Grammar-based tree compression: combinatorics and algorithms

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The size of $\mathcal{G}$ is the number of variables ($=$ number of productions).

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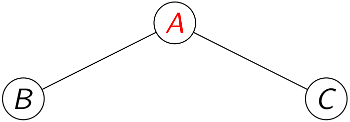

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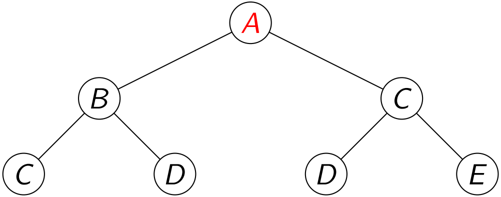

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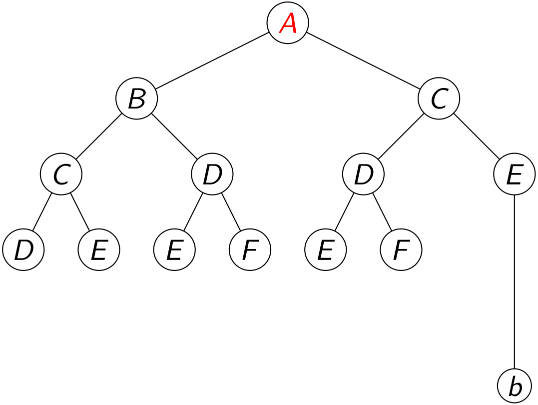

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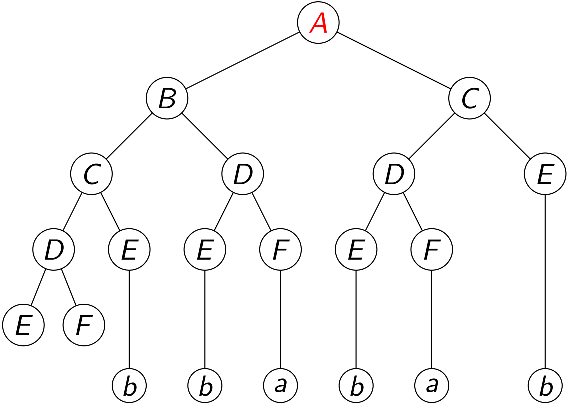

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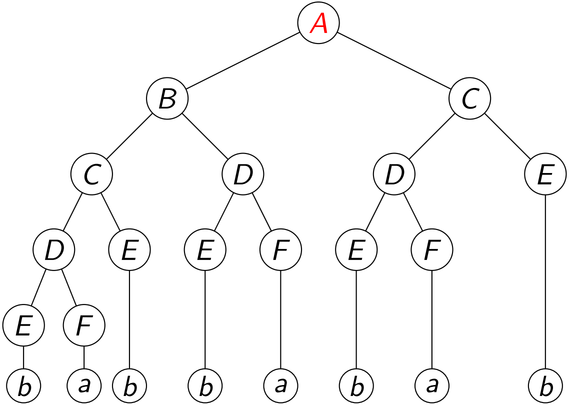

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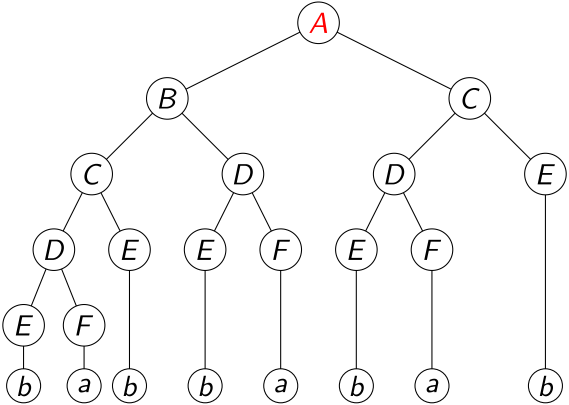

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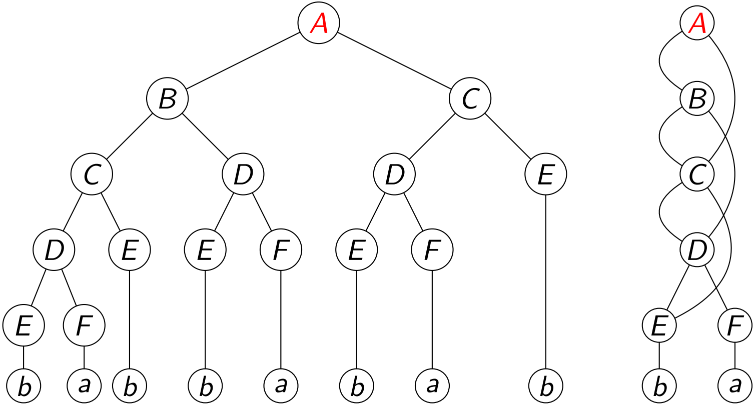

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Berstel, Brlek 1987

$\text{opt}(w) \leq \mathcal{O}\left(\frac{n}{\log_\sigma n}\right)$ (assuming $\sigma \geq 2$).

