Alternating normal form in the standard braid monoid
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Joint work with Vincent Jugé

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Braids

Objects: Braids [6]

(Upper picture source: [1])
Braids

Objects: Braids [6]

(Upper picture source: [1])

Braids

Objects: Braids [6]

Objective: Study normal forms

(Upper picture source: [1])

Braids

$n$ strings

**Generators/Letters:** $\mathbb{A}_n = \{\sigma_1, \ldots, \sigma_{n-1}\}$
Braids

$n$ strings

Generators/Letters: \( \mathbb{A}_n = \{\sigma_1, \ldots, \sigma_{n-1}\} \)

Words: \( \mathbb{A}_n^* \)
Braids

$n$ strings

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Words: $\mathbb{A}_n^*$

Relations:

\[ \sigma_i \sigma_j \equiv \sigma_j \sigma_i \text{ if } |i - j| > 1 \]

\[ \sigma_i \sigma_{i+1} \sigma_i \equiv \sigma_{i+1} \sigma_i \sigma_{i+1} \]

Figure: Commutation and braid relations
Braids

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Braids: Equivalence classes of words
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Figure: Commutation and braid relations

Braids: Equivalence classes of words
Braids

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Generators/Letters: $A_n = \{\sigma_1, \ldots, \sigma_{n-1}\}$

Words: $A_n^*$

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Figure: Commutation and braid relations

Braids: Equivalence classes of words
Braids

*n* strings

Generators/Letters: $\Sigma_n = \{\sigma_1, \ldots, \sigma_{n-1}\}$

Words: $\Sigma_n^*$

Relations:

$$\sigma_i \sigma_j \equiv \sigma_j \sigma_i \text{ if } |i - j| > 1$$

$$\sigma_i \sigma_{i+1} \sigma_i \equiv \sigma_{i+1} \sigma_i \sigma_{i+1}$$

Figure: Commutation and braid relations

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Relations:

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Figure: Commutation and braid relations

Braids: Equivalence classes of words
Figure: Representatives of the braid $\beta = \{w_1, w_2, w_3, w_4\}$
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Left-divisors $\rightarrow$ Lattice structure

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Left-divisors $\rightarrow$ Lattice structure

$w_1 = \sigma_2 \sigma_1 \sigma_2 \sigma_2 \sigma_3$

$w_2 = \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_3$

$w_3 = \sigma_1 \sigma_1 \sigma_2 \sigma_1 \sigma_3$

$w_4 = \sigma_1 \sigma_1 \sigma_2 \sigma_3 \sigma_1$

Figure: Representatives of the braid $\beta = \{w_1, w_2, w_3, w_4\}$

$\Rightarrow L(\beta) := \text{set of letters that left-divide } \beta$
Braids

Left-divisors $\rightarrow$ Lattice structure

Figure: Representatives of the braid $\beta = \{w_1, w_2, w_3, w_4\}$

$\Rightarrow L(\beta) := \text{set of letters that left-divide } \beta$

$L(\beta) = \{\sigma_1, \sigma_2\}$
I. Normal forms
Normal forms

Normal form = choice of a *unique representative* per braid
Lexicographic normal form [4]

Choose the minimal representant in the lexicographic order

\[ w_1 = \sigma_2 \sigma_1 \sigma_2 \sigma_2 \sigma_3 \]
\[ w_2 = \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_3 \]
\[ w_3 = \sigma_1 \sigma_1 \sigma_2 \sigma_1 \sigma_3 \]
\[ w_4 = \sigma_1 \sigma_1 \sigma_2 \sigma_3 \sigma_1 \]

\[ w_3 = \sigma_1 \sigma_1 \sigma_2 \sigma_1 \sigma_3 \]

Figure: Lexicographic normal form (\( \mathcal{L}\mathcal{E}\mathcal{X}_4 \)) of \( \beta \)

Lexicographic normal form [4]

Choose the minimal representant in the lexicographic order

Figure: Lexicographic normal form ($\mathcal{LEX}_4$) of $\beta$

Lexicographic normal form [4]

