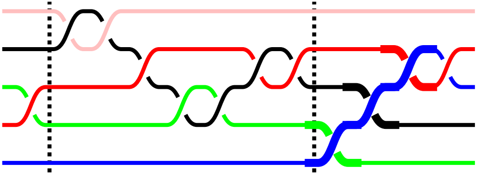
$w =$



Who is in normal form?

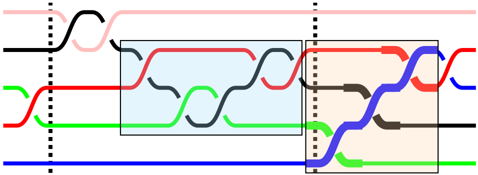
Local characterization

$v =$



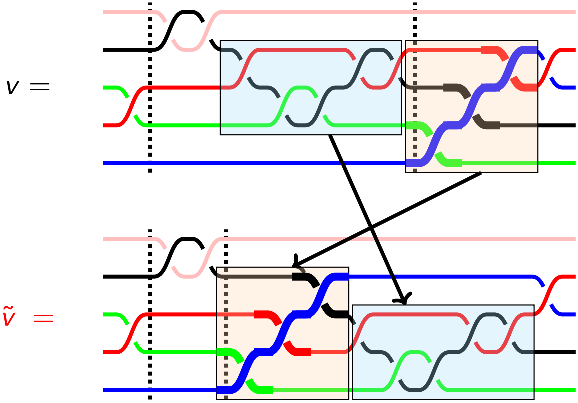
Who is in normal form?

Local characterization

$v =$ 

Who is in normal form?

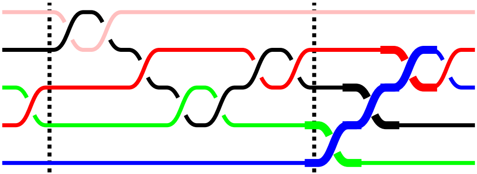
Local characterization



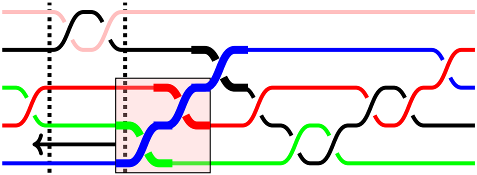
Who is in normal form?

Local characterization

$v =$



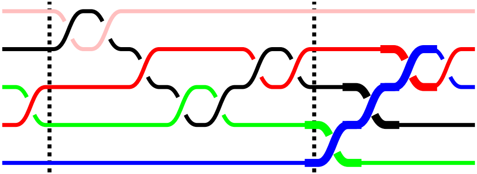
$\tilde{v} =$



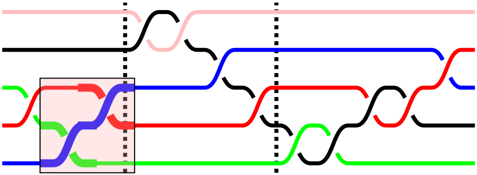
Who is in normal form?

Local characterization

$v =$

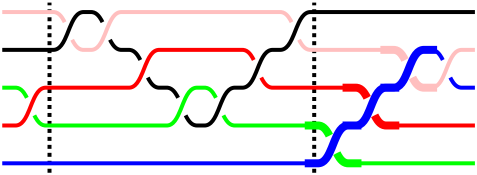


$\tilde{v}' =$



Who is in normal form?

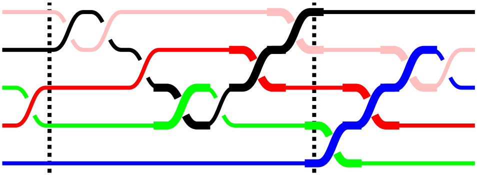
Local characterization

$w =$ 

Who is in normal form?

Local characterization

$w =$



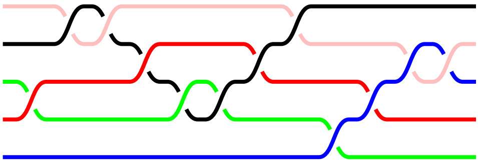
Characterization

Local characterization

Theorem (V. Jugé, J. R. 24)

$$w \in \mathcal{ANF}_n$$

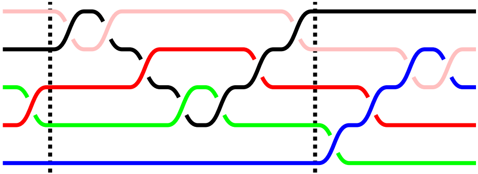
$$\Leftrightarrow$$



Theorem (V. Jugé, J. R. 24)