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Charikar et al. 2002, Rytter 2004, Jez 2013

There is a linear time algorithm that produces for every word w of length n an SLP of size at most $\mathcal{O}(\log(n) \cdot \text{opt}(w))$.

Balancing straight-line program

Ganardi, Jeż, L 2021

From a given SLP $\mathcal{G}$ of size n such that $w := \mathrm{val}(\mathcal{G})$ has length N, one can compute in time $\mathcal{O}(n)$ an SLP $\mathcal{H}$ such that:

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Has been shown by Bille et al. using several complicated data structures.

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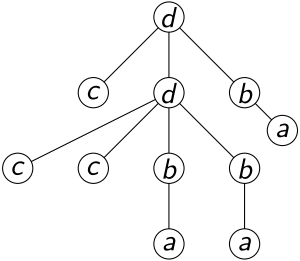

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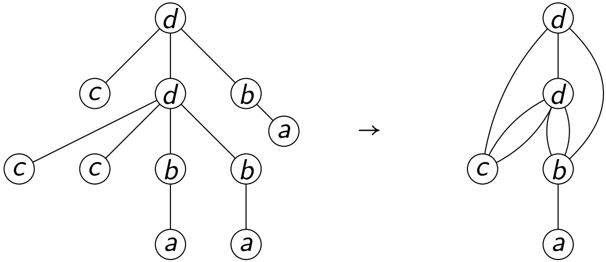

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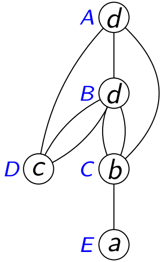

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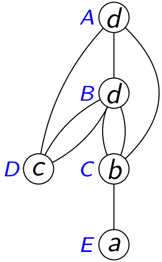

$$
\begin{aligned}
A &\to d(D, B, C) \\
B &\to d(D, D, C, C) \\
C &\to b(E) \\
D &\to c \\
E &\to a
\end{aligned}
$$

Minimal DAGs

Clearly, every tree has a unique minimal DAG: merge nodes in which isomorphic subtrees are rooted as long as possible.

Downey, Sethi, Tarjan 1980

For a given tree, its minimal DAG can be computed in linear time.

DAGs and asymptotic combinatorics

Bousquet-Mélou, L, Maneth, Noeth 2015

The average number of nodes of the minimal DAG for a uniformly chosen tree of size n with $k = |\Gamma|$ node labels is

$$\sqrt{\frac{\ln(4k)}{\pi}} \cdot \frac{n}{\sqrt{\ln n}} \cdot \big(1 + o(1)\big).$$

- Extends a result of Flajolet, Sipala and Steyaert for binary unlabelled trees.
- Similar results that apply to certain classes of random tree models were recently shown by Seelbach-Benkner and Wagner.

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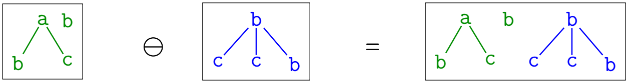

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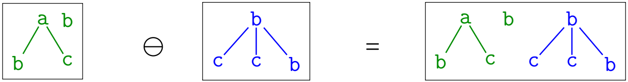

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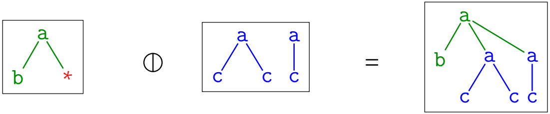

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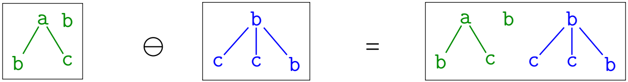

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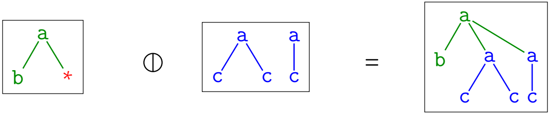

A (forest) context is a forest, where exactly one leaf is labelled with the special symbol $* \notin \Gamma$.

