AofA 2024

June 20, 2024

The Recurrence/Transience of Random Walks on a Bounded Grid in an Increasing Dimension

Shuma Kumamoto (Kyushu Univ.),

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Tomoyuki Shirai (Kyushu Univ.)

Plan of talk

1. Introduction
   - $\mathbb{Z}^3$
   - RWoGG
   - Tree
2. Related work
   - Exploration
3. Previous work
   - LHaGG
   - $\{0,1\}^n$ proof
   - Extension to $\{0,1,\ldots,N\}^n$
4. Main result
   - Weakly LHaGG
   - Recurrence
   - Transience
   - pausing coupling
5. Concluding remarks

Plan of talk $\geq$ 49 min.

1. Introduction ($\geq$ 9 min.)
   - $\mathbb{Z}^3$
   - RWoGG
   - Tree
2. Related work ($\geq$ 6 min.)
   - Exploration
3. Previous work ($\geq$ 8 min.)
   - LHaGG
   - $\{0,1\}^n$ proof
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4. Main result ($\geq$ 25 min.)
   - Weakly LHaGG
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5. Concluding remarks (1 min.)

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1. Introduction (~~≥ 9 min.~~ 6 min.)
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   - RWoGG
   - ~~Tree~~
2. Related work (~~≥ 6 min.~~ 3 min.)
   - ~~Exploration~~
3. Previous work (~~≥~~ 8 min.)
   - LHaGG
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4. Main result (~~≥ 25 min.~~ 7 min.)
   - Weakly LHaGG
   - ~~Recurrence~~
   - ~~Transience~~
   - pausing coupling
5. Concluding remarks (1 min.)

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Shuji Kijima | search

1. Introduction w/ examples

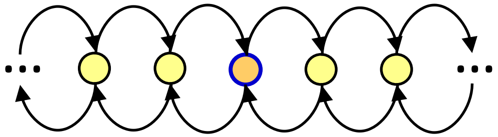
Recurrence/Transience of Random walks on *infinite* graphs

A random walk on an infinite graph is **recurrent** at vertex v if it visits v infinitely many times, i.e.,

$$\sum_{t=0}^{\infty} \Pr[X_t = v] = \infty$$

holds, otherwise it is said to be **transient**.

For instance,



RW on $\mathbb{Z}$ is recurrent at o,

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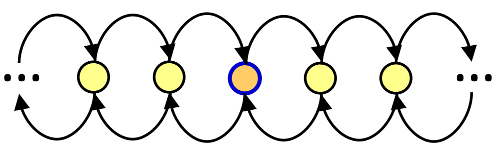

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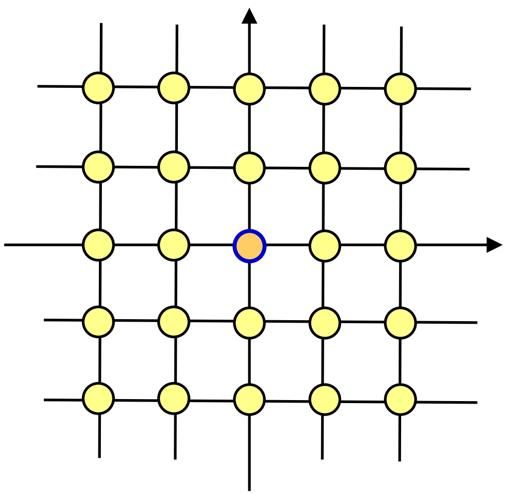

RW on $\mathbb{Z}^2$ is recurrent at o,

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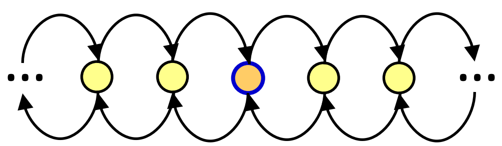

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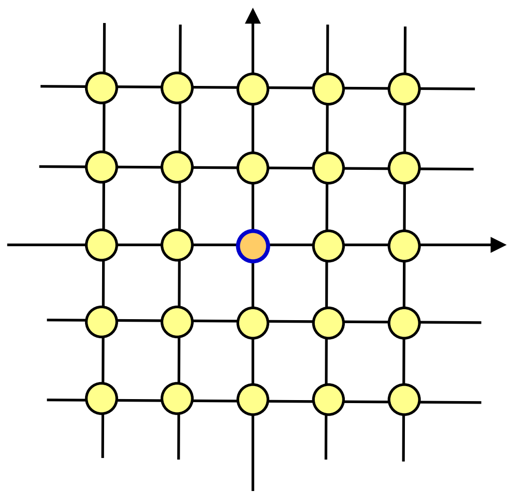

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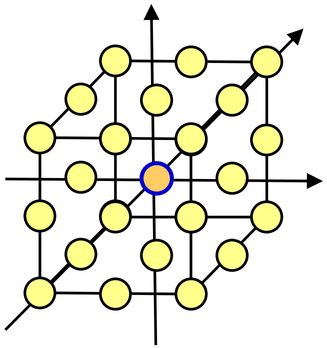


RW on $\mathbb{Z}^3$ is transient at o,

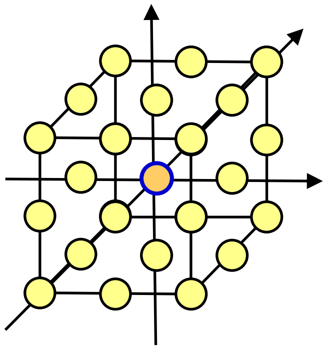
Example 1. Random walk in a growing region of $\mathbb{Z}^3$

- ✓ Random walk on $\mathbb{Z}^3$ is transient at o.
- ✓ Random walk on $\{-n, \dots, n\}^3$ is recurrent at o.

Q. Is a random walk on $\{-n, \dots, n\}^3$ recurrent or transient if n **increases** as time go on?

A. It depends on the increasing speed.

⇨ Find the phase transition point regarding the growing speed.



RW on $\mathbb{Z}^3$ is transient at o,

Model: Random Walk on a Growing Graph (RWoGG)

[K, Shimizu, Shiraga '21]

- Growing graph is a sequence of static graphs

$$\boldsymbol{\mathcal{G}} = \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \ldots$$

where each $\mathcal{G}_t = (\mathcal{V}_t, \mathcal{E}_t)$ is a static simple graph.

We assume $\mathcal{V}_t \subseteq \mathcal{V}_{t+1}$, for convenience.

Furthermore, $\mathcal{E}_t \subseteq \mathcal{E}_{t+1}$ holds in this talk.

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Furthermore, $\mathcal{E}_t \subseteq \mathcal{E}_{t+1}$ holds in this talk.

- **RWoGG** $(\mathfrak{d}, G, P)$ is a specific model:
  - $\mathfrak{d}(1), \mathfrak{d}(2), \mathfrak{d}(3), \dots \in \mathbb{Z}$ denote the duration time.
  - Growing graph is given by

$$\mathcal{G}_t = G(n) \quad \text{for} \quad t \in [T_{n-1}, T_{n-1} + \mathfrak{d}(n))$$

where $T_n = \sum_{i=1}^{n} \mathfrak{d}(n)$, i.e.,

$$\mathcal{G}_t = \begin{cases} G(1) & \text{for the first } \mathfrak{d}(1) \text{steps} \\ G(2) & \text{for the next } \mathfrak{d}(2) \text{steps} \\ G(3) & \text{for the next } \mathfrak{d}(3) \text{steps} \\ \vdots & \vdots \end{cases}$$

  - $P(n)$ denotes the transition matrix on $G(n)$.

$\mathfrak{d}$ represents (inverse) growing speed

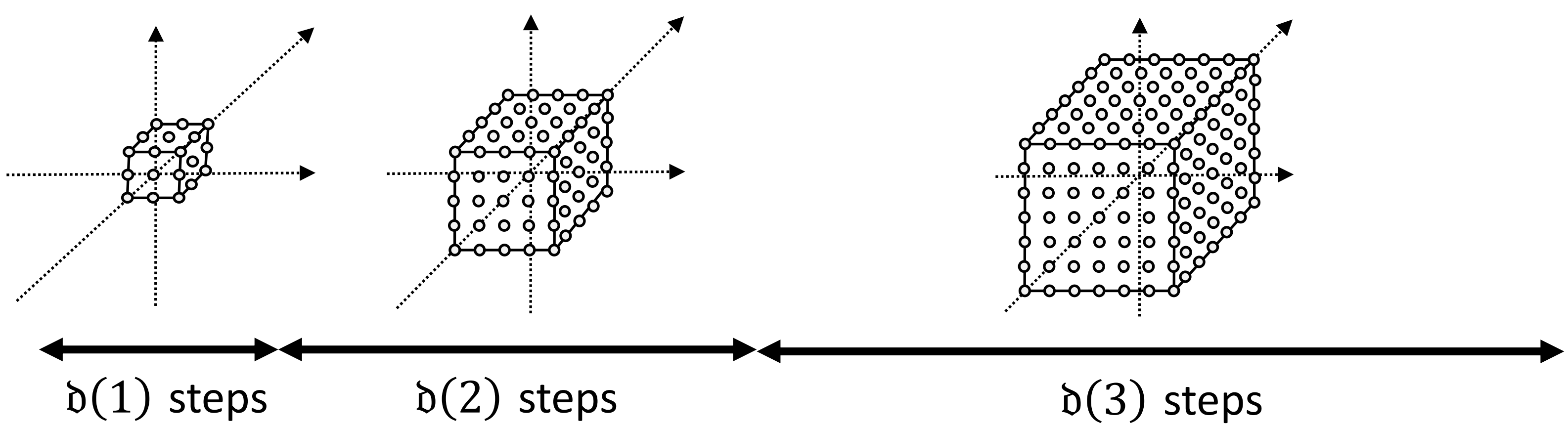
Example 1. Random walk in a growing region of $\mathbb{Z}^d$

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

- $\mathfrak{d}(n) = n^2$,
- $G(n)$ is a grid graph $\{-n, \ldots, n\}^3$,
- $P(n)$ denotes the simple random walk w/ reflection bound,

  i.e., move to a neighbor w.p. $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$ unless boundary,

for $n = 1,2, \ldots$



Thm. [Dembo et al. 2014, Kumamoto et al. 2024]

If $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{n^d} = \infty$ then recurrent, otherwise transient.

