Multi-Pivot Quicksort
and
How to Compute Precise Asymptotics

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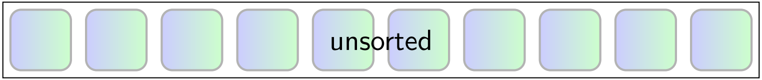
Quicksort & Quickselect



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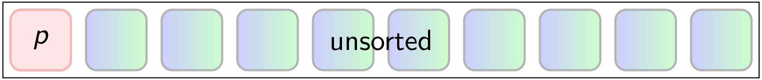

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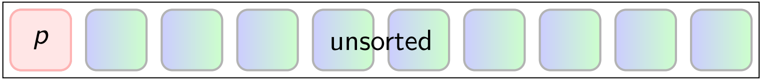

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- partition into
  - small elements
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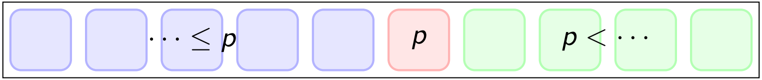

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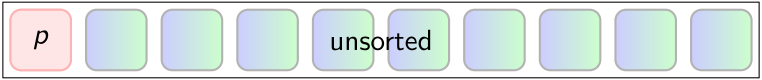

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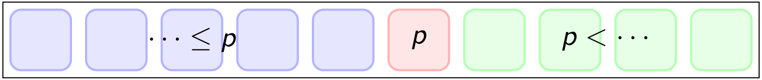

- proceed recursively

Quicksort: Recurrence Relation

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$$C_n = \quad n-1 \; + \qquad\qquad\qquad\qquad C_J \; + \; C_{n-1-J} \qquad \Big| \, \mathbb{E}$$

$$c_n = \mathbb{E}(C_n) = \quad n-1 \; + \quad \sum_{j=0}^{n-1} \mathbb{P}(J=j) \big(\mathbb{E}(C_j) + \mathbb{E}(C_{n-1-j}) \big)$$

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$$c_n = \mathbb{E}(C_n) = n - 1 + \sum_{j=0}^{n-1} \underbrace{\mathbb{P}(J=j)}_{=\frac{1}{n}} \Big(\underbrace{\mathbb{E}(C_j)}_{=c_j} + \underbrace{\mathbb{E}(C_{n-1-j})}_{=c_{n-1-j}} \Big)$$

$$c_n = n - 1 + \frac{2}{n} \sum_{j=0}^{n-1} c_j \quad \Big| \cdot n$$

$$nc_n = n(n-1) + 2 \sum_{j=0}^{n-1} c_j, \qquad c_0 = 0$$

...recurrence relation for expected value $c_n = \mathbb{E}(C_n)$

Generating Function for Quicksort

- generating function $C(z) = \sum_{n \geq 0} c_n z^n$

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$$nc_n = n(n-1) + 2\sum_{j=0}^{n-1} c_j \quad \Big| \cdot z^{n-1} \quad \Big| \sum_{n\geq 1}$$

$$\sum_{n\geq 1} nc_n z^{n-1} = \sum_{n\geq 1} n(n-1)z^{n-1} + 2\sum_{n\geq 1} z^{n-1} \sum_{j=0}^{n-1} c_j$$

$$\sum_{n\geq 1} nc_n z^{n-1} = z\sum_{n\geq 2} n(n-1)z^{n-2} + 2\sum_{n\geq 0} z^{n} \sum_{j=0}^{n} c_j$$

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C'(z) &= z\left(\frac{z}{1-z}\right)'' + 2\frac{C(z)}{1-z}
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- explicit formula

$$C(z) = \sum_{n\geq 0} c_n z^n = -\frac{2\log(1-z)}{(1-z)^2} - \frac{2}{(1-z)^2} + \frac{2}{1-z}$$

Asymptotic Analysis of Quicksort

Generating Function for Quicksort

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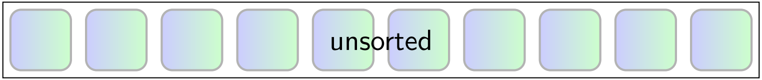
$\Downarrow$ singularity analysis $\Downarrow$

Expected Number of Key Comparisons

$$c_n = 2n\log n + (2\gamma - 4)n + 2\log n + (2\gamma + 1) + O(1/n)$$

Dual Pivot Quicksort & Quickselect



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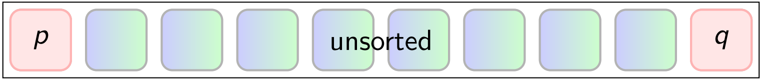

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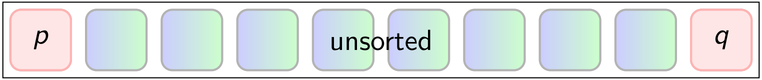

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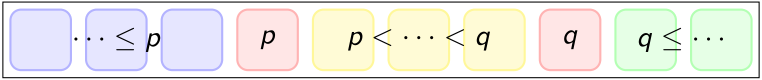

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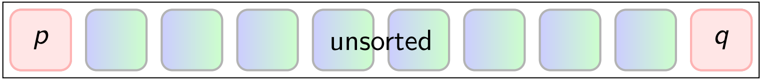

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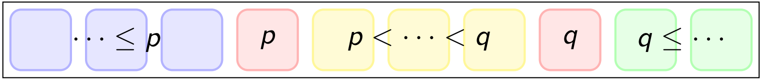