Forest algebra expressions

Forests and forest contexts can be also written as parenthesized expressions:

- A forest : $a(b\,c)\ b(b(c\,c\,a)\,a)\ a$
- A forest context: $a(b\,c)\ b(b(c\,c\,*)\,a)\ a$

Then we have

- $F \ominus G = F\,G$
- $F \varobar G = F[* \to G]$

A forest algebra expression is an expression that is built from the constants

- a and $a_* := a(*)$ for $a \in \Gamma$ and
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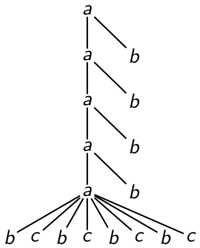

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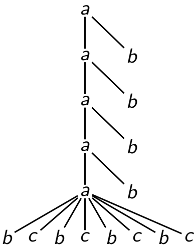

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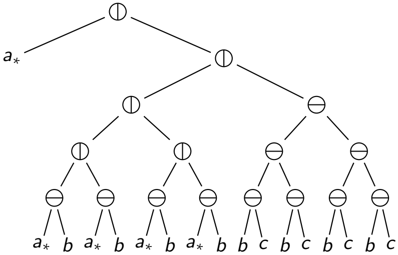

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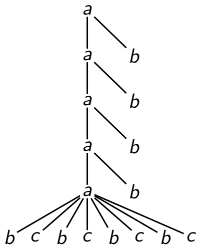

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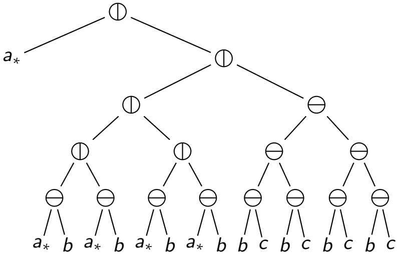

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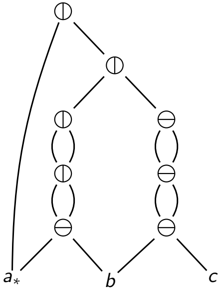

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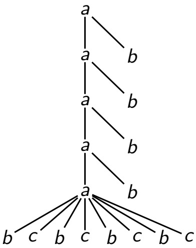

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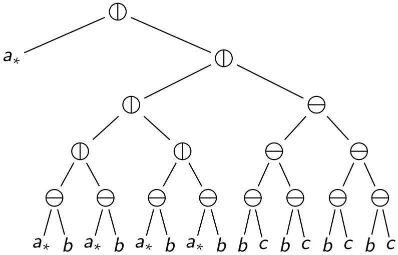

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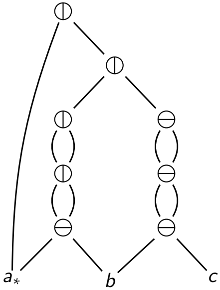

FSLP for F

For an FSLP $\mathcal{G}$ we denote with $\text{val}(\mathcal{G})$ the forest produced by $\mathcal{G}$.

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  - top DAGs (Bille, Gørtz, Landau, Weimann 2013): very similar to FSLPs
- A (string) SLP is an FSLP that only uses the constants a for $a \in \Gamma$ and the operation $\ominus$.

  Such an FSLP produces a forest consisting of a chain of singleton trees.

Small FSLPs always exist

Hucke, L, Noeth 2014

There is an algorithm that produces in linear time from a tree (or forest) of size n with k different node labels an FSLP of size $\mathcal{O}\left(\frac{n}{\log_k n}\right)$.

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3. The FSLP consists of two parts, both of size $\mathcal{O}\left(\frac{n}{\log_k n}\right)$:

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There is an algorithm that produces in linear time from a tree (or forest) of size n with k different node labels an FSLP of size $\mathcal{O}\left(\frac{n}{\log_k n}\right)$.

Proof idea:

1. Partition the input forest F of size n into $\Theta\left(\frac{n}{\ell}\right)$ many subforests and subcontexts of size in $[\ell, 2\ell]$.
2. One can choose $\ell = \Theta(\log_k n)$ such that the total number of forests and contexts of size in $[\ell, 2\ell]$ is $\mathcal{O}\left(\frac{n}{\log_k n}\right)$.
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    - A DAG producing the subforests and subcontexts from 1.

