Patricia's bad distributions

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(Goethe University Frankfurt)

joint work with **Louigi Addario-Berry** and **Pat Morin**

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University of Bath

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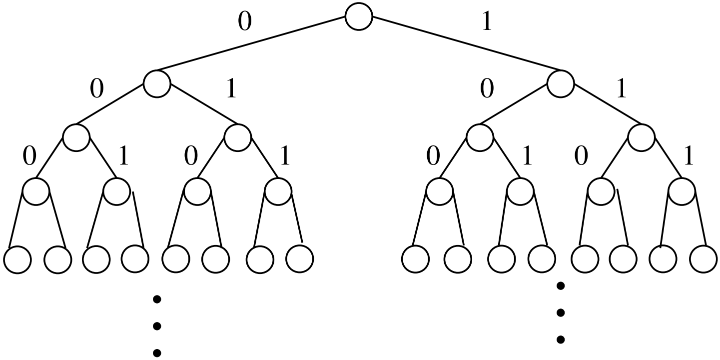

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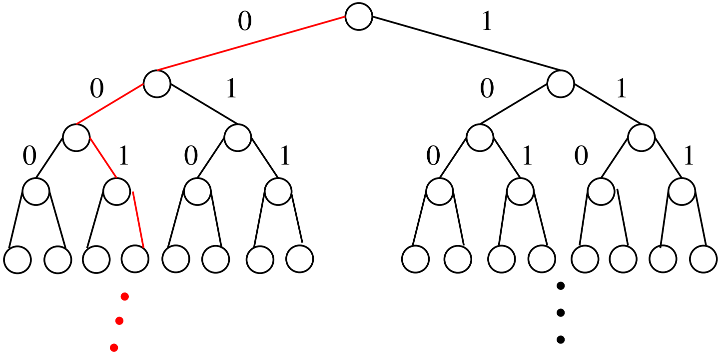

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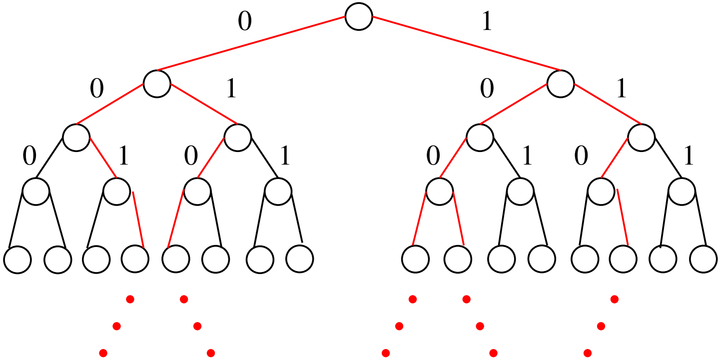

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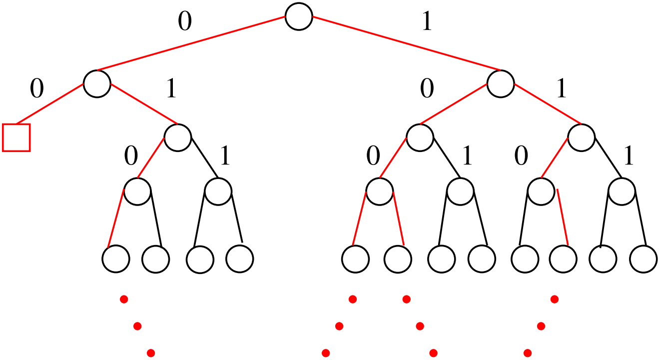

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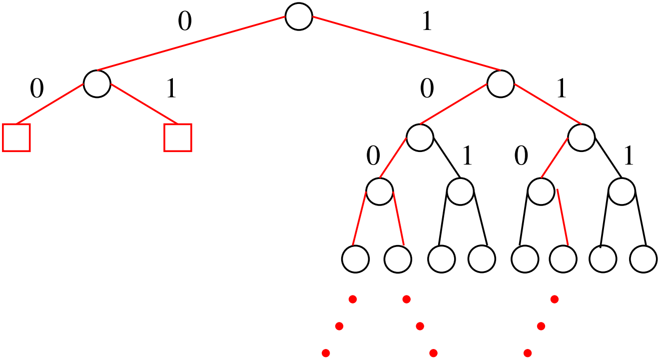