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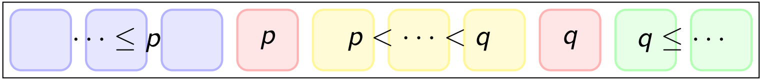
Average Number of Key Comparisons



- partitioning
  - "classical" $\rightsquigarrow n - 1$

- quicksort
  - "classical" $\rightsquigarrow 2n \log n - (2.84\ldots)n + O(\log n)$

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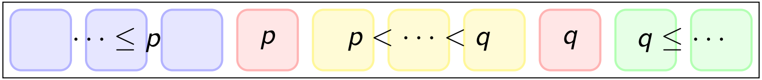

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  - “Yaroslavskiy–Bentley–Bloch” $\rightsquigarrow$ $1.9n \log n - (2.46\ldots)n + O(\log n)$
    [Wild–Nebel 2012]

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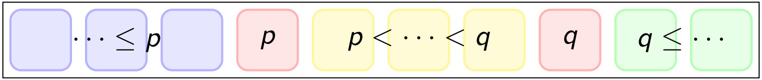


- partitioning
  - "classical" $\rightsquigarrow$ $n - 1$
  - "optimal dual pivot" $\rightsquigarrow$ $1.5n + 0.25 \log n + O(1)$
    [Aumüller–Dietzfelbinger 2014, Aumüller–Dietzfelbinger–Heuberger–K–Prodinger 2016]
- quicksort
  - "classical" $\rightsquigarrow$ $2n \log n - (2.84\ldots)n + O(\log n)$
  - "Yaroslavskiy–Bentley–Bloch" $\rightsquigarrow$ $1.9n \log n - (2.46\ldots)n + O(\log n)$
    [Wild–Nebel 2012]
  - "optimal dual pivot" $\rightsquigarrow$ $1.8n \log n - (2.38\ldots)n + O(\log n)$
    [Aumüller–Dietzfelbinger 2014, Aumüller–Dietzfelbinger–Heuberger–K–Prodinger 2016]

Overview

- C_n number of key comparisons using multi-pivot quicksort to sort a list with n elements

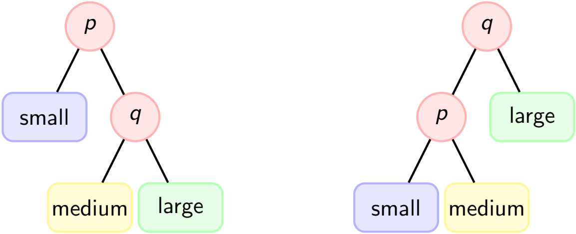
Questions

- What is the optimal multi-pivot strategy?
- Precise analysis of expected value of C_n?

?

Yaroslavskiy–Bentley–Bloch Partitioning Strategy

- comparison trees:



Yaroslavskiy–Bentley–Bloch Partitioning Strategy

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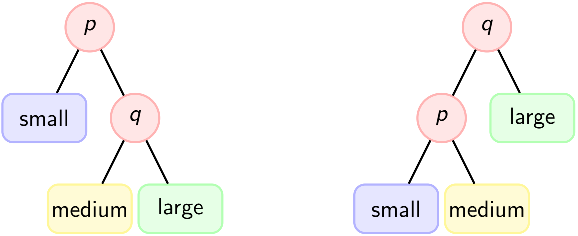

- comparison of element with pivots:
  - previously seen a small element ⇝ smaller pivot p first
  - (previously seen a medium element ⇝ smaller pivot p first)
  - previously seen a large elements ⇝ larger pivot q first

Optimal Partitioning Strategy "Count"

- comparison of element with pivots:
  - seen more small elements ⇝ smaller pivot p first
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- (equality ⇝ choice: smaller pivot p first)

Optimal Partitioning Strategy "Count"

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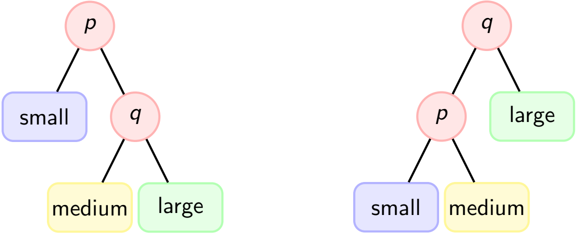
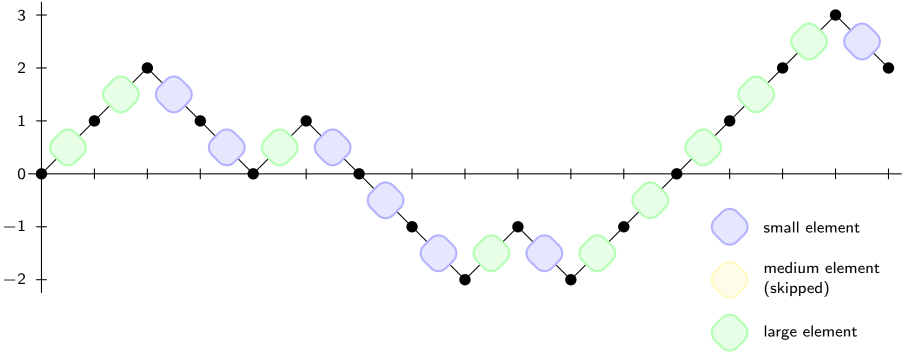