$$w \in \mathcal{ANF}_n$$

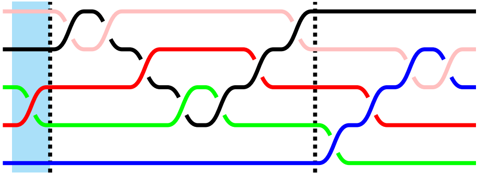
$$\Leftrightarrow$$



Theorem (V. Jugé, J. R. 24)

$$w \in \mathcal{ANF}_n$$

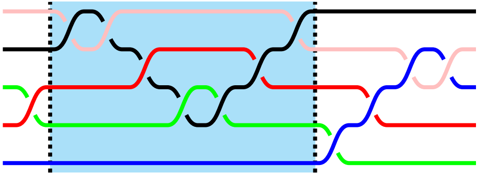
$$\Leftrightarrow$$



Theorem (V. Jugé, J. R. 24)

$$w \in \mathcal{ANF}_n$$

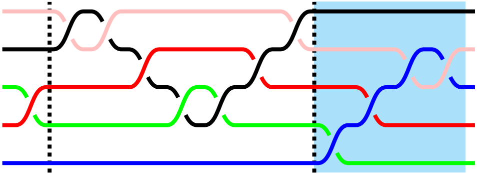
$$\Leftrightarrow$$



Theorem (V. Jugé, J. R. 24)

$$w \in \mathcal{ANF}_n$$

$$\Leftrightarrow$$



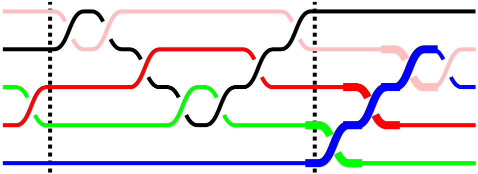
Characterization

Local characterization

Theorem (V. Jugé, J. R. 24)

$$w \in \mathcal{ANF}_n$$

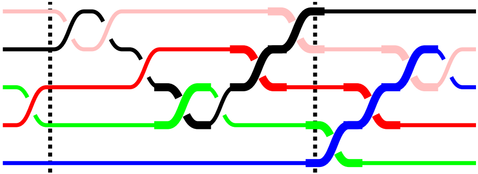
$$\Leftrightarrow$$



Theorem (V. Jugé, J. R. 24)

$$w \in \mathcal{ANF}_n$$

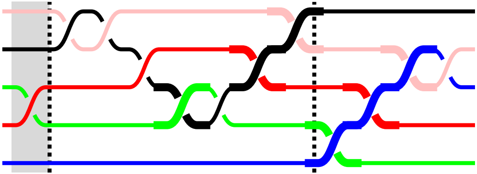
$$\Leftrightarrow$$



Theorem (V. Jugé, J. R. 24)

$$w \in \mathcal{ANF}_n$$

$$\Leftrightarrow$$



IV. Minimal automaton

for braids β such that $L(\beta) = \{\sigma_{n-1}\}$

$n = 3$

Minimal automaton

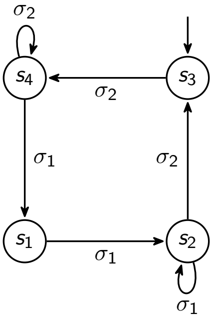


Figure: Automaton $\mathscr{A}_3'$

$n = 4$

Minimal automaton

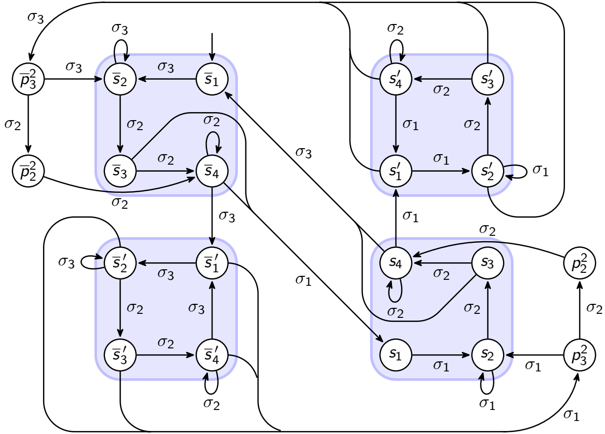
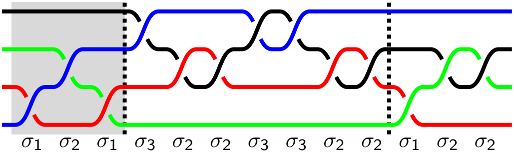
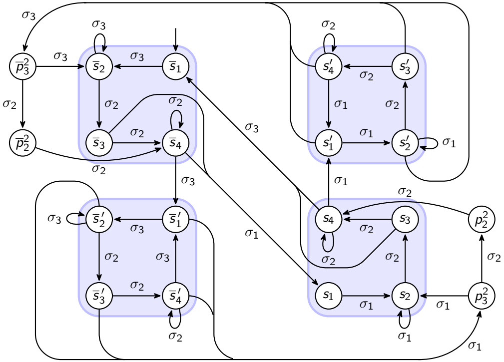