Choose the minimal representant in the lexicographic order

$\begin{align*}
  w_1 &= \sigma_2 \sigma_1 \sigma_2 \sigma_2 \sigma_3 \\
  w_2 &= \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_3 \\
  w_3 &= \sigma_1 \sigma_1 \sigma_2 \sigma_1 \sigma_3 \\
  w_4 &= \sigma_1 \sigma_1 \sigma_2 \sigma_3 \sigma_1
\end{align*}$

Figure: Lexicographic normal form ($\mathcal{LEX}_4$) of $\beta$

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$w_4 = \sigma_1 \sigma_1 \sigma_2 \sigma_3 \sigma_1$

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Lexicographic normal form [4]

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Figure: Lexicographic normal form $\mathcal{LEX}_4$ of $\beta$

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Alternating normal form [2,3]

Forbid a letter, alternatively and recursively

Figure: Alternating normal form $\mathcal{ANF}_n$

Alternating normal form [2,3]

Forbid a letter, alternatively and recursively

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Figure: Alternating normal form $\mathcal{ANF}_n$

Figure: $\beta$ in alternating normal form $(\mathcal{ANF}_4)$

Alternating normal form [2,3]

Forbid a letter, alternatively and recursively

Figure: Alternating normal form $\mathcal{ANF}_n$

Figure: $\beta$ in alternating normal form $(\mathcal{ANF}_4)$

Alternating normal form \([2,3]\]

Forbid a letter, alternatively and recursively

\[\beta_0\]
\[\beta_1\]
\[\beta_2\]

Figure: Alternating normal form \(\mathcal{ANF}_n\)

\[L(\beta') = \{\sigma_3\}\]

Figure: \(\beta\) in alternating normal form \((\mathcal{ANF}_4)\)

Alternating normal form [2,3]

Forbid a letter, alternatively and recursively

Figure: Alternating normal form $\mathcal{ANF}_n$

$L(\beta') = \{\sigma_1\}$

Figure: $\beta$ in alternating normal form ($\mathcal{ANF}_4$)

What is a good normal form?

- **Checking** that a word is in normal form is easy

- **Computing** the normal form of a word is easy
What is a good normal form?

- **Checking** that a word is in normal form is easy
  → Regularity

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  → Small minimal automaton

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What is a good normal form?

- **Checking** that a word is in normal form is easy
  - Regularity
  - Small minimal automaton

- **Computing** the normal form of a word is easy
  - Automaticity
What is a good normal form?

- Checking that a word is in normal form is easy
  - Regularity
  - Small minimal automaton

- Computing the normal form of a braid is easy
  - Automaticity
II. Regularity of normal forms
Regularity
Regularity
Regularity
Regularity

\[ L = ANF_3 = LEX_3 \]

\[ s_0 s_1 s_2 s_3 s_4 \sigma_1 \sigma_2 \sigma_1 \sigma_1 \sigma_2 \sigma_2 \sigma_2 \sigma_1 \]
Regularity

\[
L = \text{ANF}^3 = \text{LEX}^3
\]
Regularity

\[ L = \text{ANF}^3 = \text{LEX}^3 \]
Regularity

\[ L = \text{ANF}^3 = \text{LEX}^3 \]

\[ s_0, s_1, s_2, s_3, s_4 \]

\[ \sigma_1, \sigma_2 \]
Regularity

\[
\begin{align*}
L &= \text{ANF}_3 = \text{LEX}_3
\end{align*}
\]
Regularity
Regularity

$L = ANF_3 = LEX_3$
Regularity

\[ L = \mathcal{ANF}_3 = \mathcal{LEX}_3 \]
Regularity

Theorem ([3,4])

The lexicographic normal form and the alternating normal form are regular for all $n \geq 2$.

Regularity

**Theorem ([3,4])**

The **lexicographic normal form and the alternating normal form** are regular for all $n \geq 2$.

→ **Gives no explicit small automaton** for the alternating normal form!

---


Regularity

Theorem ([3,4])

The lexicographic normal form and the alternating normal form are regular for all $n \geq 2$.

→ Gives no explicit small automaton for the alternating normal form!