choose this tree if #small $\geq$ #large

choose this tree if #small $<$ #large

Optimal Partitioning Strategy "Count"–The Analysis



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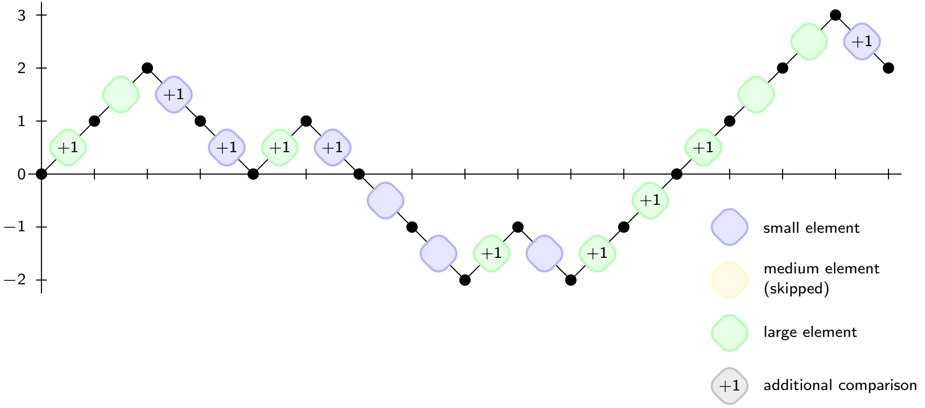

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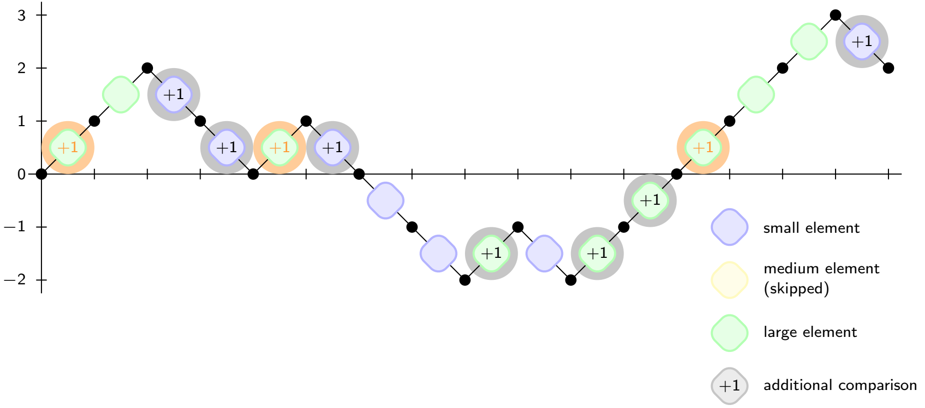

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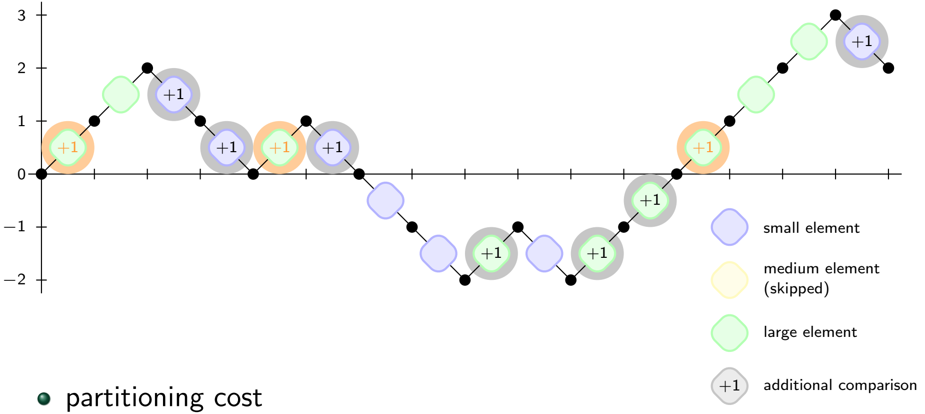
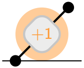


- partitioning cost

$$P_n = 1 + \frac{3}{2}(n-2) + \frac{1}{2}\,\#\text{medium} + $$  $$ - \frac{1}{2}\left|\#\text{large} - \#\text{small}\right|$$

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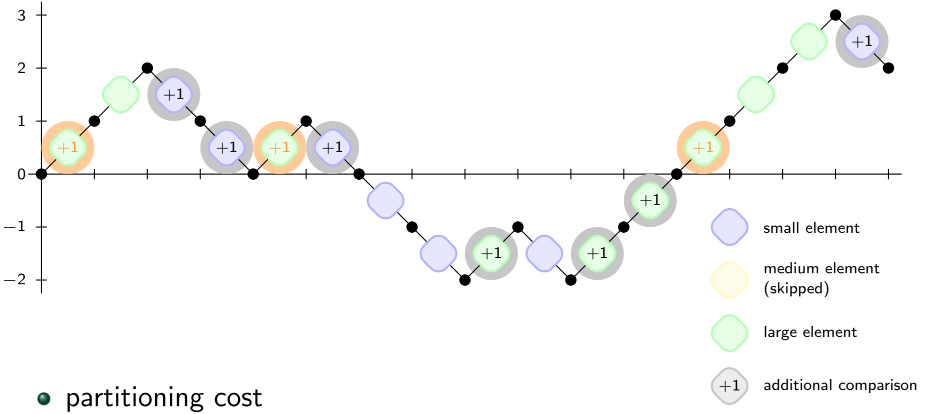