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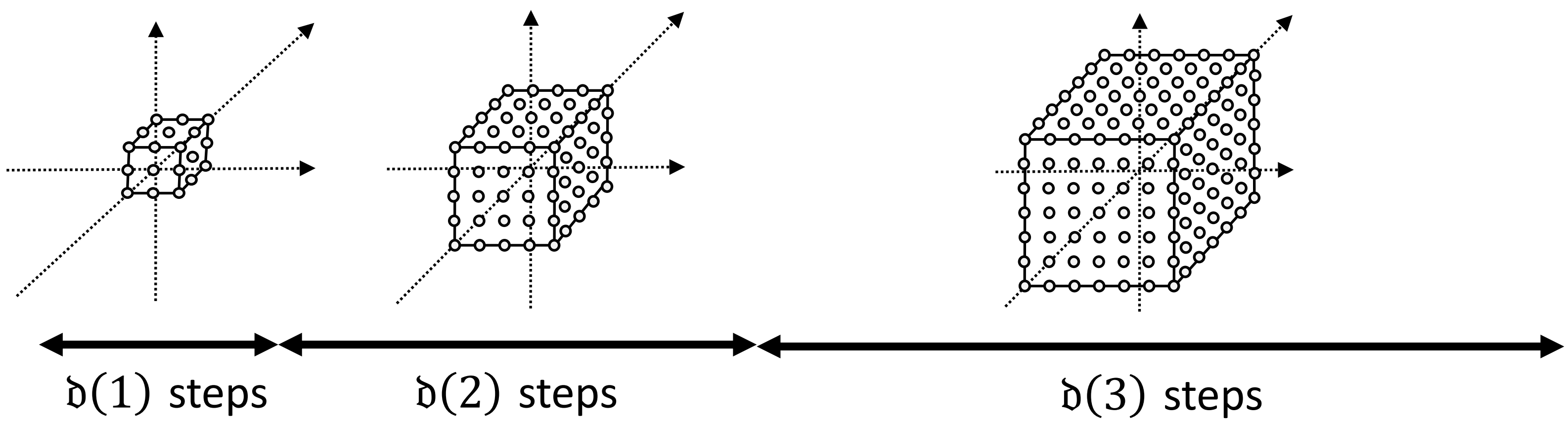
Recurrent

since $\sum_{n=1}^{\infty} \frac{n^2}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$.



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Example 1. Random walk in a growing region of $\mathbb{Z}^d$

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

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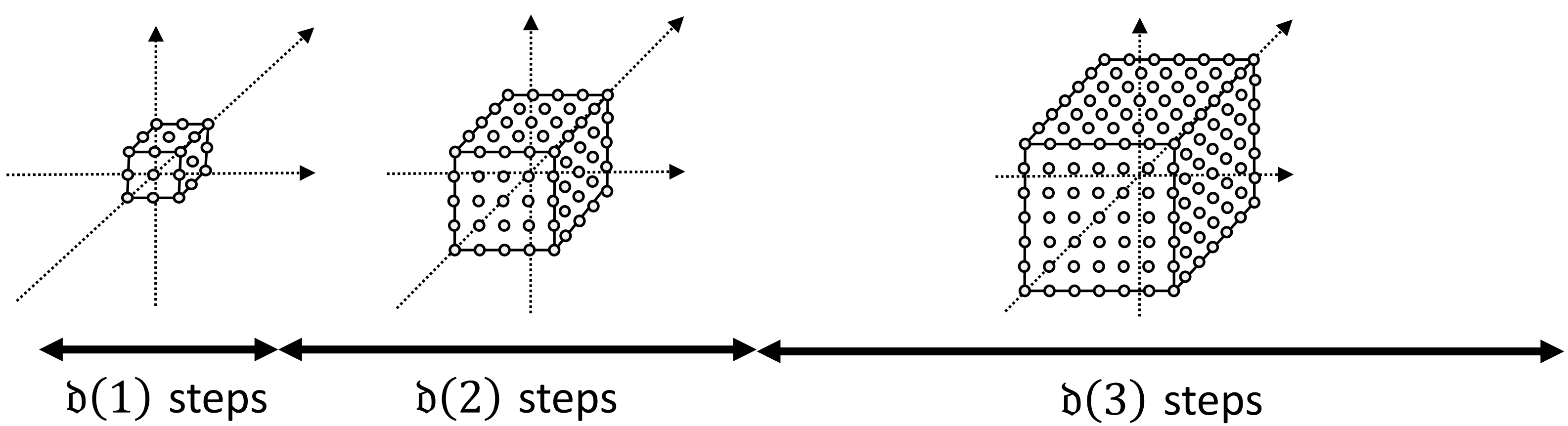

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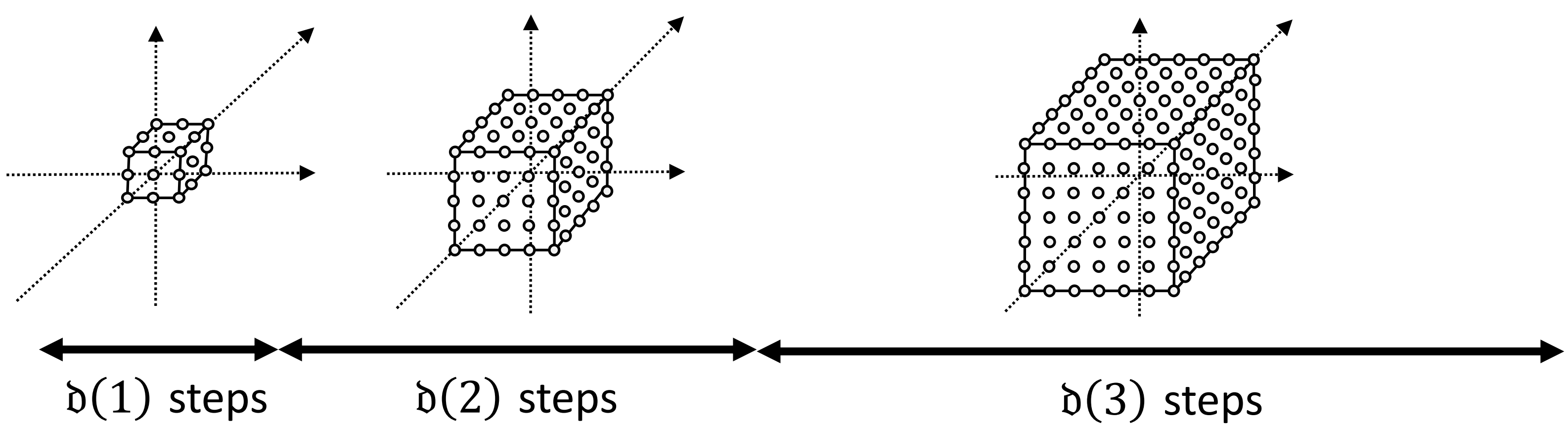
Transient

since $\sum_{n=1}^{\infty} \frac{n^{1.999}}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^{0.999}} < 1000.$



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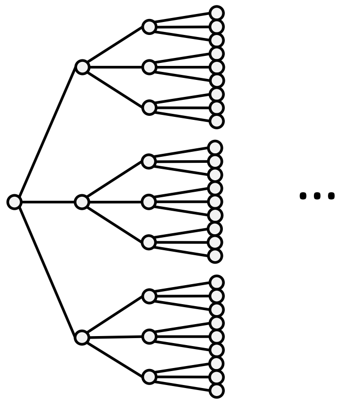
Example 2. RW on an infinite k-ary tree

✓ Random walk on an infinite k-ary tree is transient at r.



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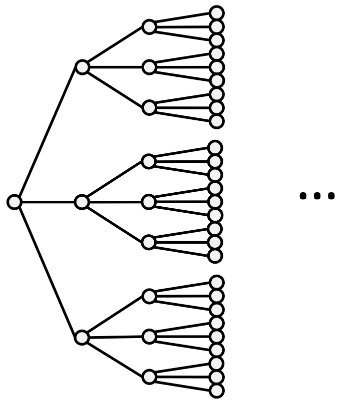
- ✓ Random walk on an infinite k-ary tree is transient at r.
- ✓ Random walk on a finite k-ary tree is recurrent at r.



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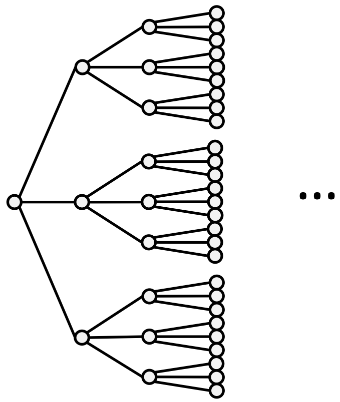

Example 2. Random walk on a growing k-ary tree

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

- $\mathfrak{d}(n) = 3^n$,
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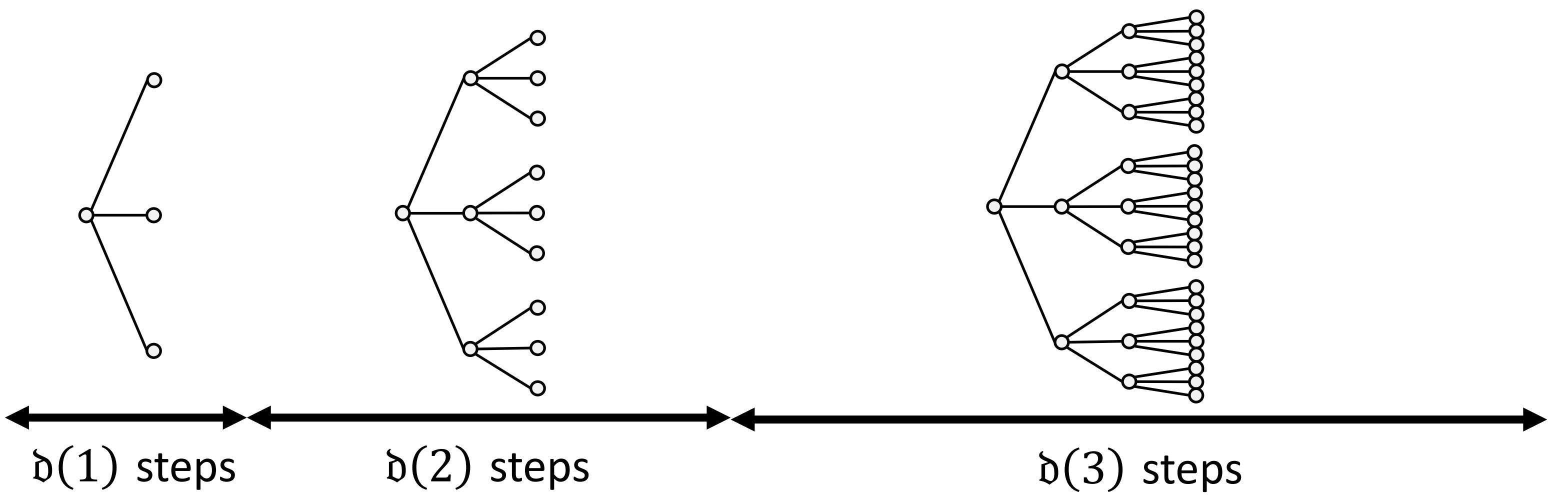

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If $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{k^n} = \infty$ then recurrent, otherwise transient.