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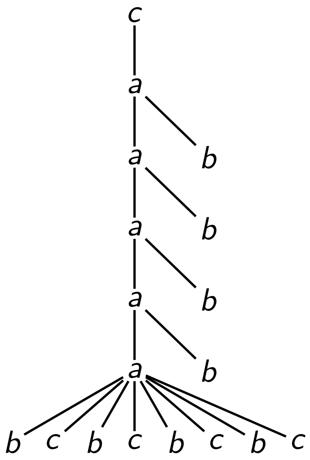

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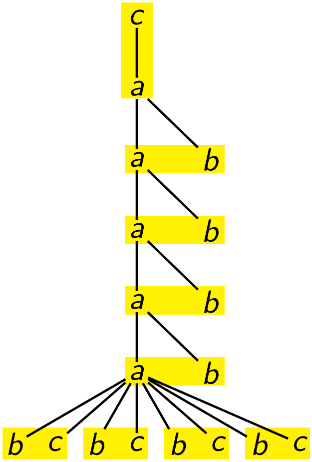

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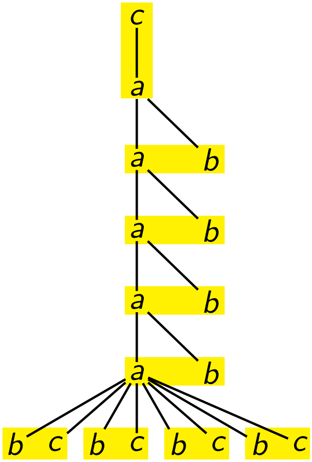

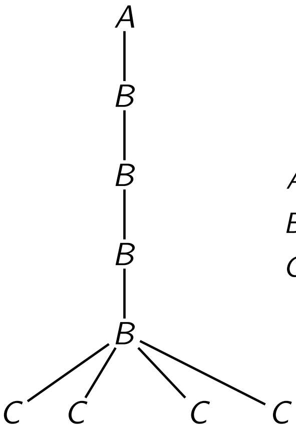

$$A \rightarrow c_* \obar a_*$$
$$B \rightarrow a_* \ominus b$$
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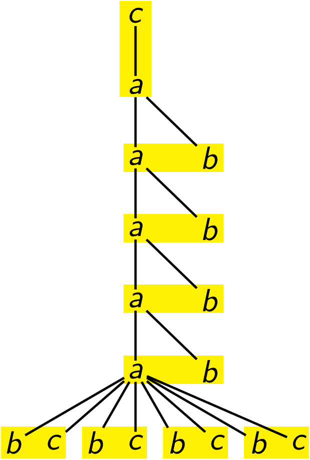

$$X_0 \to A \oplus X_1$$
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$$X_2 \to B \oplus X_3$$
$$X_3 \to B \oplus X_4$$
$$X_4 \to B \oplus X_5$$
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- L, Maneth, Mennicke 2013: TreeRePair; a practical algorithm for computing small FSLPs

The smallest grammar problem for trees

L, Jeż 2013

There is a linear time algorithm that produces for every forest F of size n an FSLP of size $O(\log(n) \cdot \mathrm{opt}(F))$.

Balancing forest straight-line program

Ganardi, Jeż, L 2021

From a given FSLP $\mathcal{G}$ of size n such that $F := \mathrm{val}(\mathcal{G})$ has size N, one can compute in time $\mathcal{O}(n)$ an FSLP $\mathcal{H}$ such that:

- $\mathrm{val}(\mathcal{H}) = \mathrm{val}(\mathcal{G})$
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Corollary: random access in logarithmic time on compressed forests.

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Corollary: random access in logarithmic time on compressed forests.

From a given FSLP $\mathcal{G}$ one can built in linear time a data structure that allows to solve for $F = \mathsf{val}(\mathcal{G})$ the following problem in time $\mathcal{O}(\log |F|)$:

- Input: a preorder number of a node v in F
- Output: the label of the node v.

FSLPs in database theory

Goal: For a given

- huge tree (e.g. an XML tree structure) that is stored compressed as an FSLP and
- a query formulated in a suitable query language

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We assume that queries are formulated in MSO (monadic 2nd order logic):

- there are two types of variables:
  - x, y, z, x' etc. for tree nodes
  - X, Y, Z, Z' etc. for sets of tree nodes

FSLPs in database theory

- atomic formulas (x, y are node variables, X is a node set variable):
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Consider now a forest F and an MSO formula $\phi(X)$ where X is the only free variable X in ϕ.

Then $\text{query}(\phi(X), F)$ is the sets $A \subseteq \text{nodes}(F)$ such that $F \models \phi(A)$.