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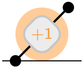

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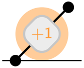
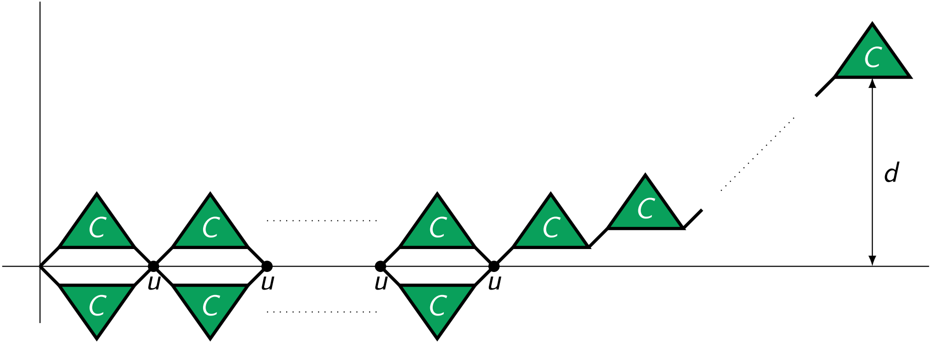
- 

$\rightsquigarrow$ analyze up-from-zero situations

Symbolic Decomposition of Our Lattice Paths

- symbolic equation



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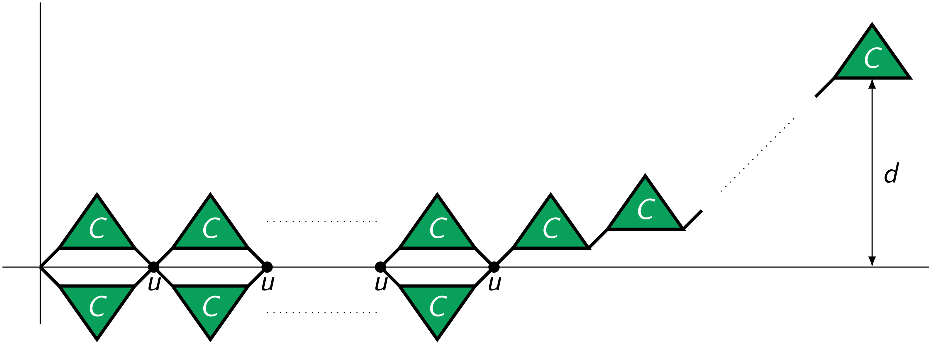

- translates to generating function

$$Q_d(z,u) = \frac{C(z)^{|d|}\, z^{|d|}}{1 - 2uz^2\, C(z)} = \frac{v^{|d|}(1+v^2)}{1 - v^2(2u-1)} \qquad \text{with } z = \frac{v}{1+v^2}$$

Getting the Double Sum

- expected number of zeros

$$\mu_{n,d} = \frac{[z^n] \left.\frac{\partial}{\partial u} Q_d(z,u)\right|_{u=1}}{[z^n] Q_d(z,1)} = \frac{2}{\binom{n}{\ell}} \sum_{k=0}^{\ell-1} \binom{n}{k}$$

with $\ell = \frac{1}{2}(n - |d|)$

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- sum over all d
- expected number of zeros

$$\mathbb{E}(X_n) = \frac{4}{n+1} \sum_{0 \le k < \ell < \lceil n/2 \rceil} \frac{\binom{n}{k}}{\binom{n}{\ell}} + [n \text{ even}]\, \frac{1}{n+1} \left(\frac{2^n}{\binom{n}{n/2}} - 1 \right)$$

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What now?

Simplify?—Asymptotic behavior?

The Identity

$$\frac{4}{n+1} \sum_{0 \le k < \ell < \lceil n/2 \rceil} \frac{\binom{n}{k}}{\binom{n}{\ell}} + \frac{[n \text{ even}]}{n+1} \left(\frac{2^n}{\binom{n}{n/2}} - 1 \right) + 1 = ???$$

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- computational proof with Sigma

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Theorem (ADH*K*P 2016)

$$\frac{4}{n+1} \sum_{0 \le k < \ell < \lceil n/2 \rceil} \frac{\binom{n}{k}}{\binom{n}{\ell}} + \frac{[n \text{ even}]}{n+1} \left(\frac{2^n}{\binom{n}{n/2}} - 1 \right) + 1 = \sum_{i=1}^{n+1} \frac{[i \text{ odd}]}{i} = H_{n+1}^{\mathrm{odd}}$$

- computational proof with Sigma
  - $\rightsquigarrow$ returns a single sum
  - $\rightsquigarrow$ proof certificate: yes, but ...