Figure: Automaton $\mathscr{A}'_4$

Small walk in $\mathscr{A}'_4$

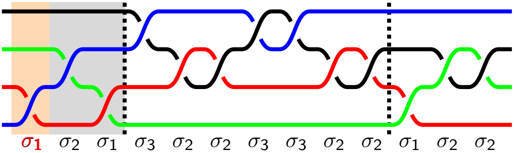
Minimal automaton

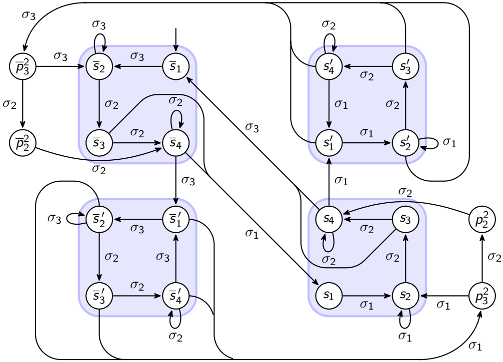




Small walk in $\mathscr{A}_4'$

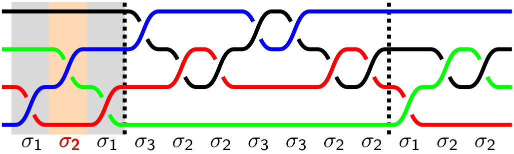
Minimal automaton

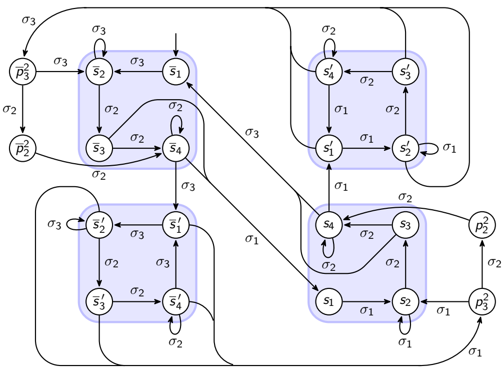




Small walk in $\mathscr{A}_4'$

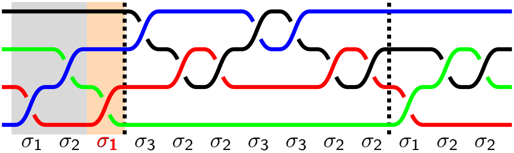
Minimal automaton

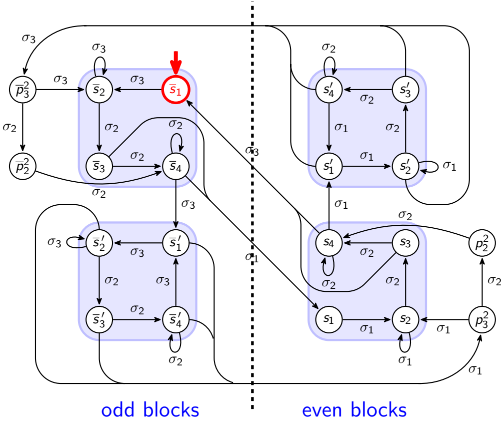




Small walk in $\mathscr{A}_4'$

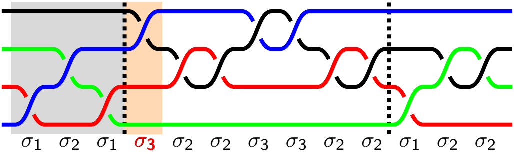