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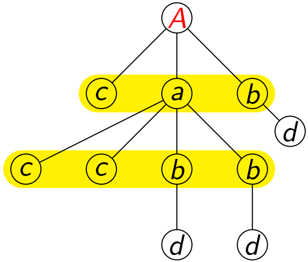

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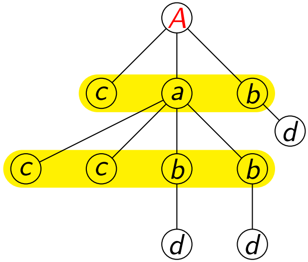

Note: $\text{query}(\phi(X), F)$ may have size $2^{|F|}$, i.e., size $2^{2^{\mathcal{O}(|\mathcal{G}|)}}$ if F is given by the FSLP $\mathcal{G}$.

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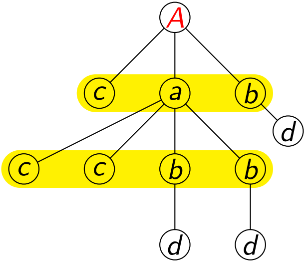

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What does it mean to enumerate efficiently $\text{query}(\phi(X), \text{val}(\mathcal{G}))$?

Enumeration problems

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- $\mathcal{A}$ works in output-linear delay if $t_i - t_{i-1} \leq \mathcal{O}(|y_i|)$ for all $1 \leq i \leq k$.

FSLPs in database theory

L, Schmid 2024

Fix a query $\phi(X)$. One can enumerate $\text{query}(\phi(X), \text{val}(\mathcal{G}))$ for a given FSLP $\mathcal{G}$ in

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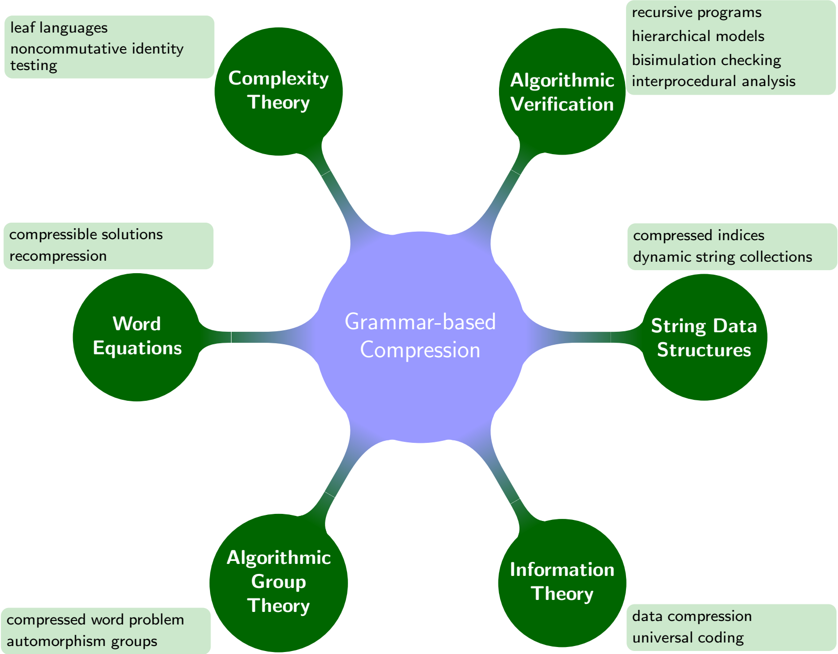
Proof strategy: Let Γ be the set node labels of our trees.

1. Translate the MSO-query $\phi(X)$ into a node-selecting tree automaton $\mathcal{A}$ (a tree automaton working on the label set $\Gamma \times \{0, 1\}$).
2. Reduce enumeration of $\text{query}(\mathcal{A}, \text{val}(\mathcal{G}))$ to the enumeration of $\text{query}(\mathcal{B}, \text{unfold}(\mathcal{G}))$, where
   - $\text{unfold}(\mathcal{G})$ is the forest algebra expression obtained by unfolding $\mathcal{G}$ and
   - $\mathcal{B}$ is a leaf-selecting tree automaton.
3. Bagan solved the previous enumeration problem for the case where the tree $\text{unfold}(\mathcal{G})$ is given explicitly.

   We extend Bagan's algorithm to DAG-compressed trees.


Grammar-based Compression
Complexity Theory
leaf languages
noncommutative identity testing
Algorithmic Verification
recursive programs
hierarchical models
bisimulation checking
interprocedural analysis
Word Equations
compressible solutions
recompression
String Data Structures
compressed indices
dynamic string collections
Algorithmic Group Theory
compressed word problem
automorphism groups
Information Theory
data compression
universal coding