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  (via Vandermonde's convolution)

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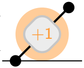
- combinatorial proof

Summing Up

- up-form-zero situations

$\mathbb{E}\Big($  $\Big) = \dfrac{1}{2\binom{n}{2}} \displaystyle\sum_{m=0}^{n-2} (m+1) H_m^{\mathrm{odd}}$

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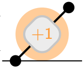
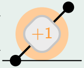
$\mathbb{E}\big($  $\big) = \dfrac{1}{2\binom{n}{2}} \displaystyle\sum_{m=0}^{n-2} (m+1) H_m^{\mathrm{odd}}$

with $H_m^{\mathrm{odd}} = \sum_{i=1}^{m} \frac{[i \text{ odd}]}{i}$

- $\sum_{n>0} \frac{[n \text{ odd}]}{n} z^n = \operatorname{artanh}(z) \quad \rightsquigarrow \quad \sum_{n\geq 0} H_n^{\mathrm{odd}} z^n = \frac{\operatorname{artanh}(z)}{1-z}$

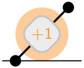
Generating function

$\displaystyle\sum_{n\geq 0} \mathbb{E}\big($  $\big) z^n = \dfrac{\operatorname{artanh}(z)}{2(1-z)} - \dfrac{z^2}{8(1-z)} - \dfrac{3z+5}{8} \operatorname{artanh}(z) + \dfrac{1}{8} z$

Partitioning & Generating Function

- partitioning cost of strategy "Count"

$P_n =$ "necessary" comparisons $+$ 

$+$ 

Generating function

$$P(z) = \sum_{n \geq 0} \mathbb{E}(P_n)\, z^n = \frac{3}{2(1-z)^2} + \frac{\operatorname{artanh}(z)}{2(1-z)} - \frac{31z^2}{8(1-z)}$$
$$- \frac{3+z}{8} \operatorname{artanh}(z) - \frac{3}{2} - \frac{25z}{8}$$

Solving the Dual-Pivot Quicksort Recurrence

Recurrence

- C_n cost dual-pivot quicksort
- P_n cost for partitioning

$$\mathbb{E}(C_n) = \mathbb{E}(P_n) + \frac{3}{\binom{n}{2}} \sum_{k=1}^{n-2} (n-1-k)\,\mathbb{E}(C_k)$$

Solution [Hennequin 1991, Wild 2013]

- $C(z) = \sum_{n\geq 0} \mathbb{E}(C_n)\, z^n$
- $P(z) = \sum_{n\geq 0} \mathbb{E}(P_n)\, z^n$

$$C(z) = (1-z)^3 \int_0^z (1-t)^{-6} \int_0^t (1-s)^3 P''(s)\, ds\, dt$$

The Result

Theorem (ADH*K*P 2016)

average number of key comparisons
in dual pivot quicksort
with the **optimal** *partitioning strategy "Count" is*

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$$\frac{9}{5}nH_n - \frac{1}{5}nH_n^{\mathrm{alt}} - \frac{89}{25}n + \frac{67}{40}H_n - \frac{3}{40}H_n^{\mathrm{alt}} - \frac{83}{800} + \frac{(-1)^n}{10} + \cdots$$

- *harmonic numbers*
  - $H_n = \sum_{i=1}^{n} 1/i$
  - $H_n^{\mathrm{alt}} = \sum_{i=1}^{n} (-1)^i/i$

The Result

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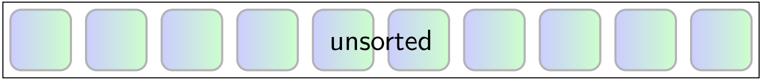
$$= \frac{9}{5}n\log n + An + B\log n + C + \frac{D}{n} + \frac{E}{n^2} + \frac{(-1)^nF+G}{n^3} + O\Big(\frac{1}{n^4}\Big)$$

asymptotically as $n \to \infty$

- *harmonic numbers*
  - $H_n = \sum_{i=1}^{n} 1/i$
  - $H_n^{\mathrm{alt}} = \sum_{i=1}^{n} (-1)^i/i$
- *constant of linear term*
  $A = \frac{9}{5}\gamma + \frac{1}{5}\log 2 - \frac{89}{25}$
  $= -2.3823823670652\dots$
- *explicit constants* B, C, …

Dual Pivot Quicksort & Quickselect



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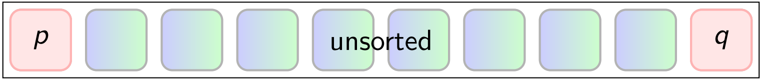

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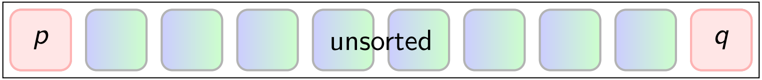
Dual Pivot Quicksort & Quickselect



- choose pivot elements p and q
- partition into
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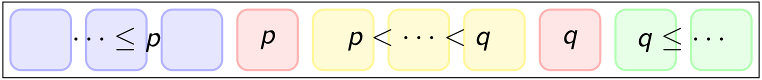

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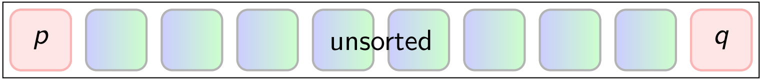

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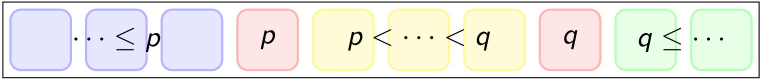