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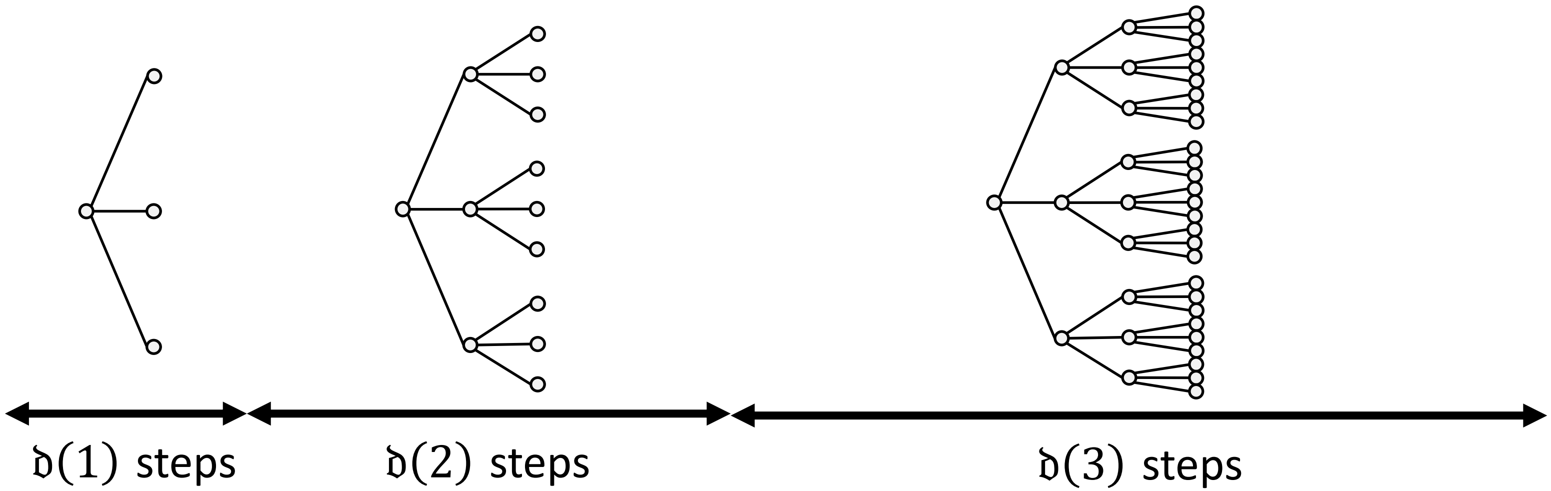
Recurrent

since $\sum_{n=1}^{\infty} \frac{3^n}{3^n} = \sum_{n=1}^{\infty} 1 = \infty$.



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Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

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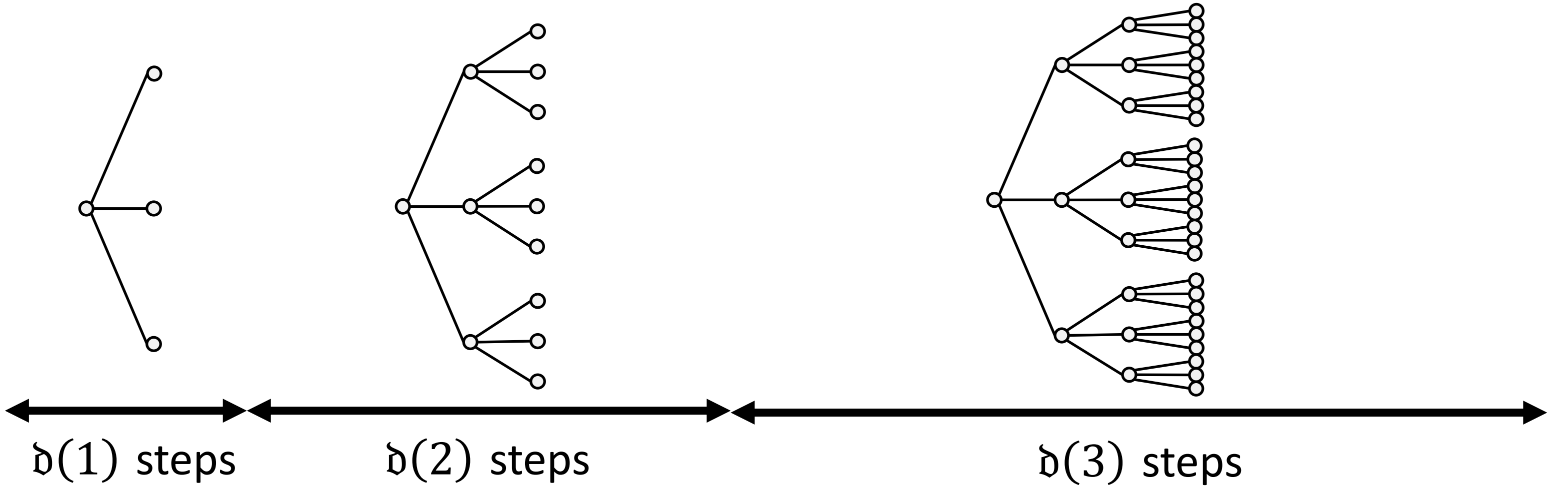

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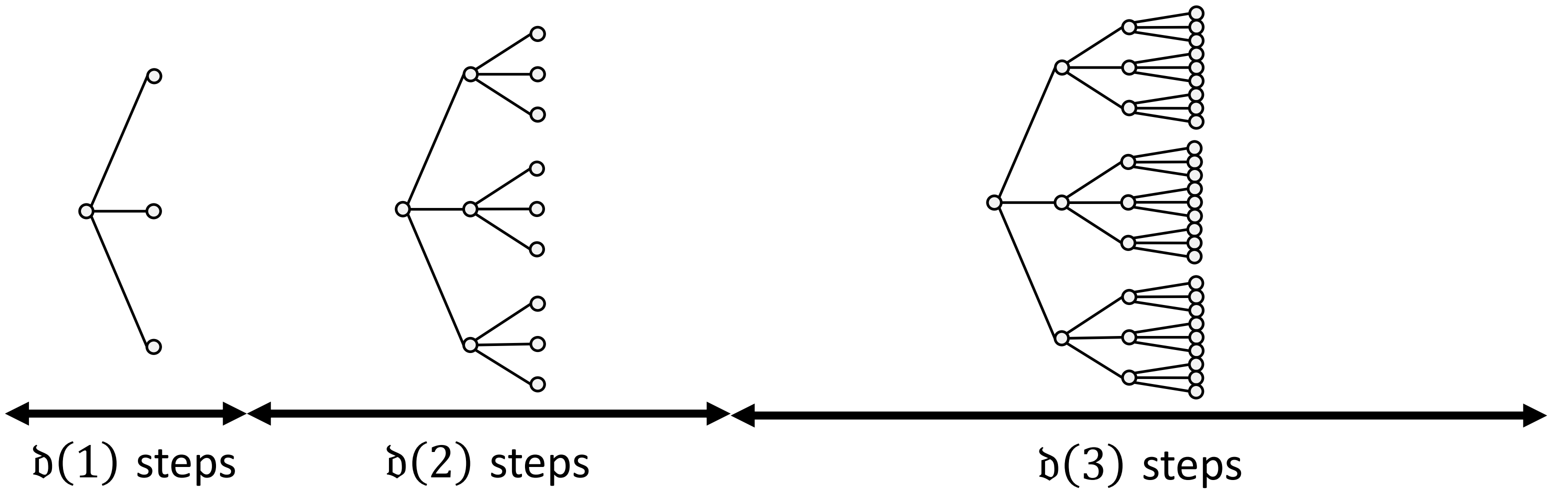
Transient

since $\sum_{n=1}^{\infty} \frac{2.999999^n}{3^n} < 1{,}000{,}000$.



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2. Related work

About analysis of algorithms in dynamic environment

Related work (1/2): Random walks on dynamic graphs

Graph search by RW --- related to cover time

- Copper and Frieze (2003): Crawling on simple models of web graphs.
- Avin, Koucky and Lotker (2008): a bad example for hitting-time (cover time as well) w/ $\Omega(2^n)$ for the number of vertices n.
- Denysyuk and Rodrigues (2014): cover time under some fairness condition.
- Lamprou, Martin and Spirakis (2018): edge-uniform stochastically graphs.
- Sauerwald and Zanetti (2019): $O(n^2)$ cover time for d-regular graphs.
- K, Shimizu, Shiraga (2021): cover ratio of **RWoGG**

Mixing time

- Saloff-Coste and Zuniga (2009,2011): mixing time for time-inhomogeneous Markov chains w/ an invariant stationary distribution.
- Dembo, Huang and Zheng (2019) analyzed the conductance of growing subregion of $\mathbb{Z}^d$.
- Cai, Sauerwald and Zanetti (2020): mixing time for edge-Markovian graph.