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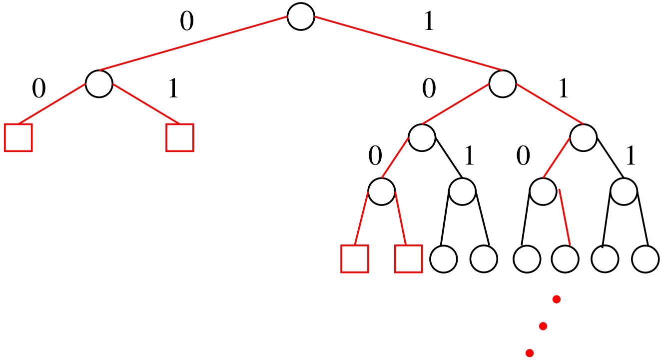

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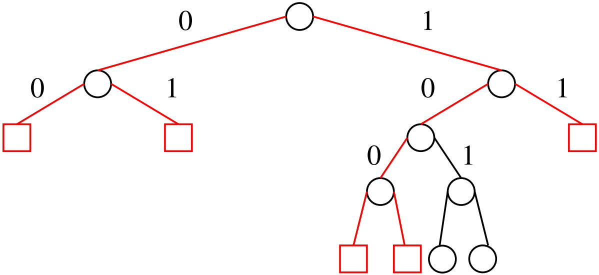

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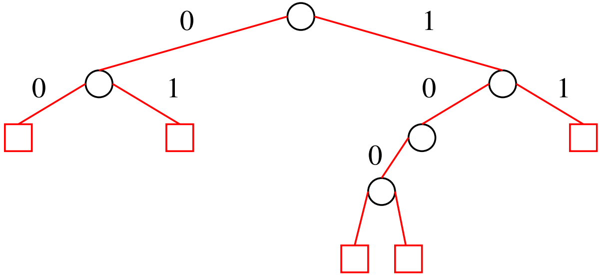

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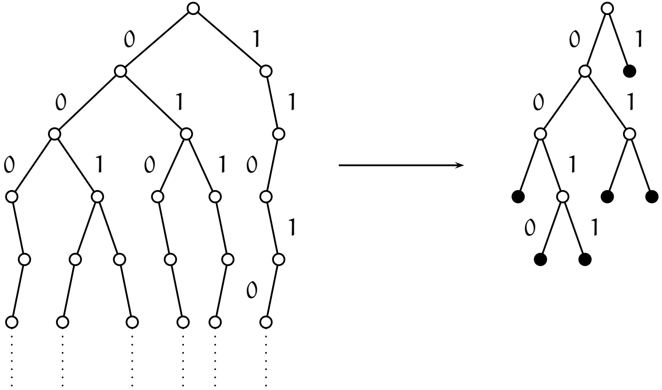

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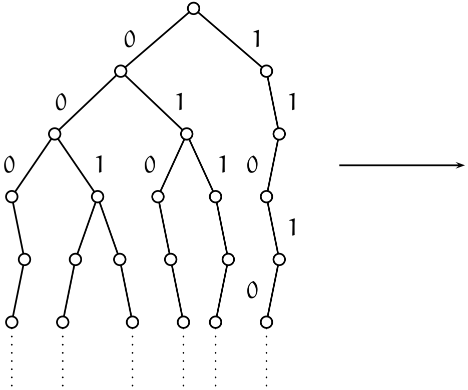

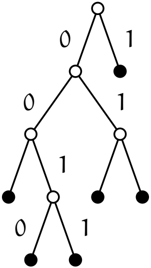

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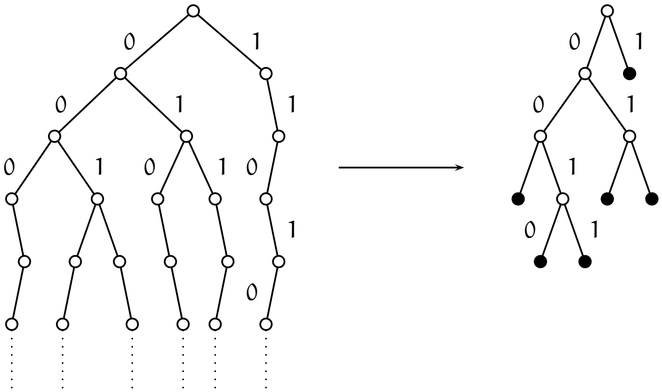
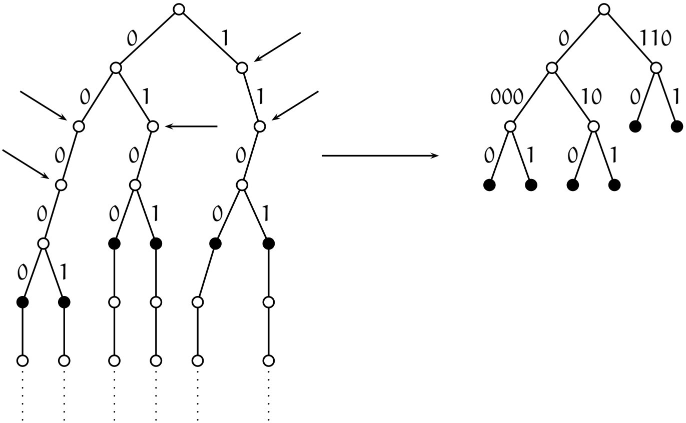