- proceed recursively

The Dual-Pivot Quickselect Recurrence

Recurrence Relation

- $C_{n,j}$ cost dual-pivot quickselect (select jth rank)
- P_n cost for partitioning
- then

$$C_{n,j} = P_n + S_{n,j} + M_{n,j} + L_{n,j}$$

with

$$S_{n,j} = \frac{1}{\binom{n}{2}} \sum_{s=j}^{n-2} (n-1-s) C_{s,j}$$

$$M_{n,j} = \frac{1}{\binom{n}{2}} \sum_{m=1}^{n-2} \sum_{s=\max\{0,j-m-1\}}^{\min\{j-2,n-m-2\}} C_{m,j-s-1}$$

$$L_{n,j} = \frac{1}{\binom{n}{2}} \sum_{\ell=n-j+1}^{n-2} (n-1-\ell) C_{\ell,n-j+1}$$

Solving the Dual-Pivot Quickselect Recurrence

- $C(z, u) = \sum_{n,j} C_{n,j} z^n u^j$
- functional equation $u\, C(zu, 1/u) = C(z, u)$

Differential Equation

$$\frac{\partial^2}{\partial z^2} C(z, u) = \frac{u}{1-u} \big(P''(z) - u^2\, P''(zu)\big) + 2\, C(z, u)\, r(z, u)$$

with $r(z, u) = \frac{1}{(1-z)^2} + \frac{u}{(1-z)(1-zu)} + \frac{u^2}{(1-zu)^2}$

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- grand averages:

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} C(z,u)\Big|_{u=1} = \frac{1}{z}\left(z^2\, P''(z)\right)' + \frac{6}{(1-z)^2}\, C(z,1)$$

- with solution

$$C(z,1) = (1-z)^3 \int_0^z (1-t)^{-6} \int_0^t (1-s)^3 \frac{1}{s}\left(s^2\, P''(s)\right)'\, ds\, dt$$

Quickselect: Grand Averages

- selecting the jth smallest element (the jth rank) with $j \in \{1, \dots, n\}$ uniformly at random
- average number of key comparisons
  - "classical" $\rightsquigarrow$ $3n - 8H_n + 13 - 8n^{-1}H_n$

$$= 3n - 8\log n - 8\gamma + 13 + O\big(n^{-1}\log n\big)$$

[Mahmoud–Modarres–Smythe 1995]

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  - "Yaroslavskiy" $\rightsquigarrow$ $$\frac{19}{6}n - \frac{37}{5}H_n + \frac{1183}{100} - \frac{37}{5}n^{-1}H_n - \frac{71}{300}n^{-1} = \frac{19}{6}n - \frac{37}{5}\log n - \frac{37}{5}\gamma + \frac{1183}{100} + O\big(n^{-1}\log n\big)$$ [Wild–Nebel–Mahmoud 2016]

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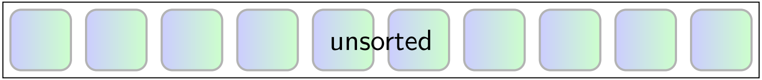
Theorem (K 2017)

"optimal dual pivot" $\rightsquigarrow$

$$\mathbb{E}(C_n) = 3n + \frac{3}{20n}\sum_{k=1}^{n-1} H_k H_{n-k} - \frac{3}{10n}\sum_{k=1}^{n} \frac{H_{k-1}^{\mathrm{alt}}}{k}(n-k+1) + \cdots$$

$$= 3n + \frac{3}{20}(\log n)^2 + \left(\frac{\gamma + \log 2}{10} + \frac{319}{50}\right)\log n + O(1)$$

Dual Pivot Quicksort & Quickselect



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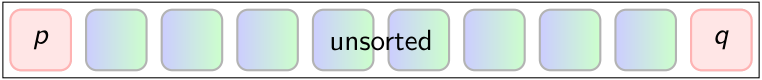

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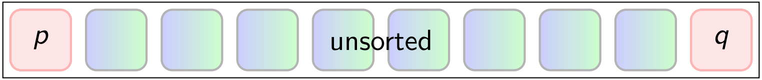

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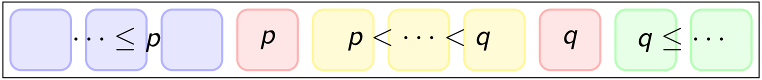

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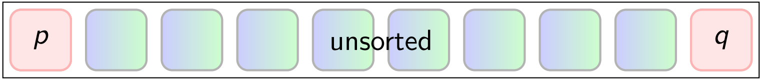

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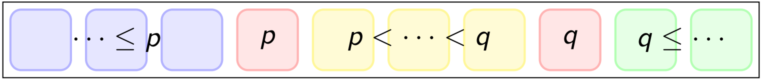

- proceed recursively

Optimal Partitioning Strategy "Count"

- comparison of element with pivots:
  - seen more small elements $\rightsquigarrow$ smaller pivot p first
  - seen more large elements $\rightsquigarrow$ larger pivot q first
  - equality $\rightsquigarrow$ choice: smaller pivot p first

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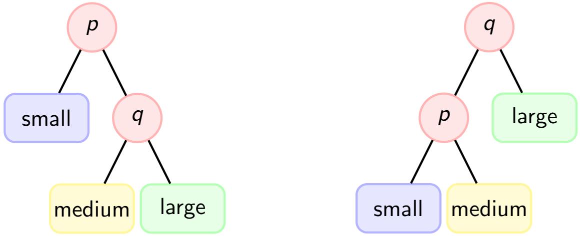
- comparison trees:



choose this tree if #small $\geq$ #large

choose this tree if #small $<$ #large

Optimality

Theorem (ADHKP 2016, ..., HK 2018)