Minimal automaton

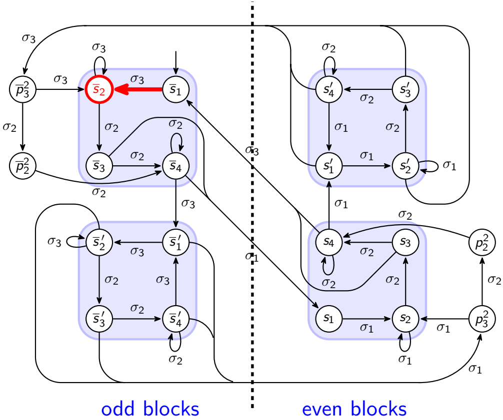




Small walk in $\mathscr{A}_4'$

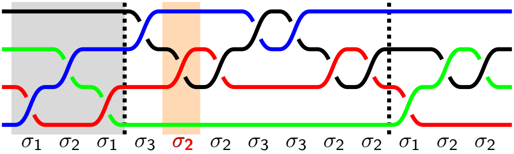
Minimal automaton

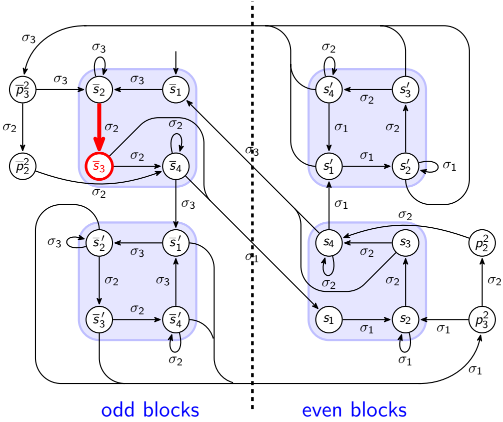




Small walk in $\mathscr{A}_4'$

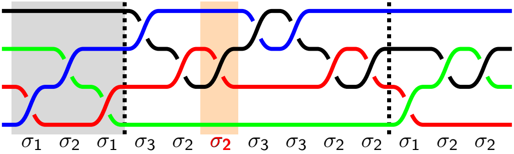
Minimal automaton

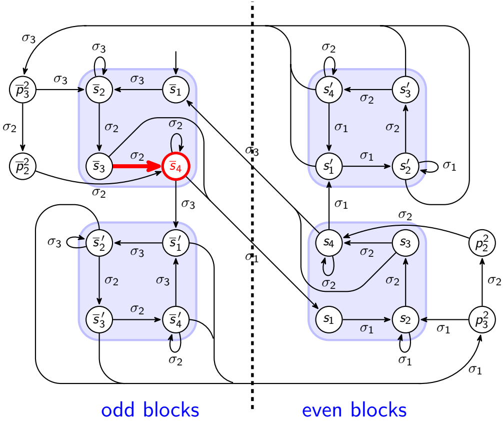




Small walk in $\mathscr{A}_4'$

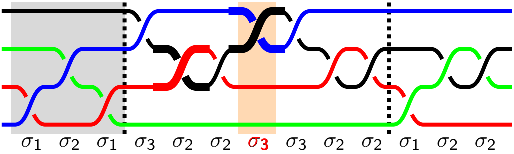
Minimal automaton

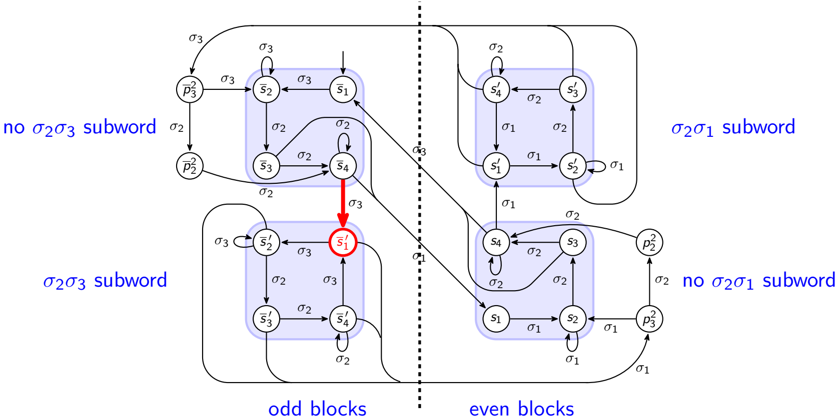




Small walk in $\mathscr{A}_4'$