⇒ Need another characterization

III. Local characterization of the alternating normal form
Who is in normal form?

Local characterization

\[ v = w = \tilde{v} = \tilde{v}' \]
Who is in normal form?

Local characterization
Who is in normal form?

Local characterization

\[ v \equiv w = \tilde{v} = \tilde{v}' = 14/23 \]
Who is in normal form?

Local characterization
Who is in normal form?

Local characterization

\[ \tilde{v} = \]

\[ v = n = \tilde{v} = \tilde{v}' \]
Who is in normal form?

Local characterization

$v = w = \tilde{v} = \tilde{v}' = 14/23$
Who is in normal form?

Local characterization

$w =$

$\tilde{v} = \tilde{v}' = 14/23$
Who is in normal form?

Local characterization
Characterization

Local characterization

Theorem (V. Jugé, J. R. 24)

\[ w \in ANF_n \]

\[ \iff \]
Theorem (V. Jugé, J. R. 24)

\[ w \in ANF_n \]
Characterization

Local characterization

Theorem (V. Jugé, J. R. 24)

\[ w \in ANF_n \]

[Diagram of interwoven lines with a blue shaded area and dashed vertical lines]
Theorem (V. Jugé, J. R. 24)

\[ w \in ANF_n \]

\[ \Leftrightarrow \]

Characterization

Local characterization
Characterization

Local characterization

Theorem (V. Jugé, J. R. 24)

\[ w \in \mathcal{ANF}_n \]
Characterization

Local characterization

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Characterization
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Theorem (V. Jugé, J. R. 24)

\[ w \in ANF_n \]

\[ \iff \]
Characterization

Local characterization

Theorem (V. Jugé, J. R. 24)

\[ w \in AN\mathcal{F}_n \iff \]

\[
\]

\[
\]
IV. Minimal automaton

for braids $\beta$ such that $L(\beta) = \{\sigma_{n-1}\}$
$n = 3$
Minimal automaton

Figure: Automaton $\mathcal{A}_3'$
$n = 4$

Minimal automaton

Figure: Automaton $\mathcal{A}_4'$
Small walk in $A'_4$

Minimal automaton
Small walk in $\mathcal{A}_4'$

Minimal automaton

\[ \begin{array}{cccccccccccc}
\sigma_1 & \sigma_2 & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_2 & \sigma_2 & \sigma_1 & \sigma_1 & \sigma_2 & \sigma_2 & \sigma_2 & \sigma_1 \\\n\hline
s_1 & s_2 & s_3 & s_4 & s'_1 & s'_2 & s'_3 & s'_4 & s_1 & s_2 & s_3 & s_4 & s'_4 \end{array} \]
Small walk in $A'_4$

Minimal automaton
Small walk in $\mathcal{A}_4'$

Minimal automaton

Odd blocks

Even blocks
Small walk in $A_4'$

Minimal automaton

odd blocks
even blocks
Small walk in $A'_4$

Minimal automaton

Odd blocks

Even blocks
Small walk in $A'_4$

Minimal automaton
Small walk in $\mathcal{A}_4'$

Minimal automaton
Small walk in $\mathcal{A}_4'$

Minimal automaton

![Diagram showing a small walk in $\mathcal{A}_4'$](image)

- Odd blocks
- Even blocks

- No $\sigma_2\sigma_3$ subword
- $\sigma_2\sigma_3$ subword
- No $\sigma_2\sigma_1$ subword
- $\sigma_2\sigma_1$ subword
Small walk in $\mathcal{A}_4'$

Minimal automaton
Small walk in $A_4'$

Minimal automaton

Odd blocks

Even blocks

no $\sigma_2\sigma_3$ subword

$\sigma_2\sigma_3$ subword

$\sigma_2\sigma_1$ subword

no $\sigma_2\sigma_1$ subword
Small walk in $\mathcal{A}_4'$
Minimal automaton

Odd blocks
- $\sigma_2 \sigma_3$ subword
- $\sigma_2 \sigma_3$ subword
- no $\sigma_2 \sigma_1$ subword