Recurrence/transience

… Continued

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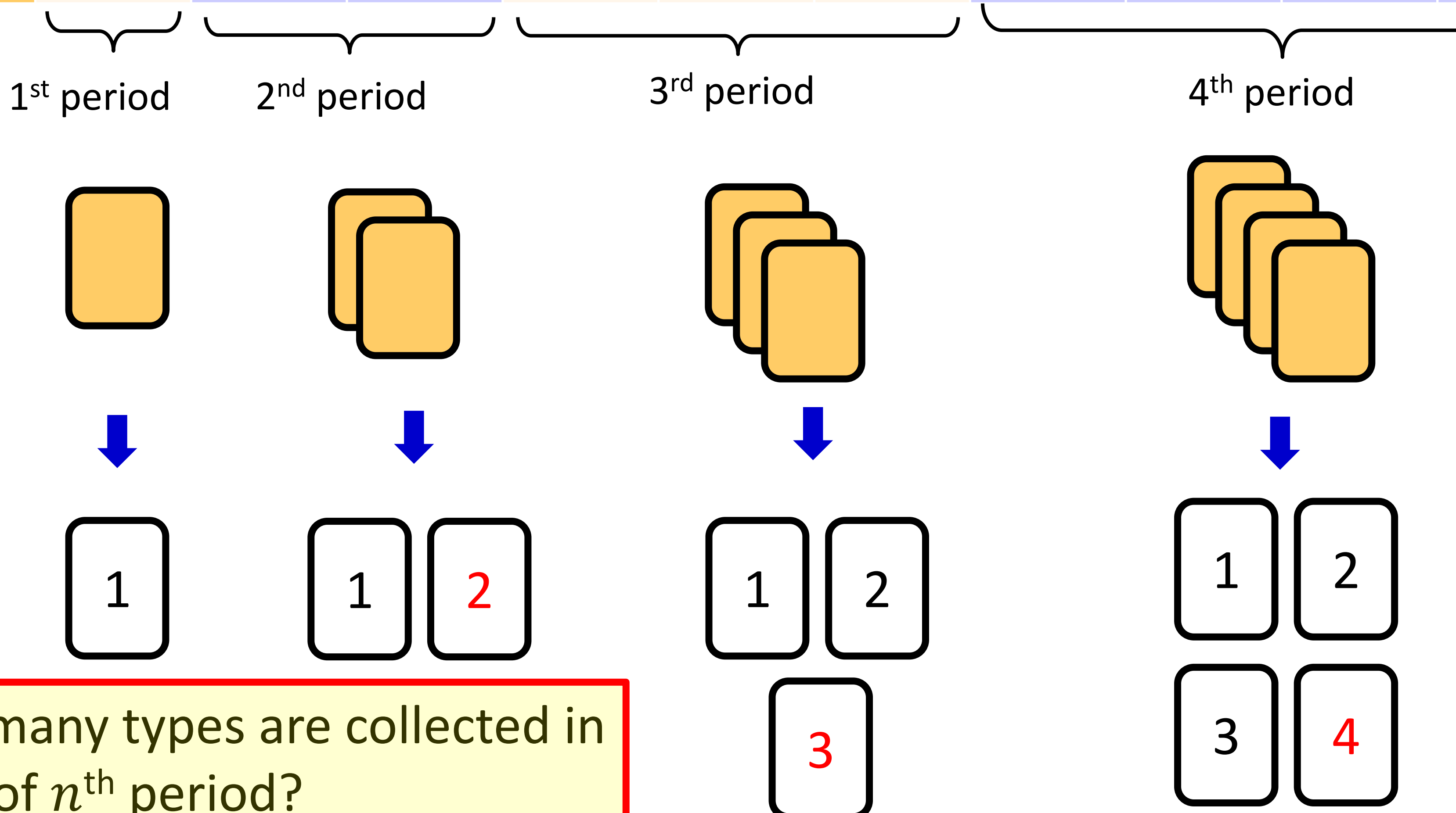
… Continued

Collecting an increasing number of coupons [K, Shimizu, Shiraga '21]

Day	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7	Day 8	Day 9
# types	1	2	2	3	3	3	4	4	4
$\Pr[X_t = k]$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$



Q. How many types are collected in the end of n^{th} period?

Collecting an increasing number of coupons [K, Shimizu, Shiraga '21]

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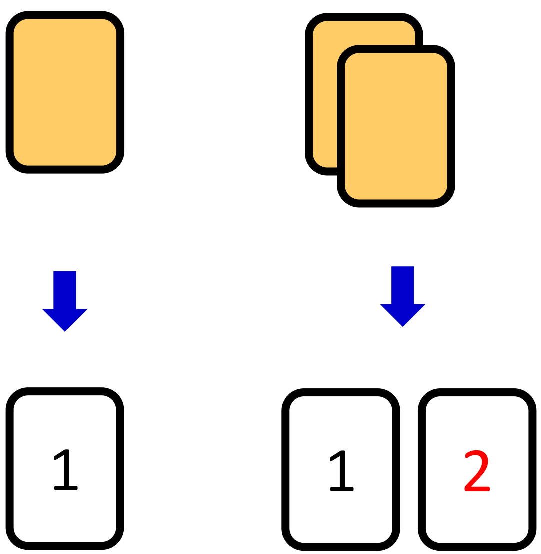
1st period   2nd period   3



Q. How many types are collected in the end of n^{th} period?

1. $O(\log n)$
2. $O(\sqrt{n})$
3. $\frac{n}{2}$
4. $.99n$

Collecting an increasing number of coupons [K, Shimizu, Shiraga '21]

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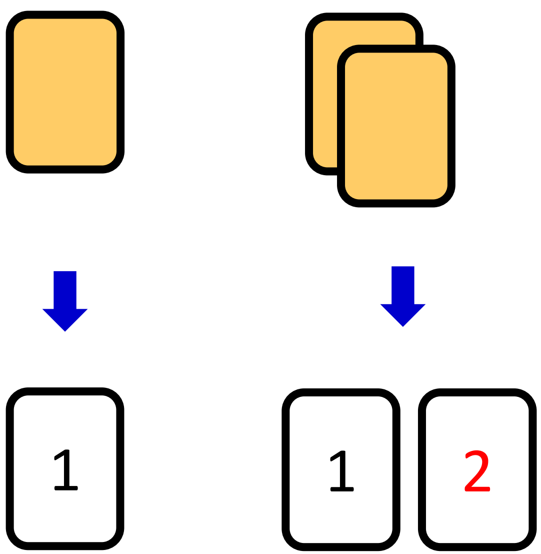


1. $O(\log n)$
2. $O(\sqrt{n})$
3. $\frac{n}{2}$
4. $.99n$
5. **at least $n-1$ in expectation**

Q. How many types are collected in the end of n^{th} period?

Collecting an increasing number of coupons

Draw a coupon everyday

$\mathfrak{d}(n)$: #days of the n^{th} period

U_n: #items uncollected in the end of n^{th} period

[K, Shimizu, Shiraga '21]

Prop.

If $\mathfrak{d}(n) = n$ then $\mathrm{E}[U_n] < \frac{1}{\mathrm{e}-1}$.

Proof.

- ✓ $\mathcal{E}_{i,n} := \begin{cases} 1 & (\text{item } i \text{ is uncollected in the end of the } n^{\mathrm{th}} \text{ period}) \\ 0 & (\text{item } i \text{ is collected by the end of the } n^{\mathrm{th}} \text{ period}) \end{cases}$

  for $i = 1,2,\ldots,n$.
- ✓ $U_n = \sum_{i=1}^{n} \mathcal{E}_{i,n}$
- ✓ Prob. that item n is uncollected in the end of the nth period:

$$\Pr[\mathcal{E}_{n,n} = 1] = \left(1 - \frac{1}{n}\right)^n < \mathrm{e}^{-1}$$

- ✓ Prob. that item i ($i \le n$) is uncollected in the end of the n^{th} period:

$$\Pr\left[\mathcal{E}_{i,n} = 1\right] = \left(1 - \frac{1}{i}\right)^i \left(1 - \frac{1}{i+1}\right)^{i+1} \cdots \left(1 - \frac{1}{n}\right)^n < \left(\frac{1}{\mathrm{e}}\right)^{n+1-i}$$

- ✓ $\mathrm{E}[U_n] = \sum_{i=1}^{n} \Pr\left[\mathcal{E}_{i,n}\right] < \sum_{i=1}^{n} \left(\frac{1}{\mathrm{e}}\right)^{n+1-i} = \frac{1}{\mathrm{e}} + \frac{1}{\mathrm{e}^2} + \cdots + \frac{1}{\mathrm{e}^n} < \frac{\frac{1}{\mathrm{e}}}{1-\frac{1}{\mathrm{e}}} = \frac{1}{\mathrm{e}-1} < 0.582.$

RWoGG $(\mathfrak{d}, G\ P)$

Coupon collector is often regarded as a RW on the complete graph, and we can extend the arguments to RWoGG for general graphs.

Thm. (general upper bound)

If $\mathfrak{d}(i) \geq c t_{\mathrm{hit}}(i)$ ($c \geq 1$) then $\mathrm{E}[U] = O(1)$.

Particularly, if $\frac{\mathfrak{d}(i)}{t_{hit}(i)} \xrightarrow{i\to\infty} \infty$ then $\mathrm{E}[U_n] \xrightarrow{n\to\infty} 0$.

Thm. (upper bound for lazy and reversible walk)

Suppose $P^{(i)}$ is lazy and reversible.

If $\frac{t_{\mathrm{hit}}(i)}{t_{\mathrm{mix}}(i)} \geq \frac{i^{\gamma}}{c}$ and $\mathfrak{d}(i) \geq \frac{3c t_{\mathrm{hit}}(i)}{i^{\gamma}}$ ($c > 0$) then $\mathrm{E}[U_n] \leq \frac{8n^{\gamma}}{c} + 32$.

S. Kijima, N. Shimizu, T. Shiraga, How many vertices does a random walk miss in a network with moderately increasing the number of vertices?, in Proc. SODA 2021, 106—122.

Related work (2/2): recurrence/transience of RW

- Much work about the recurrence/transience on growing graphs exist in the context of self-interacting random walks including reinforced random walks, excited random walks, etc. since 1990s, or before.
- Dembo, Huang and Sidoravicius (2014× 2): recurrent $\Leftrightarrow \sum_{t=0}^{\infty} \pi_t(0) = \infty$ for growing subregion of $\mathbb{Z}^d$ (fixed d), by conductance argument.
  - See also Huang and Kumagai (2016), Dembo, Huang, Morris and Peres (2017), Dembo, Huang and Zheng (2019), etc. about heat kernel, evolving set arguments.
- Amir, Benjamini, Gurel-Gurevich and Kozma (2015): random walk on growing tree. (random walk in changing environment).
- Huang (2017): growing graph w/ *uniformly bounded degrees*.
- Kumamoto, K. and Shirai (2024): k-ary tree, $\{0,1\}^n$ w/ an increasing n under **RWoGG** model by coupling.
- This work (2024): $\{0,1,\ldots,N\}^n$ (fixed N, increasing n) by pausing coupling.

3. Our previous work [SAND '24]

About the recurrence/transience of RWoGG,

for an introduction of the basic technique and its issue.