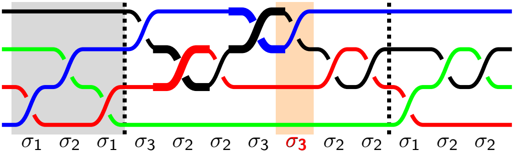
Minimal automaton

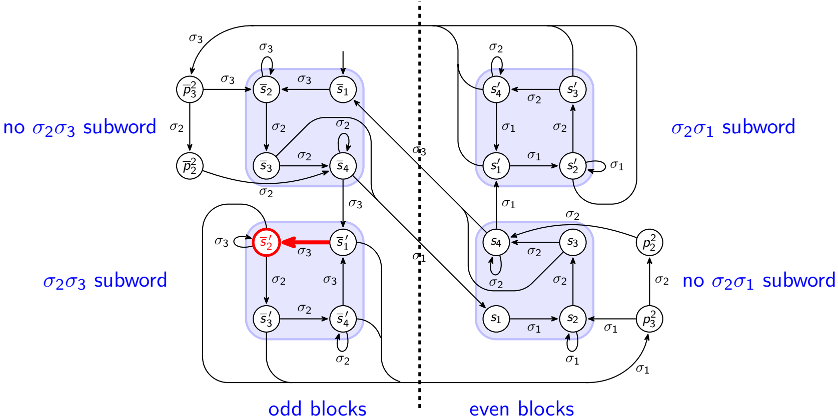




Small walk in $\mathscr{A}_4'$

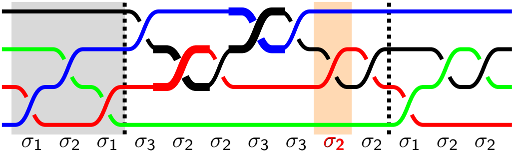
Minimal automaton

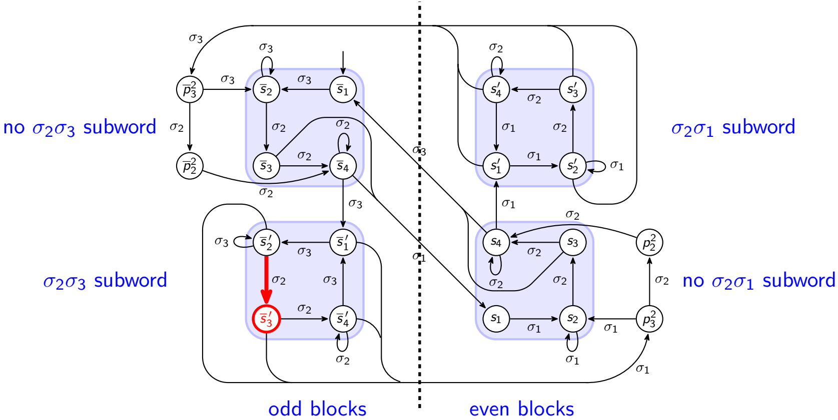




Small walk in $\mathscr{A}_4'$

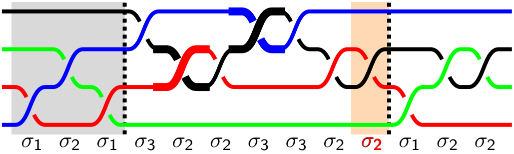
Minimal automaton

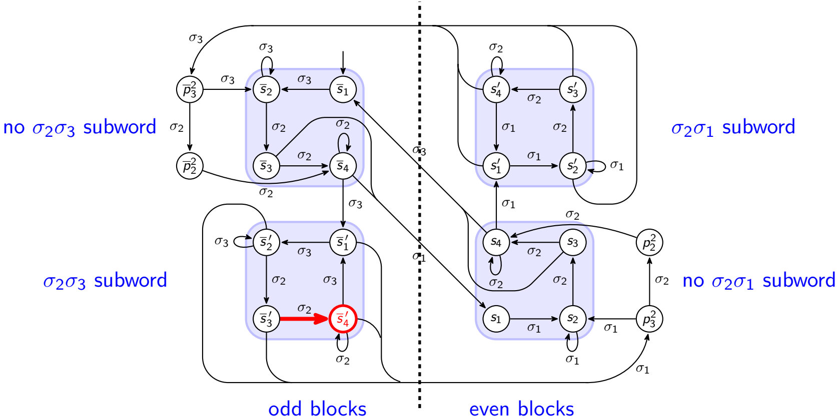




Small walk in $\mathscr{A}_4'$

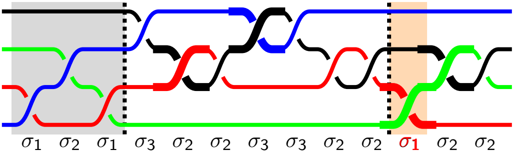
Minimal automaton

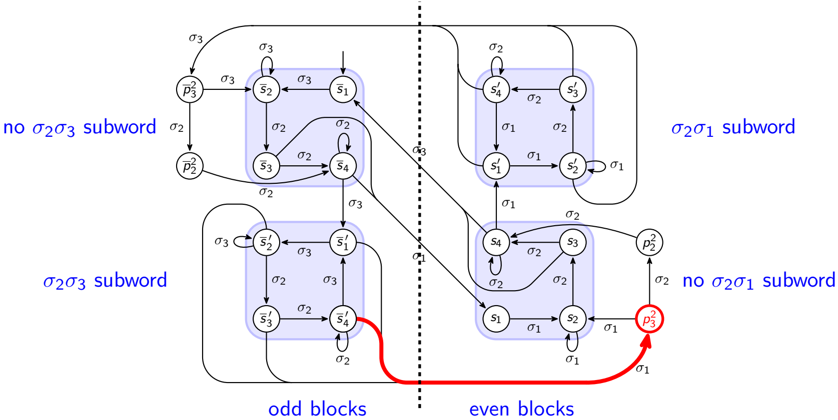