- *cost $P_n^\star$ of strategy "Count"*
- *then $\mathbb{E}(P_n^\star)$ is minimal among all strategies*

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- urn model:
  uniformly distributed

$$S_0 + \cdots + S_d$$

- in step $k = \underbrace{s_0 + \cdots + s_d}_{\text{already classified elements}}$
- e.g. $d = 2$ (dual-pivot)
  - $s_0 = \#\text{small}$
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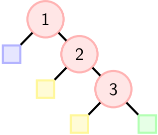
  - $s_1 = \#\text{medium}$
  - $s_2 = \#\text{large}$

- expected number of comparisons to classify next element:

$$\sum_{i=0}^{d} \overbrace{h_i(t)}^{\substack{\#\text{comparisons} \\ = \text{depths } h_i(t) \\ \text{in comparison tree } t}} \underbrace{\frac{s_i + 1}{k + d + 1}}_{\substack{\text{conditional probability that} \\ \text{next element is of type } i \\ (\text{given } s_0, \ldots, s_d)}} = \frac{\overbrace{\ell_t(s)}^{\substack{\text{affine linear} \\ \text{function } \ell_t(s)}}}{k + d + 1}$$

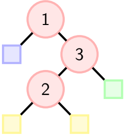
- choose tree $T_k^\star = t$ such that $\ell_t(s)$ is minimal

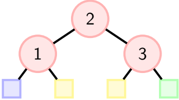
Optimal Partitioning Strategy with Three Pivots



$$s_1 \geq s_3$$
$$s_0 \geq s_2 + s_3 + 1$$

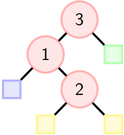
$$s_0 \geq s_1 + s_2 + 1$$
$$s_3 \geq s_1$$
$$s_0 \geq s_3$$

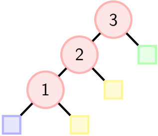




$$s_2 + s_3 + 1 \geq s_0$$
$$s_1 + s_2 + 1 \geq s_0$$
$$s_1 + s_2 + 1 \geq s_3$$
$$s_0 + s_1 + 1 \geq s_3$$

$$s_0 \geq s_2$$
$$s_3 \geq s_0$$
$$s_3 \geq s_1 + s_2 + 1$$





$$s_3 \geq s_0 + s_1 + 1$$
$$s_2 \geq s_0$$

comparison trees & polyhedra minimizing $\ell_t(s) = \sum_{i=0}^{d} h_i(t)(s_i + 1)$

Expected Partitioning Cost

- from optimality

$$\mathbb{E}(P_n) \geq \mathbb{E}(P_n^\star) = \mathbb{E}(Q_n) + \sum_{k=0}^{n-d-1} \frac{1}{(k+d+1)\binom{k+d}{d}} \underbrace{\sum_{s \in \mathcal{N}_k} \ell_{T_k^\star}(s)}_{= \sum_{t \in \mathcal{T}} \sum_{s \in \mathcal{N}_k} [s \in \mathcal{C}_t] \underbrace{\ell_t(s)}_{= \sum_{i=0}^{d} h_i(t)(s_i+1)}}$$

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- generating function $P(z) = \sum_{n\geq 0} \mathbb{E}(P_n)\, z^n$ satisfies

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{d+1} ((1-z)\,P(z)) = d! \sum_{t\in\mathcal{T}} \sum_{i=0}^{d} h_i(t) \sum_{k\geq 0} \sum_{s\in\mathcal{N}_k} [s \in \mathcal{C}_t]\,(s_i+1) z^k = R(z)$$

Expected Partitioning Cost

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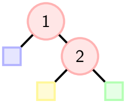
- $G_{t,i}(y,z) = y\, H_t(z, \dots, z, zy, z, \dots, z)$
  with $H_t(y_0, \dots, y_d) = \sum_{s \in \mathcal{C}_t} y^s$ and polyhedron $\mathcal{C}_t$

How to Compute Generating Function of Polyhedron?

- $d = 2$, $\mathcal{C}_{t_1} = \{(s_0, s_1, s_2) \in \mathbb{N}_0^3 \mid s_0 \geq s_2\}$

$$H_{t_1}(y_0, y_1, y_2) = \sum_{s_0 \geq s_2 \geq 0,\ s_1 \geq 0} y_0^{s_0} y_1^{s_1} y_2^{s_2} = \sum_{s_1,\ s_2,\ u \geq 0} y_0^{s_2+u} y_1^{s_1} y_2^{s_2}$$

$$= \sum_{s_1,\ s_2,\ u \geq 0} y_0^{u} y_1^{s_1} (y_0 y_2)^{s_2} = \frac{1}{(1-y_0)(1-y_1)(1-y_0 y_2)}$$



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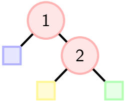

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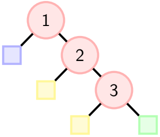
- $d = 3$, $\mathcal{C}_{t_1} = \{(s_0, s_1, s_2, s_3) \in \mathbb{N}_0^4 \mid s_1 \geq s_3,\ s_0 \geq s_2 + s_3 + 1\}$



$$H_{t_1}(y_0, y_1, y_2, y_3) = \sum_{s_1 \geq s_3 \geq 0,\ s_0 \geq s_2+s_3+1,\ s_2 \geq 0} y_0^{s_0} y_1^{s_1} y_2^{s_2} y_3^{s_3}$$
$$= \sum_{s_2,\ s_3,\ u,\ v \geq 0} y_0^{s_2+s_3+v+1} y_1^{s_3+u} y_2^{s_2} y_3^{s_3}$$
$$= \frac{y_0}{1-y_0} \frac{1}{1-y_1} \frac{1}{1-y_0y_2} \frac{1}{1-y_0y_1y_3}$$