S. Kumamoto, S. Kijima and T. Shirai, An analysis of the recurrence/transience of random walks on growing trees and hypercubes, Proc. SAND 2024, 17:1-17:17

Example 3. Random walk on $\{0,1\}^n$ w/ an increasing n

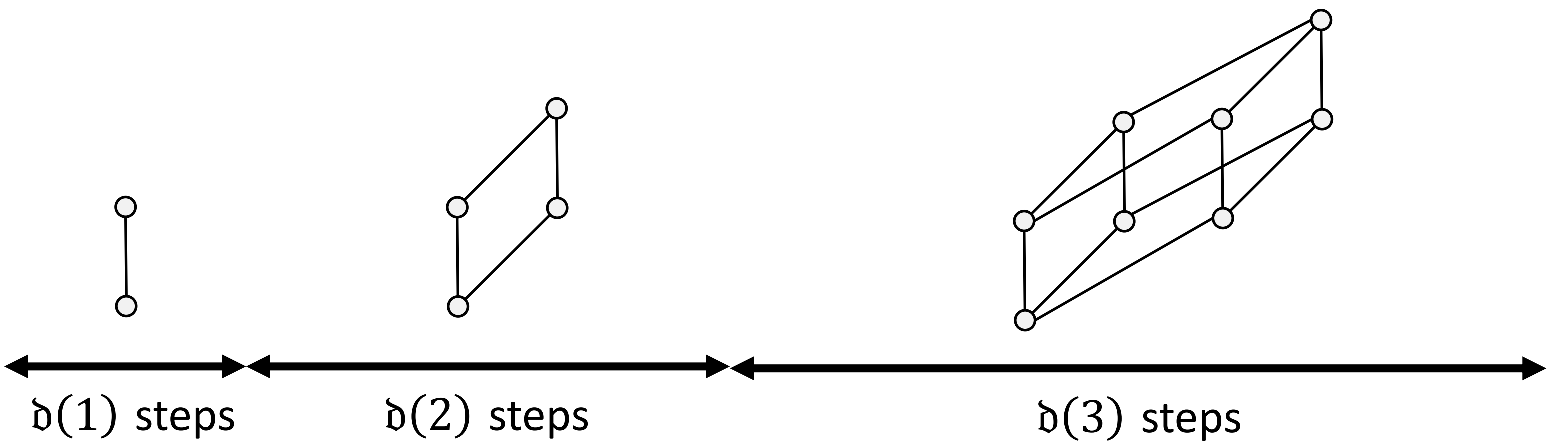
[SAND '24]

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

- $\mathfrak{d}(n) = 2^n$,
- $G(n)$ is a $\{0,1\}^n$ skeletone,
- $P(n)$ denotes the simple random walk,

  i.e., move to a neighbor w.p. $1/n$,

for $n = 1,2,\ldots$



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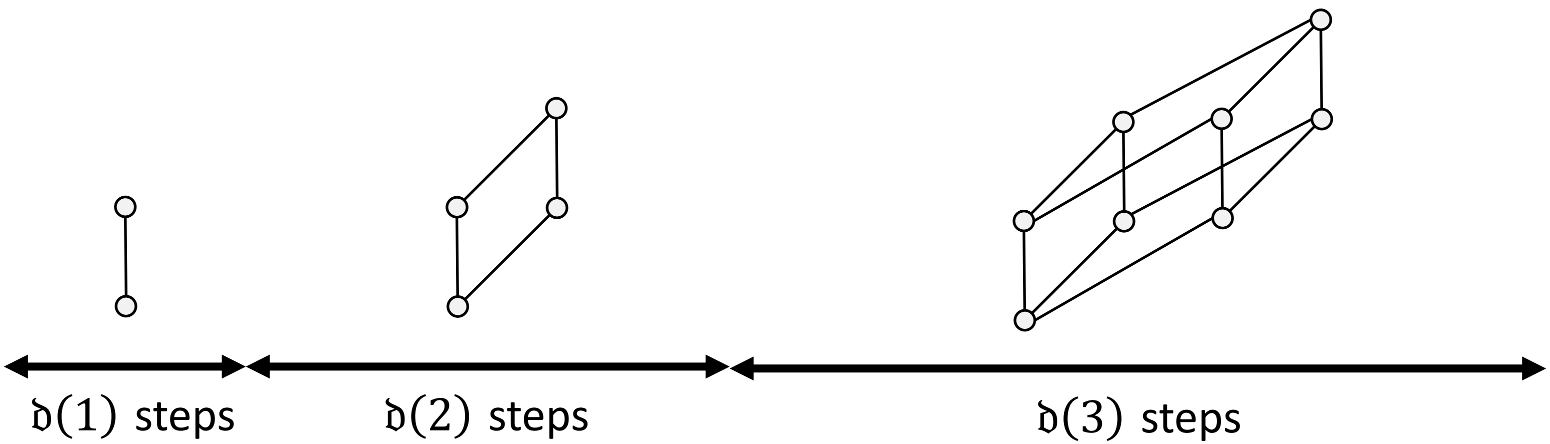
Recurrent

since $\sum_{n=1}^{\infty} \frac{2^n}{2^n} = \infty$.



Thm. [Kumamoto et al. 2024]

If $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{2^n} = \infty$ then recurrent, otherwise transient.

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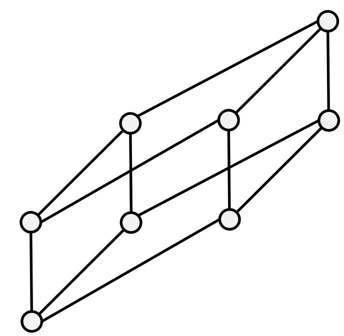
Lem. [Kumamoto et al. 2024]

Random walk on $\{0,1\}^n$ is ***LHaGG***.



$\mathfrak{d}(1)$ steps $\mathfrak{d}(2)$ steps $\mathfrak{d}(3)$ steps

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If $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{2^n} = \infty$ then recurrent, otherwise transient.

LHaGG [SAND '24]

Defs.

- $\mathcal{D}_1 = (f_1, G_1, P_1)$ is less homesick than $\mathcal{D}_2 = (f_2, G_2, P_2)$ if $R_1(t) \le R_2(t)$ for any t where $R_1(t)$ and $R_2(t)$ respectively denote the return probabilities of $\mathcal{D}_1$ and $\mathcal{D}_2$ at time t.

- $\mathcal{D} = (f, G, P)$ is less homesick as graph growing (**LHaGG**) if $\mathcal{D}$ is less homesick than $\mathcal{D}' = (g, G, P)$ for any g satisfying that $\sum_{k=1}^{n} f(k) \le \sum_{k=1}^{n} g(k)$ for any n,
  i.e., $\mathcal{D}$ and $\mathcal{D}'$ grows similarly, but $\mathcal{D}$ grows *faster.*

The faster a graph grows,
the smaller the return probability.

Theorems by LHaGG

The faster a graph grows,
the smaller the return probability.

Under the condition of LHaGG, we can prove the following sufficient conditions of recurrence/transience, respectively.

Thm. [Kumamoto, K., Shirai '24]
Suppose $\mathcal{D} = (\mathfrak{d}, G, P)$ is LHaGG. If

$$\sum_{n=1}^{\infty} \mathfrak{d}(n) p(n) = \infty$$

then $\mathcal{D}$ is recurrent at v, where $p(n) = \pi_n(v)$.

Thm. [Kumamoto, K., Shirai '24]
Suppose $\mathcal{D} = (\mathfrak{d}, G, P)$ is LHaGG. If

$$\sum_{n=1}^{\infty} \max\{\mathfrak{d}(n), \mathfrak{t}(n)\}\, p(n) < \infty$$

then $\mathcal{D}$ is transient at v, where $\mathfrak{t}(n)$ represents the mixing time.

Example 3. Random walk on $\{0,1\}^n$ w/ an increasing n

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

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The faster a graph grows,
the smaller the return probability?

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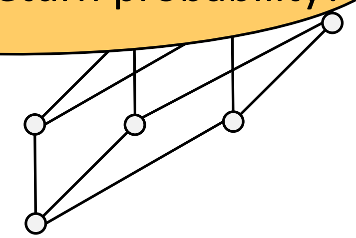

$\mathfrak{d}(1)$ steps | $\mathfrak{d}(2)$ steps | $\mathfrak{d}(3)$ steps

Thm. [Kumamoto et al. 2024]

If $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{2^n} = \infty$ then recurrent, otherwise transient.

FAQ: Any example for ***not*** LHaGG?

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Isn't it trivial?

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A (lazy) simple random walk on

The faster a graph grows,
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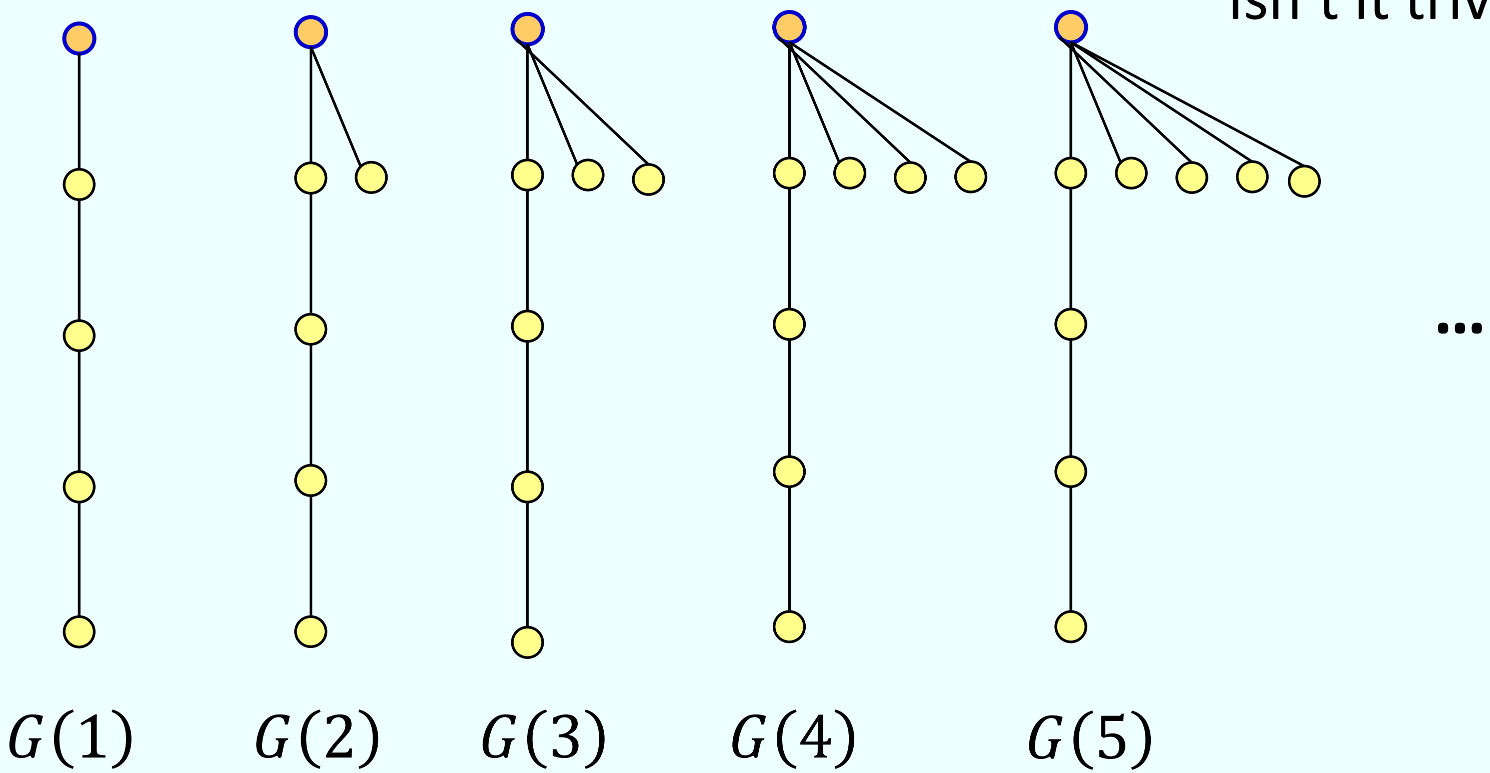
Isn't it trivial?



is ***not*** LHaGG.