Small walk in $\mathscr{A}_4'$

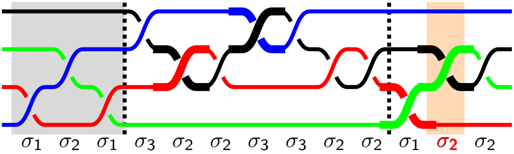
Minimal automaton

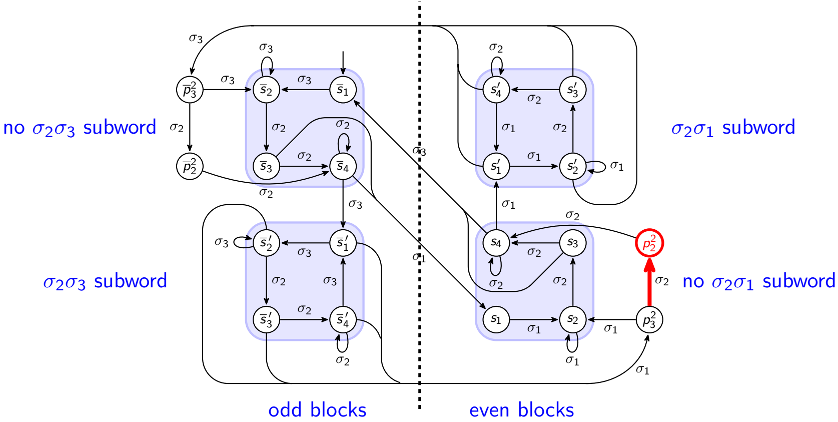




Small walk in $\mathscr{A}_4'$

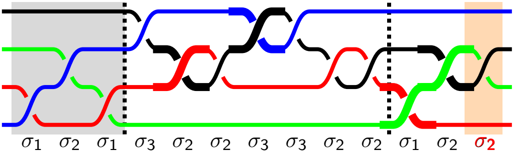
Minimal automaton

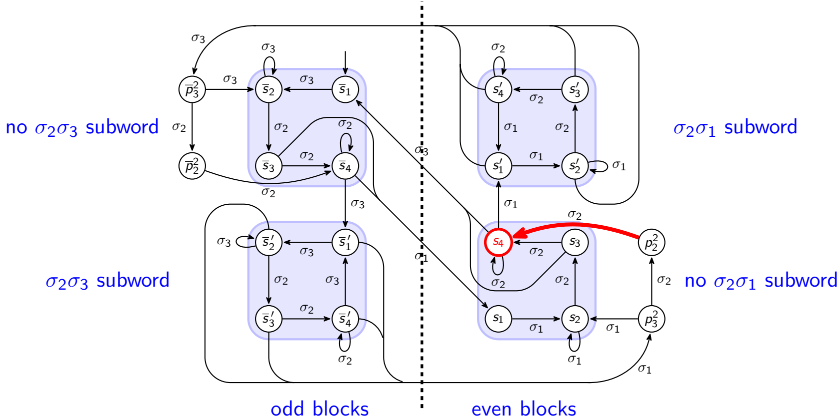




Small walk in $\mathscr{A}_4'$

Minimal automaton





Properties

Minimal automaton

- Minimal
- # States
- # Transitions

Properties

Minimal automaton

- Minimal
- $\#$ States $= \frac{25\times 2^{2n-3} - 9n^2 + 3n + 7}{27}$
- $\#$ Transitions