Even blocks
- no $\sigma_2 \sigma_3$ subword
- $\sigma_2 \sigma_1$ subword
- no $\sigma_2 \sigma_1$ subword

The automaton transitions are depicted with arrows.
Small walk in $A'_4$

Minimal automaton

no $\sigma_2\sigma_3$ subword

$\sigma_2\sigma_3$ subword

$\sigma_2\sigma_1$ subword

no $\sigma_2\sigma_1$ subword
Small walk in $A'_4$

Minimal automaton

Odd blocks

Even blocks

no $\sigma_2\sigma_3$ subword

$\sigma_2\sigma_3$ subword

$\sigma_2\sigma_1$ subword

no $\sigma_2\sigma_1$ subword
Properties
Minimal automaton

- Minimal
- # States
- # Transitions
Properties

Minimal automaton

- Minimal
- # States \( \frac{25 \times 2^{2n-3} - 9n^2 + 3n + 7}{27} \)
- # Transitions
Properties

Minimal automaton

- Minimal

- # States = \(\frac{25 \times 2^{2n-3} - 9n^2 + 3n + 7}{27} = \Theta(4^n)\)

- # Transitions
Properties
Minimal automaton

- Minimal

- # States = \( \frac{25 \times 2^{2n-3} - 9n^2 + 3n + 7}{27} = \Theta(4^n) \)

- # Transitions = \( \frac{(225n - 290)2^{2n-5} - 9n^3 - 9n^2 + 93n - 77}{81} \)
Properties

Minimal automaton

► Minimal

► # States = \( \frac{25 \times 2^{2n-3} - 9n^2 + 3n + 7}{27} = \Theta(4^n) \)

► # Transitions = \( \frac{(225n - 290)2^{2n-5} - 9n^3 - 9n^2 + 93n - 77}{81} = \Theta(n \times 4^n) \)
V. Conclusion
Perspectives

▸ Automaticity

▸ Rotating normal form in the dual monoid [5]

▸ Generalisation to other monoids? [2]

▸ Random generation [4]

▸ Links to the ordering of braids [3]

▸ New normal form in the group

Perspectives

- Automaticity → redaction in progress

- Rotating normal form in the dual monoid [5]

- Generalisation to other monoids? [2]

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Perspectives

- Automaticity → redaction in progress

- Rotating normal form in the dual monoid [5] → redaction in progress

- Generalisation to other monoids? [2]

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Perspectives

- Automaticity → redaction in progress
- Rotating normal form in the dual monoid [5] → redaction in progress
- Generalisation to other monoids? [2] → work in progress with Jean Fromentin
- Random generation [4]
- Links to the ordering of braids [3]
- New normal form in the group

Thank you!
Sources

Appendix

Theorem (Characterization of the alternating normal form)

*The word* $w \in \mathbb{A}_n^*$ *is in* $\mathbb{ANF}_n$ *with* $L(w) = \{\sigma_1\}$ *if*

$w = w_0 \Phi_n(w_1)w_2 \ldots \Phi^n(\ell)(w_\ell)$ *and:

(C1) $\forall i \geq 0$: $w_i$ *is in* $\mathbb{ANF}_{n-1}$ *with* $L(w_i) = \{\sigma_1\}$.

(C2) $\forall i \in [0, \ell - 1]$: $\Phi_n(w_{i+1})w_{i+2}$ *starts with* $\sigma_{n-1} \rightarrow k_i$

$\Rightarrow w_i$ *contains a* $(n - 2, k_i - 1)$-*chain.*
Appendix

Minimal automaton

Get $\mathcal{A}'_3$

Figure: Construction of $\mathcal{A}'_4$
Appendix

Minimal automaton

Make a copy

Figure: Construction of $A_4'$
Appendix

Minimal automaton

Control the exit

Figure: Construction of $\mathcal{A}_4'$
Appendix

Minimal automaton

Duplicate and connect

Figure: Construction of $A_4'$
Appendix

Minimal automaton

Add $A_3$

Figure: Construction of $A_4$
Appendix

Minimal automaton

Connect $A_3$ to $A_4'$

Figure: Construction of $A_4$