MacMahon's Omega Calculus

Ω-Operator

$$\underset{\geq}{\Omega} \sum_{s \in \mathbb{N}_0^{d+1}} \sum_{r \in \mathbb{Z}^m} c_{sr} y^s \lambda^r = \sum_{s \in \mathbb{N}_0^{d+1}} \sum_{r \in \mathbb{N}_0^m} c_{sr} y^s$$

- remove summands corresponding to negative exponents of λ

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Result for $d = 4$

$$\begin{aligned}(1-z)^4 R(z) &= \frac{19072}{75(1-z)^2} + \frac{1744}{75(1-z)} \\ &\quad - \frac{48(3z^2 - z + 3)}{5(1+z+z^2+z^3+z^4)^2} + \frac{48(51z^3 + 14z^2 + 14z + 51)}{25(1+z+z^2+z^3+z^4)} \\ &\quad + \frac{24}{(1+z+z^2)^3} + \frac{8(3z-2)}{(1+z+z^2)^2} - \frac{8(19z+16)}{3(1+z+z^2)} - \frac{24}{1+z}\end{aligned}$$

with $R(z) = \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{d+1}((1-z)\,P(z))$

Solving the Multi-Pivot Quicksort Recurrence

- expected values

$$C_n = P_n^\star + \sum_{i=0}^{d} C_{S_i} \quad \Longrightarrow \quad \mathbb{E}(C_n) = \mathbb{E}(P_n^\star) + \sum_{i=0}^{d} \sum_{s_i=0}^{n-d} \mathbb{E}(C_{s_i})\, \mathbb{P}(S_i = s_i)$$

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- differential equation

$$(1-z)^d \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^d C(z) - (d+1)!\, C(z) = (1-z)^d \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^d P(z)$$

$$\frac{\mathrm{d}}{\mathrm{d}z}$$

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- solution

$$C(z) = Q(z) + (I_{\alpha_1} \circ I_{\alpha_2} \circ \cdots \circ I_{\alpha_d})\left((d+1)!Q(z) + (1-z)^d \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^d P(z)\right)$$

with

- $(I_\alpha f)(z) = (1-z)^{-\alpha} \int_0^z (1-t)^{\alpha-1} f(t)\,dt$
- polynomial $Q(z)$
- $\sum_{k=1}^{d} \left[{d \atop k}\right] X^k - (d+1)! = \prod_{i=1}^{d} (X - \alpha_i)$

Singularity Analysis & Transfers

- operator I_α for $z \to 1$
  - $I_\alpha \dfrac{1}{(1-z)^\beta} = \dfrac{1}{\alpha - \beta}\left(\dfrac{1}{(1-z)^\alpha} - \dfrac{1}{(1-z)^\beta}\right)$
    for $\alpha \neq \beta$
  - $\left(I_{\alpha_1} \circ \dots \circ I_{\alpha_k}\right)(1-z)^{-\beta}\left(-\log(1-z)\right)^\ell = \dots$

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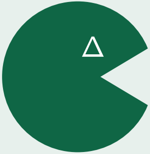
Transfer

- $f(z) = O\bigl((1-z)^{-\beta}\bigr)$ for $z \to 1$



with $f^\star(s) = \int_0^1 (1-t)^{s-1} f(t)\,\mathrm{d}t$

$$(I_\alpha f)(z) = \frac{f^\star(\alpha)}{(1-z)^\alpha} + O\bigl((1-z)^{-\beta}\bigr)$$

for $\operatorname{Re}\beta < \operatorname{Re}\alpha$

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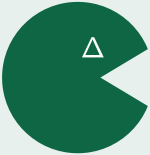

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- operator I_α for $z \to \rho \neq 1$, e.g. $d = 4$, $\rho = \zeta_{60}^k$
  - $(I_{\alpha_1} \circ \dots \circ I_{\alpha_k})\,\big(1-\tfrac{z}{\rho}\big)^{-\beta}\big(-\log\big(1-\tfrac{z}{\rho}\big)\big)^\ell = \dots$

Optimal Multi-Pivot Quicksort

Theorem (Heuberger–K 2017–2024)

- *expected number of key comparisons*
  in trial pivot quicksort
  with the **optimal** *partitioning strategy is*

  $$\frac{133}{78} n \log n + An + B \log n + O(1)$$

  - $A = \frac{133}{78}\gamma - \frac{2}{117}\sqrt{3}\pi + \frac{4}{39}\log 3 + \frac{3}{26}\log 2 - \frac{6761}{2028} = -2.24995\ldots$
  - $B = \frac{707}{468}$

- *expected number of key comparisons*
  in quadral pivot quicksort
  with the **optimal** *partitioning strategy is*

  $$\frac{9536}{5775} n \log n + An + B \log n + O(1)$$

  - $A = \textit{skipped} = -2.20515\ldots$
  - $B = \frac{48823}{34650}$