Properties

Minimal automaton

- Minimal
- $\#$ States $= \frac{25\times 2^{2n-3} - 9n^2 + 3n + 7}{27} = \Theta(4^n)$
- $\#$ Transitions

Properties

Minimal automaton

- Minimal
- # States $= \frac{25\times 2^{2n-3}-9n^2+3n+7}{27} = \Theta(4^n)$
- # Transitions $= \frac{(225n-290)2^{2n-5}-9n^3-9n^2+93n-77}{81}$

Properties

Minimal automaton

- Minimal
- # States $= \frac{25 \times 2^{2n-3} - 9n^2 + 3n + 7}{27} = \Theta(4^n)$
- # Transitions $= \frac{(225n - 290)2^{2n-5} - 9n^3 - 9n^2 + 93n - 77}{81} = \Theta(n \times 4^n)$

V. Conclusion

Perspectives

- Automaticity
- Rotating normal form in the dual monoid [5]
- Generalisation to other monoids? [2]
- Random generation [4]
- Links to the ordering of braids [3]
- New normal form in the group

[2] P. Dehornoy, 2008
[3] S. Burkel, 1997
[4] V. Gebhardt and J. González-Meneses, 2012
[5] J. Fromentin, 2018

Perspectives

- Automaticity $\rightarrow$ redaction in progress
- Rotating normal form in the dual monoid [5]
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[5] J. Fromentin, 2018

Perspectives

- Automaticity → redaction in progress
- Rotating normal form in the dual monoid [5] → redaction in progress
- Generalisation to other monoids? [2]
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[2] P. Dehornoy, 2008
[3] S. Burkel, 1997
[4] V. Gebhardt and J. González-Meneses, 2012
[5] J. Fromentin, 2018

Perspectives

- Automaticity → redaction in progress
- Rotating normal form in the dual monoid [5] → redaction in progress
- Generalisation to other monoids? [2] → work in progress with Jean Fromentin
- Random generation [4]
- Links to the ordering of braids [3]
- New normal form in the group

[2] P. Dehornoy, 2008
[3] S. Burkel, 1997
[4] V. Gebhardt and J. González-Meneses, 2012
[5] J. Fromentin, 2018

Thank you!

Sources

[1] `https://fr.wikipedia.org/wiki/Tresse_(mathématiques)`

[2] Patrick Dehornoy, *Alternating Normal Forms for Braids and Locally Garside Monoids*, 2008

[3] Serge Burckel, *The wellordering on positive braids*, 1997

[4] Volker Gebhardt and Juan González-Meneses, *Generating random braids*, 2012

[5] Jean Fromentin, *The rotating normal form is regular*, 2018

[6] Emil Artin, *Theorie der Zöpfe*, 1925

[7] David B.A. Epstein, *Word processing in groups*, 1992

[8] Vincent Jugé and June Roupin, *The alternating normal form of braids and its minimal automaton*, 2024

Appendix

Theorem (Characterization of the alternating normal form)

The word $w \in \mathbb{A}_n^*$ is in $\mathcal{ANF}_n$ with $L(w) = \{\sigma_1\}$ if $w = w_0\Phi_n(w_1)w_2 \dots \Phi_n^{\ell}(w_\ell)$ and:

(C1) $\forall i \geqslant 0$: *w_i is in $\mathcal{ANF}_{n-1}$ with $L(w_i) = \{\sigma_1\}$.*

(C2) $\forall i \in [\![0, \ell - 1]\!]$: *$\Phi_n(w_{i+1})w_{i+2}$ starts with $\sigma_{n-1 \to k_i}$ $\Rightarrow$ w_i contains a $(n-2, k_i - 1)$-chain.*

