Multicoloured Hardcore Model: Fast Mixing and Queueing

Sam Olesker-Taylor
University of Warwick

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- Our objective is to *properly colour* a (large) subset U of vertices
- If $u, u' \in U$ and $\{u, u'\} \in E$, then u and u' have different colours
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Examples. Frequency-limited communication

- Nearby users of short-range radio
- Fibreoptic routing (more on this later)

Model and Main Result

Glauber-Type Dynamics

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Let $K \in \mathbb{N}$ and $\Omega := \{\omega \in \{0, 1, \ldots, K\}^V \mid \omega \text{ proper}\}$, where

$$\omega \text{ is } \textit{proper} \quad \text{if} \quad \omega_u \neq \omega_v \quad \text{whenever} \quad \{u, v\} \in E \quad \text{and} \quad \omega_u + \omega_v > 0.$$

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- When vertex $v \in V$ is chosen, toss a p_v-biassed coin: $C \sim \mathrm{Bern}(p_v)$.
  - If $C = 1$, choose a (non-zero) colour $k \in \{1, ..., K\}$ uniformly. If k is *available* for v, then paint v with colour k.
  - If $C = 0$, give it 'colour' 0—ie, *deactivate* it.

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Denote the equilibrium distribution π and the *equilibrium service rates*

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If preferred, at each discrete time-step, choose $v \in V$ with probability $\propto \lambda_v$.

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Let X and Y be a coupling of μ and π: ie, $X \sim \mu$ and $Y \sim \pi$. Then,

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Path coupling only requires contraction of adjacent configurations in expectation.

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Theorem (Fast Mixing for MCH)

Let $n := |V|$ *and* $X \sim \mathsf{MCH}_\Omega(\boldsymbol{\lambda}, \boldsymbol{p})$*. Suppose that there exists* $\beta > 0$ *such that*

$$\tfrac{1}{K} \textstyle\sum_{u \in V : \{u,v\} \in E} p_u \lambda_u / \lambda_v \le 1 - \beta \quad \text{for all} \quad v \in V.$$

Then,

$$\max_{x \in \Omega} \| \mathbb{P}_x[X^t \in \cdot] - \pi \|_{TV} \le \min\{2ne^{-\beta \lambda_{\min} t}, 1\} \quad \text{for all} \quad t \ge 0.$$

In particular, $t_{mix}(\varepsilon) \le (\beta \lambda_{\min})^{-1} \log(2n/\varepsilon) \quad \text{for all} \quad \varepsilon \in (0,1).$

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Remark (Heuristic-Driven Choice of Parameters $(\boldsymbol{\lambda}, \boldsymbol{p})$)

The graph G and number K of colours are prescribed by the application.

- High-degree vertices have more impact, so update them faster: $\lambda_v \propto d_v$.
- If v is 'on' with prob p_v, then it removes $p_v d_v$ colour choices: $p_v d_v \propto K$.

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The conditions are satisfied with $\beta = \frac{1}{3}$ if $\lambda_v = d_v / \bar{d}$ and $p_v \le \frac{2}{3} K / d_v$.
Further, under these conditions, $\frac{1}{3} p_v \le s_v \le p_v$ for all $v \in V$.

Application as a Queueing System

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The MCH dynamics provide a decentralised way to share the resource.

- It is fast and easy to test if a given colour is available along the route
- *Optical switches* can be configured rapidly to set-up a *light path*
- The particular light path is used to transmit data until it is refreshed

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$$q \to \begin{cases} q^{v,+} & \text{at rate} \quad \nu_v \\ q^{v,-} & \text{at rate} \quad \mu_v \mathbf{1}\{x_v \neq 0\} \end{cases} \quad \begin{array}{l} \text{where} \quad q_u^{v,\pm} := q_u \pm \mathbf{1}\{u = v\} \\ \text{given MCH configuration } X = x \end{array}$$

In other words, *jobs* arrive to vertex $v \in V$ at rate ν_v and are processed at rate μ_v *provided* vertex v is *active* in the underlying MCH configuration (ie, $x_v \neq 0$).

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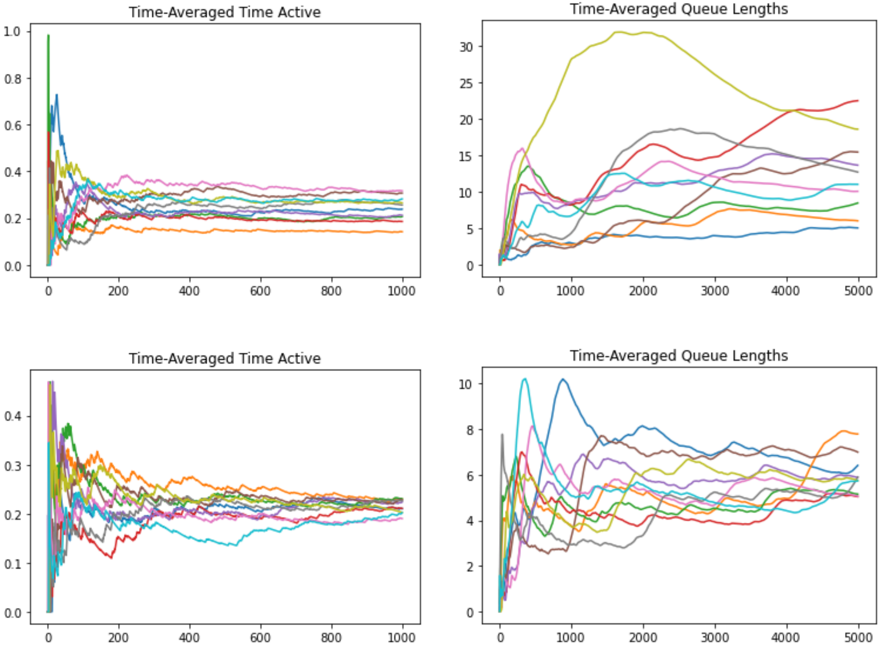
$$\mathbb{E}[Q_v^0] \le \frac{18(\bar{d}/d_{\min}) n \log(2n/e)}{(s_v - \nu_v)^2} \quad \textit{for all} \quad v \in V.$$

Simulations

Graphs: Erdős–Rényi (top) and random-regular (bottom) graphs

- $n = 500$ vertices and average degree $\bar{d} = 40$
- $K = 10$ colours and probabilities $p_v = \frac{4}{5} eK/d_v \approx 0.5$


Time-Averaged Time Active
1.0
0.8
0.6
0.4
0.2
0.0
0
200
400
600
800
1000
Time-Averaged Queue Lengths
30
25
20
15
10
5
0
0
1000
2000
3000
4000
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