Lazy RW on $\{0,1\}^n$ w/ increasing n is LHaGG

[SAND '24]

Proof.

The proof is a **monotone coupling**.

- Let $X_t \sim \mathcal{D}_f = (f, G, P)$ and $Y_t \sim \mathcal{D}_g = (g, G, P)$ where $\sum_{i=1}^{n} f(i) \geq \sum_{i=1}^{n} g(i)$,
  - i.e., the graph of $\mathcal{D}_g$ grows faster than that of $\mathcal{D}_f$.
- Let $|X_t|, |Y_t|$ denote the number of 1s in $X_t \in \{0,1\}^{n_t}$, $Y_t \in \{0,1\}^{m_t}$ where notice that $n_t \leq m_t$. Then,

$$\Pr[|X_{t+1}| - 1 = |X_t|] = \frac{1}{2}\frac{|X_t|}{n_t},\ \Pr[|X_{t+1}| = |X_t|] = \frac{1}{2},\ \Pr[|X_{t+1}| + 1 = |X_t|] = \frac{1}{2}\left(1 - \frac{|X_t|}{n_t}\right)$$
$$\Pr[|Y_{t+1}| - 1 = |Y_t|] = \frac{1}{2}\frac{|Y_t|}{m_t},\quad \Pr[|Y_{t+1}| = |Y_t|] = \frac{1}{2},\quad \Pr[|Y_{t+1}| + 1 = |Y_t|] = \frac{1}{2}\left(1 - \frac{|Y_t|}{m_t}\right)$$

- if $|X_t| < |Y_t|$ then we can couple so that $|X_{t+1}| \leq |Y_{t+1}|$
  - thanks to the self-loop w.p. $\frac{1}{2}$.
- If $|X_t| = |Y_t|$ then we can couple so that $|X_{t+1}| \leq |Y_{t+1}|$ since $n_t \leq m_t$.

Thus, $X_t = o$ if $Y_t = o$,
meaning that $\Pr[X_t = o] \geq \Pr[Y_t = o]$. □

It looks a very simple exercise if you are familiar with **coupling**, but $n_t \neq m_t$ makes some trouble, in general.

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Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

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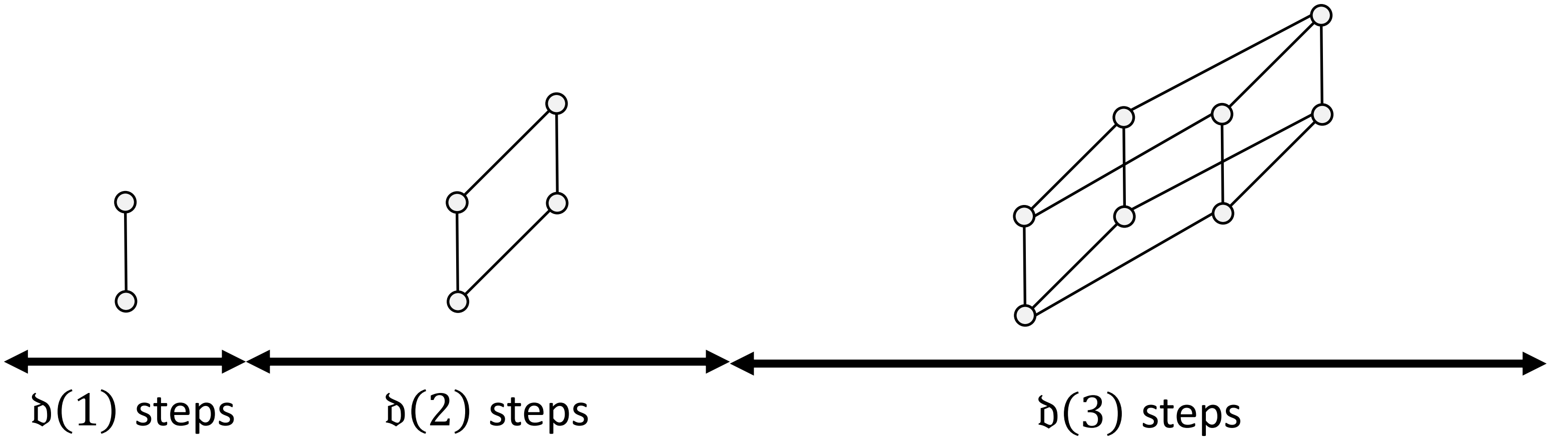

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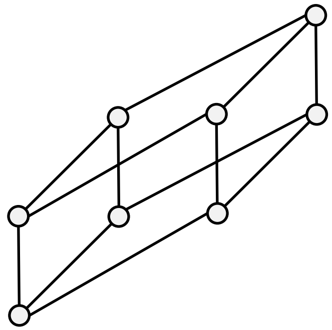

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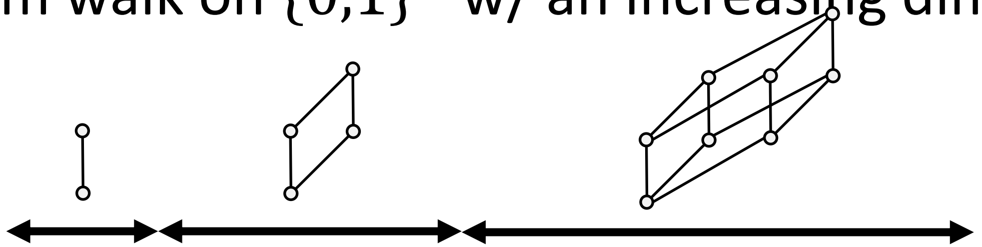
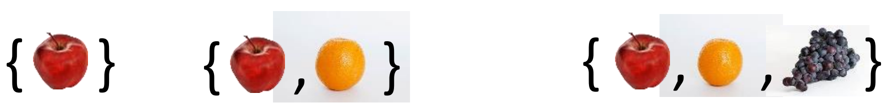
Three representations (or "applications"?) of $\{0,1\}^n$

- Random walk on $\{0,1\}^n$ w/ an increasing dimensions



$\mathfrak{d}(1)$ steps $\mathfrak{d}(2)$ steps $\mathfrak{d}(3)$ steps

- Random pick/drop items w/ an increasing number of items



$\mathfrak{d}(1)$ steps $\mathfrak{d}(2)$ steps $\mathfrak{d}(3)$ steps

- Random bit flip of binary w/an increasing bit length

.1 .10 .101

$\mathfrak{d}(1)$ steps $\mathfrak{d}(2)$ steps $\mathfrak{d}(3)$ steps

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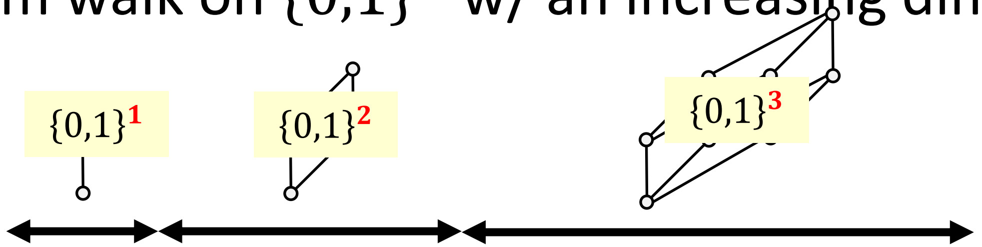

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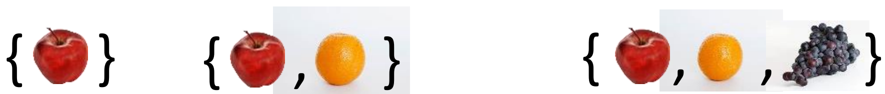

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.1 .10 .101

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Extension from $\{0,1\}^n$ to $\{0,1,\dots,9\}^n$

- Random walk on $\{0,1,\dots,9\}^n$ w/ an increasing n

$\{0,1,\dots,9\}^1$ $\{0,1,\dots,9\}^2$ $\{0,1,\dots,9\}^3$

$\mathfrak{d}(1)$ steps $\mathfrak{d}(2)$ steps $\mathfrak{d}(3)$ steps

- Random buy/sell stocks w/ an increasing #brands

$\times 3$ $\times 2 +$ $\times 4$ $\times 3 +$ $\times 0+$ $\times 2$

$\mathfrak{d}(1)$ steps $\mathfrak{d}(2)$ steps $\mathfrak{d}(3)$ steps

- Random up/down digits w/ an increasing digit length

.3 .24 .302

$\mathfrak{d}(1)$ steps $\mathfrak{d}(2)$ steps $\mathfrak{d}(3)$ steps

Target. Random walk on $\{0,1,\dots,N\}^n$ w/ an increasing n

w/ a fixed N.

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

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$\{0,1,\dots,N\}^{\mathbf{1}}$ $\{0,1,\dots,N\}^{\mathbf{2}}$ $\{0,1,\dots,N\}^{\mathbf{3}}$

$\mathfrak{d}(1)$ steps $\mathfrak{d}(2)$ steps $\mathfrak{d}(3)$ steps

Q.
Is random walk on $\{0,1,\dots,N\}^n$ LHaGG?

A. We can't prove it.

4. Main Result

Target. Random walk on $\{0,1,\dots,N\}^n$ w/ an increasing n

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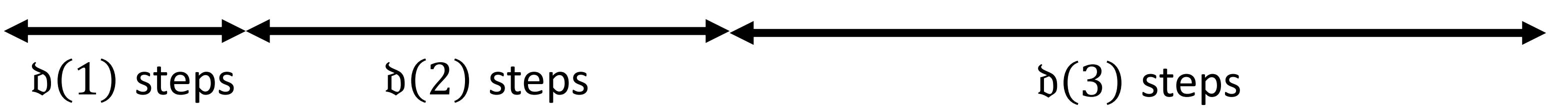
Lem. 7.