PATRICIA tree

From Trie to PATRICIA tree:



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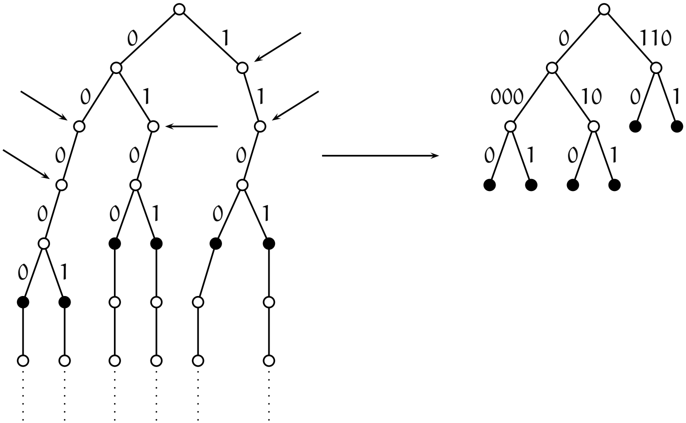

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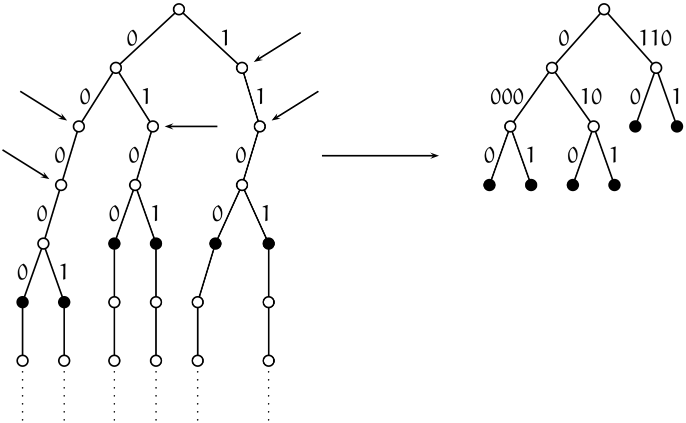
PATRICIA: Practical Algorithm To Retrieve Information Coded In Alphanumeric (D. Morrison 1968, G. Gwehenberger 1968)



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Dynamical sources: Clément, Flajolet & Vallée (2001)

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Theorem: (Devroye, 2005)
For all $t > 0$ we have

$$\mathbb{P}(H_n \geq \mathbb{E}[H_n] + t) \leq \exp\left(-\frac{t^2}{2(\mathbb{E}[H_n]+1) + 2t/3}\right)$$

$$\mathbb{P}(H_n \leq \mathbb{E}[H_n] - t) \leq \exp\left(-\frac{t^2}{2(\mathbb{E}[H_n]+1)}\right)$$

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Remark: Devroye (1992) has Theorem 1 for the density model and Theorem 2 for $\alpha_n = n^\varepsilon$.

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Lemma: For all $n \in \{1, \dots, N\}$ and μ_N we have

$$\mathbb{E}[H_n] \geq n - 2.$$

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For all large n we have $\lceil 2n/\alpha_n \rceil \leq n \leq B(\beta_n)$.

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They have suffixes $(\Xi_1^{(j)}, \Xi_2^{(j)}, \ldots)$ drawn ind. from $\mu_{B(\beta_n)}$

For all large n we have $\lceil 2n/\alpha_n \rceil \leq n \leq B(\beta_n)$.

The Lemma implies the assertion. $\square$

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Borel–Cantelli Lemma implies

$$\frac{H_n}{n/\alpha_n} \to \infty \text{ almost surely.}$$

THANK YOU!

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Then, by continuity of measure

$$\mathbb{P}(\Xi = v) = \lim_{k\to\infty} \mathbb{P}(\xi_1\ldots\xi_k = v_1\ldots v_k) \geq \varepsilon.$$

Contradiction to diffuse.

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Tail bound for Binomial plus Borel–Cantelli.