Get $\mathscr{A}_3'$

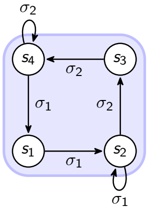

Figure: Construction of $\mathscr{A}_4'$

Make a copy

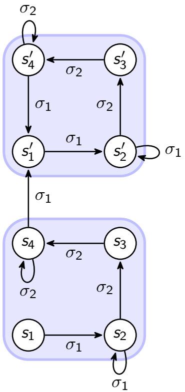

Figure: Construction of $\mathscr{A}'_4$

Appendix

Minimal automaton

Control the exit

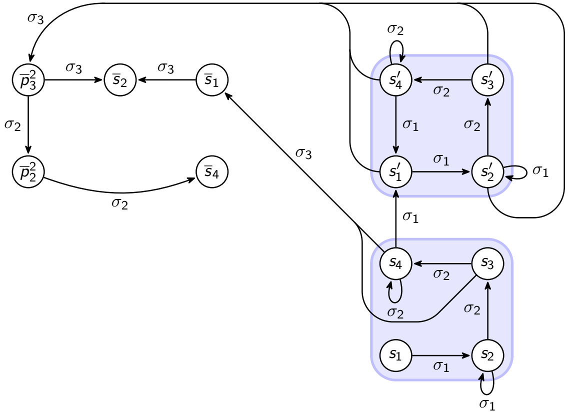

Figure: Construction of $\mathscr{A}'_4$

Appendix

Minimal automaton

Duplicate and connect

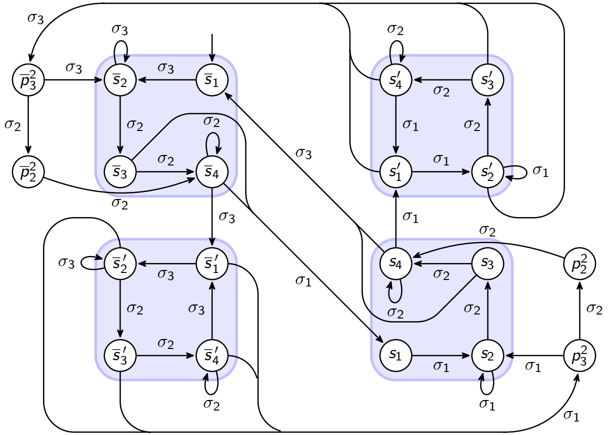

Figure: Construction of $\mathscr{A}_4'$

Appendix

Minimal automaton

Add $\mathscr{A}_3$

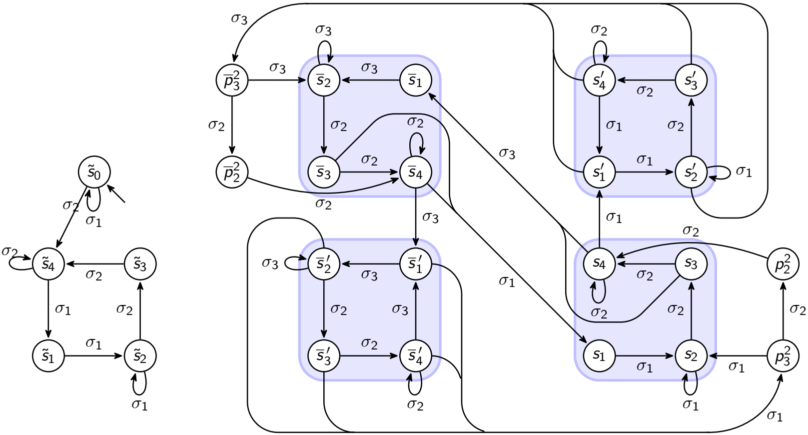

Figure: Construction of $\mathscr{A}_4$

Appendix

Minimal automaton

Connect $\mathscr{A}_3$ to $\mathscr{A}_4'$

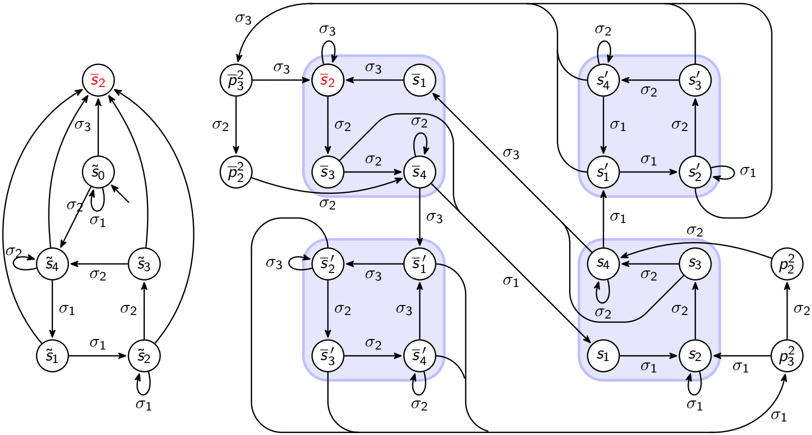

Figure: Construction of $\mathscr{A}_4$