Random walk on $\{0,1,\ldots,N\}^n$ is ***weakly*** LHaGG.

$N\}^{3}$



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$N\}^3$

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Thm. 6. If $\mathcal{D} = (\mathfrak{d}, G, P)$ satisfies
$$\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{(2N)^n} = \infty$$
then o is recurrent, otherwise o is transient.

Recall: LHaGG [SAND '24]

Defs.

- $\mathcal{D}_1 = (f_1, G_1, P_1)$ is less homesick than $\mathcal{D}_2 = (f_2, G_2, P_2)$
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The faster a graph grows,
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Recall: LHaGG

We replace the condition about the return prob. with

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The faster a graph grows,
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wLHaGG

We replace the condition about the return prob. with a condition of the **sum of** return prob.

Defs.

- $\mathcal{D}_1 = (f_1, G_1, P_1)$ is weakly less homesick than $\mathcal{D}_2 = (f_2, G_2, P_2)$ if $\sum_{t=1}^{T} R_1(t) \leq \sum_{t=1}^{T} R_2(t)$ for any T where $R_1(t)$ and $R_2(t)$ respectively denote the return probabilities of $\mathcal{D}_1$ and $\mathcal{D}_2$ at time t.

- $\mathcal{D} = (f, G, P)$ is weakly less homesick as graph growing (**wLHaGG**) if $\mathcal{D}$ is weakly less homesick than $\mathcal{D}' = (g, G, P)$ for any g satisfying that $\sum_{k=1}^{n} f(k) \leq \sum_{k=1}^{n} g(k)$ for any n,
  i.e., $\mathcal{D}$ and $\mathcal{D}'$ grows similarly, but $\mathcal{D}$ grows *faster.*

The faster a graph grows,
the smaller the **expected number** of returns.

= sum of return prob.

General theorems

Condition 0. (ergodic). In $\mathcal{D} = (\mathfrak{d}, G, P)$, every transition matrix $P(n)$ is ergodic.

Condition 1. (mixing time). $\mathcal{D} = (\mathfrak{d}, G, P)$ satisfies

$$\sum_{k=1}^{\infty} \tau^*(k) p(k) < \infty$$

where $p(k) = \pi_k(o)$ and $\tau^*(k) = t_{\text{mix}}^{k}\left(\frac{p(k)}{4}\right)$.

Mixing time is not very big.
E.g., $O\left(\frac{1}{\pi_k(o)} \frac{1}{k \log k}\right)$

Thm. 2. (Recurrence).
Suppose $(\mathfrak{d}, G, P)$ satisfies Conditions 0 and 1.
If $\sum_{k=1}^{\infty} \mathfrak{d}(k) p(k) = \infty$ then the initial vertex v is recurrent.

Thm. 4. (Transience).
Suppose $(\mathfrak{d}, G, P)$ satisfies Conditions 0 and 1, and it is wLHaGG.
If $\sum_{k=2}^{\infty} \mathfrak{d}(k) p(k-1) < \infty$ then the initial vertex v is transient.

Recurrence

Thm. 2. (Recurrence).

Suppose $(\mathfrak{d}, G, P)$ satisfies Conditions 0 and 1.

If $\sum_{k=1}^{\infty} \mathfrak{d}(k)p(k) = \infty$ then the initial vertex v is recurrent.

Proof. Let X_t follow $(\mathfrak{d}, G, P)$, and let $R(t) = \Pr[X_t = o]$. We claim

Lem. 3. $\sum_{t=1}^{T_n} R(t) \geq \frac{1}{2} \sum_{k=1}^{n} \big(\mathfrak{d}(k) - \tau^*(k)\big)p(k)$

Proof of Lem. 3.

- Notice that X_t follows P_n for $t \in \big[T_{n-1}, T_{n-1} + \mathfrak{d}(n)\big)$.
- If $\mathfrak{d}(n) > t_{\mathrm{mix}}(\epsilon)$ then $R(t) \geq \pi_n(\mathrm{o}) - \epsilon$ for $t \geq T_{n-1} + t_{\mathrm{mix}}(\epsilon)$ where π_n is the stationary distribution of P_n.
- Thus, $R(t) \geq \pi_n(\mathrm{o}) - \frac{1}{2}p(n) = \frac{1}{2}p(n)$ since $\tau^*(k) = t_{\mathrm{mix}}\left(\frac{1}{2}p(n)\right)$ and $p(n) = \pi_n(o)$.
- $\sum_{t=1}^{T_n} R(t) = \sum_{k=1}^{n} \sum_{s=1}^{\mathfrak{d}(k)} R(T_{n-1} + s) \geq \sum_{k=1}^{n} \sum_{s=\tau^*(n)}^{\mathfrak{d}(k)} R(T_{n-1} + s) \geq \sum_{k=1}^{n} \sum_{s=\tau^*(n)}^{\mathfrak{d}(k)} \frac{1}{2} p(n) = \frac{1}{2} \sum_{k=1}^{n} \big(\mathfrak{d}(k) - \tau^*(k)\big)p(k)$

Once we obtain Lem. 3, Thm. 2 is easy: $\sum_{t=1}^{\infty} R(t) = \infty$ holds if $\sum_{k=1}^{\infty} \mathfrak{d}(k)p(k) = \infty$ and $\sum_{k=1}^{\infty} \tau^*(k)p(k) < \infty$. ← Mixing time condition

Transience

Thm. 4. (Transience).
Suppose $(\mathfrak{d}, G, P)$ satisfies Conditions 0 and 1, and it is wLHaGG.
If $\sum_{k=2}^{\infty} \mathfrak{d}(k) p(k-1) < \infty$ then the initial vertex v is transient.

Proof. Let $f(k) = \max\{\mathfrak{d}, \tau^*(k)\}$.

By wLHaGG, $\sum_{t=1}^{T_n} R_{\mathfrak{d}}(t) \le \sum_{t=1}^{T_n} R_g(t)$.

Lem. 5. $\sum_{t=1}^{T_n} R_g(t) \le g(1) + \frac{3}{2} \sum_{k=2}^{n} g(k) p(k-1)$

Proof of Lem. 5.

Let $f(k) = \begin{cases} g(k) & k \le n-1 \\ \infty & k = n. \end{cases}$ Then, $\sum_{k=1}^{m} g(k) \le \sum_{k=1}^{m} g(k)$ for any m.

Let $X_t \sim \mathcal{D}_g = (g, G, P)$ and $Y_t \sim \mathcal{D}_f = (f, G, P)$.

- Notice that Y_t follows P_{n-1} for $t \ge T_{n-2}$.
- By wLHaGG, $\sum_{t=1}^{T} \Pr[X_t = o] \le \sum_{t=1}^{T} \Pr[Y_t = o]$ for any T.
- $R_f(t) \le \pi_n(o) + \frac{1}{2} p(n-1) = \frac{3}{2} p(n-1)$ for $t \ge T_{n-1}$
- $\sum_{t=1}^{T_n} R_g(t) = \sum_{k=1}^{n} \sum_{s=1}^{g(k)} R_g(T_{k-1} + s) \le g(1) + \sum_{k=2}^{n} \sum_{s=1}^{g(k)} R_f(T_{k-1} + s) \le g(1) + \sum_{k=2}^{n} \sum_{s=1}^{g(k)} \frac{3}{2} p(k-1) = g(1) + \frac{3}{2} \sum_{k=2}^{n} g(k) p(k-1)$

Particularly, remark $X_t \sim P_n$ but $Y_t \sim P_{n-1}$ for $t \in [T_{n-1}, T_n)$

Once we obtain Lem. 5, Thm. 4 is clear.

Target. Random walk on $\{0,1,\dots,N\}^n$ w/ an increasing n

w/ a fixed N.

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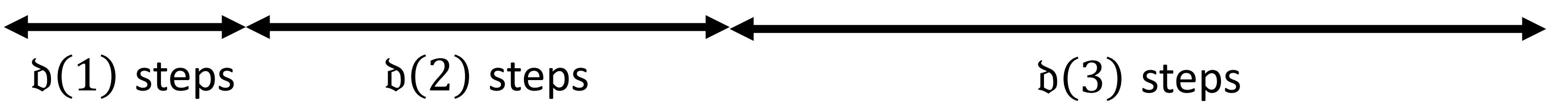
- $G(n)$ is a $\{0,1,\dots,N\}^n$ skeletone,
- $P(n)$ denotes the lazy simple random walk w/ reflection bound,

  i.e., move to a neighbor w.p. $1/4n$, unless boundary

for $n = 1,2,\dots$

> Lem. 7.
>
> Random walk on $\{0,1,\dots,N\}^n$ is ***weakly*** LHaGG.

$\}^3$



It looks a very simple exercise if you are familiar with **coupling**, but $n_t \neq m_t$ makes some trouble, in general.

Target. Random walk on $\{0,1,\ldots,N\}^n$ w/ an increasing n

w/ a fixed N.

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

- $\mathfrak{d}(n) = N^n$,
- $G(n)$ is a $\{0,1,\ldots,N\}^n$ skeletone,
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Random walk on $\{0,1,\ldots,N\}^n$ is ***weakly*** LHaGG.

$\}^3$

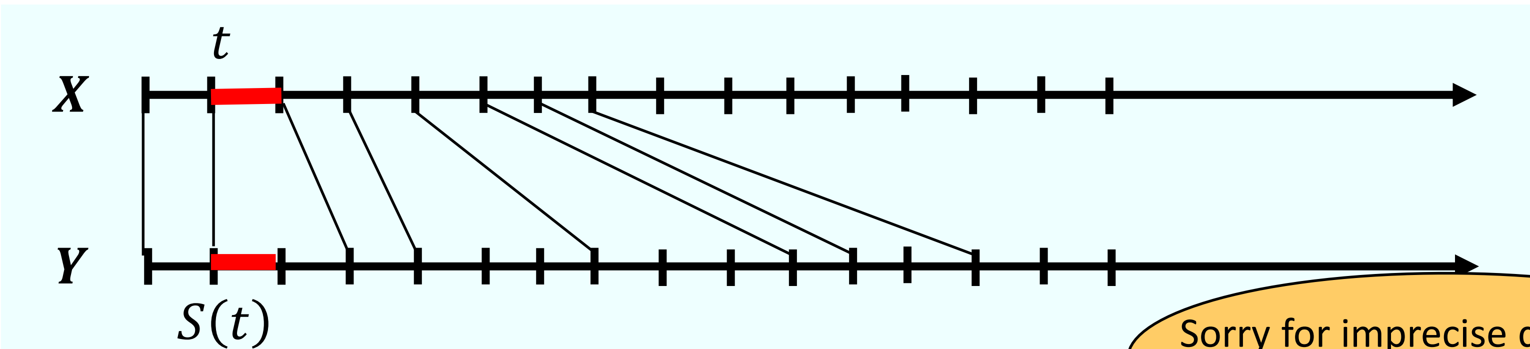
$\mathfrak{d}(1)$ steps $\mathfrak{d}(2)$ steps $\mathfrak{d}(3)$ steps

We develop "pausing coupling"

It looks a very simple exercise if you are familiar with **coupling**, but $n_t \neq m_t$ makes some trouble, in general.

Figure of pausing coupling

- Let $\boldsymbol{X} = X_0, X_1, X_2, \ldots \sim \mathcal{D}_f$ and $\boldsymbol{Y} = Y_0, Y_1, Y_2, \ldots \sim \mathcal{D}_g$ where let $\mathcal{D}_g$ grow faster than $\mathcal{D}_f$.
- We couple $\boldsymbol{X}$ and $\boldsymbol{Y}$, instead of X_t and Y_t.



Sorry for imprecise description to avoid bothering notation.

We define time correspondence $t \mapsto S(t)$ depending on $\boldsymbol{Y}$ so that

1. **if $\boldsymbol{Y_t}$ does self-loop then so does $\boldsymbol{X_{S^{-1}(t)}}$,**
2. if Y_t updates Y_t^i for $i \leq \dim\left(X_{S^{-1}(t)}\right)$ then $\boldsymbol{X}$ updates $X_{S^{-1}(t)}^i$,
3. if Y_t updates Y_t^i for $i > \dim\left(X_{S^{-1}(t)}\right)$ then $\boldsymbol{X}$ pauses.

⇨ We need to check “measure conservation” of the coupling.

Figure of pausing coupling

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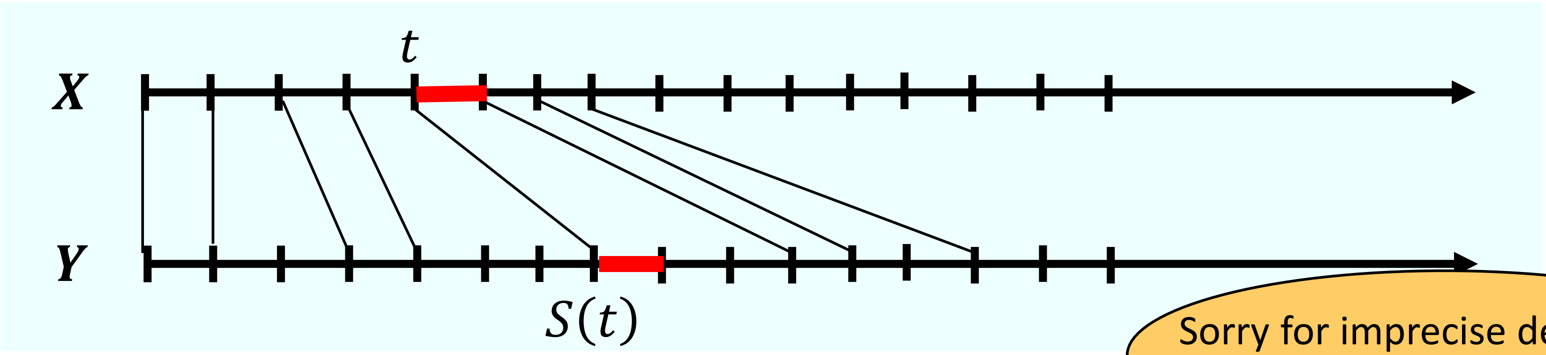

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1. if Y_t does self-loop then so does $X_{S^{-1}(t)}$,
2. **if $\boldsymbol{Y_t}$ updates $\boldsymbol{Y_t^i}$ for $\boldsymbol{i \le \dim\left(X_{S^{-1}(t)}\right)}$ then $\boldsymbol{X}$ updates $\boldsymbol{X^i_{S^{-1}(t)}}$,**
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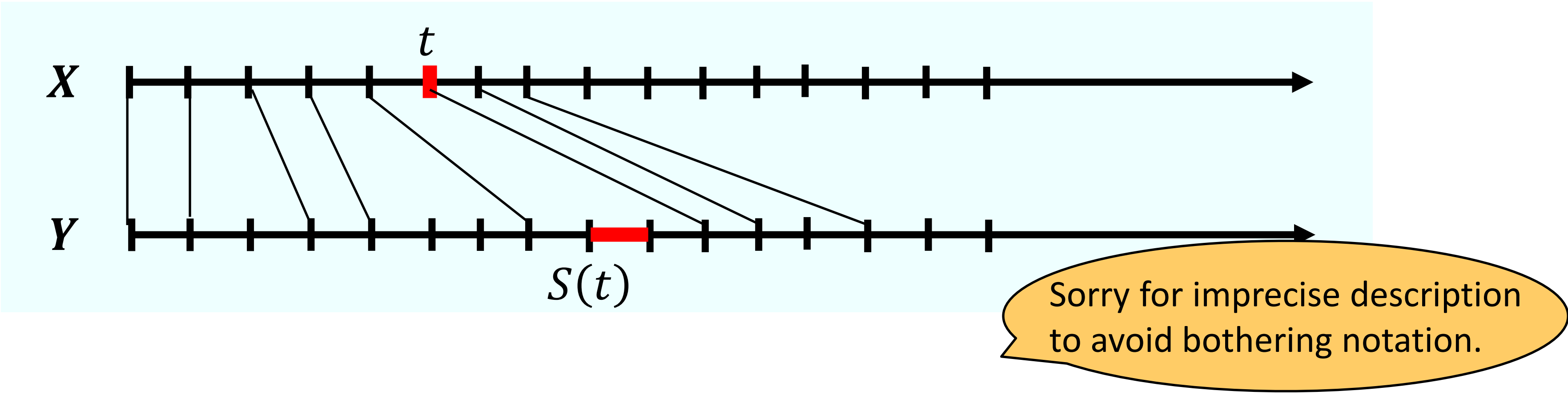

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We need to check "measure conservation" of the coupling.

Outline of the proof

Lem. 7.

Random walk on $\{0,1,\dots,N\}^n$ w/ increasing n is wLHaGG.

Let η: $\boldsymbol{Y} \mapsto \boldsymbol{X} = \eta(\boldsymbol{Y})$ denote the coupling described in the previous slide.

We prove two things:

- The coupling η preserves the measure, i.e.,
$$\Pr[\boldsymbol{Y} = \boldsymbol{y}] = \Pr[\boldsymbol{X} = \eta(\boldsymbol{y})]$$
- The coupling η preserves $|X_t| \leq |Y_s|$ (meaning "$|\eta(y_s)| \leq |y_s|$") for any s satisfying $S(t) \leq s < S(t+1)$.
  - This implies $\#\{t \leq T \mid X_t = o\} \geq \#\{t \leq T \mid Y_t = o\}$ for any T.

Def. $S(t)$

Proof.

Suppose $\boldsymbol{Y} = Y_0, Y_1, Y_2, Y_3, \ldots$ is represented by

$$\boldsymbol{\theta}_{\boldsymbol{Y}} = (\lambda_1, j_1, \rho_1), (\lambda_2, j_2, \rho_2), (\lambda_3, j_3, \rho_3), \ldots$$

We define $S: \mathbb{Z} \to \mathbb{Z}$ according to $\boldsymbol{\theta}$.

Let $S(1) = \min\{\min\{t > 0 | \lambda_t = 0\}, \min\{t > 0 | j_t \in n_0\}\}$.

Recursively, let

$$S(k) = \min\{\min\{t > S(k-1) | \lambda_t = 0\}, \min\{t > S(k-1) | j_t \in n_{k-1}\}\}$$

where let $\min\{\emptyset\} = \infty$.

If $S(k) = \infty$ then let $S(k+1) = \infty$.

For convenience, let $S^{-1}(t) = k$ for $t = S(k) < \infty$ $(k = 1,2, \ldots)$.

Then, we define $\boldsymbol{X} = X_0, X_1, X_2, \ldots$ by

$$\boldsymbol{\theta}_{\boldsymbol{X}} = \left(\left(\lambda_{S^{-1}(k)}, j_{S^{-1}(k)}, \rho_{S^{-1}(k)}\right)\right)_{k=1,2,\ldots}$$

$$= \left(\lambda_{S^{-1}(1)}, j_{S^{-1}(1)}, \rho_{S^{-1}(1)}\right), \left(\lambda_{S^{-1}(2)}, j_{S^{-1}(2)}, \rho_{S^{-1}(2)}\right), \ldots$$

as far as $S(k) < \infty$.

If $S(k) = \infty$ then generate $(\lambda'_k, j'_k, \rho'_k)$ and transit to X_{k+1} according to it.

Def. $S(t)$

Proof.

Suppose $\boldsymbol{Y} = Y_0, Y_1, Y_2, Y_3, \ldots$ is represented by

$$\boldsymbol{\theta}_{\boldsymbol{Y}} = (\lambda_1, j_1, \rho_1), (\lambda_2, j_2, \rho_2), (\lambda_3, j_3, \rho_3), \ldots$$

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For convenience, let $S^{-1}(t) = k$ for $t = S(k) < \infty$ $(k = 1,2, \ldots)$.

Then, we define $\boldsymbol{X} = X_0, X_1, X_2, \ldots$

$\boldsymbol{\theta}$

Time up…

$_{(2)}), \ldots$

as far as $S(k) <$

If $S(k) = \infty$ then generate $(\lambda'_k, j'_k, \rho'_k)$ and transit to X_{k+1} according to it.

5. Concluding remarks

Final slide

Result

- ■ Recurrence/transience of wLHaGG RWoGG.
- ■ Random walk on $\{0,1,\ldots,N\}^n$ w/ increasing n is wLHaGG.
  - ➢ Proof by pausing coupling.

Future work

- ☐ Simplify the proof
  - ➢ Extension to other RWoGGs
    - E.g., GW tree, PA graph, and more general graphs,
    - Edge dynamics, e.g., growing + edge Markovian.
- ☐ Analysis of RWoGG beyond recurrence/transience
  - ➢ Hitting time, meeting time, gathering time, etc.
  - ➢ Find a new limit, undefined for an infinite graph.

The end

Thank you for the attention.

Lazy simple random walk on $\{0,1,\dots,N\}^n$ w/ increasing n

Current state $X_t = \left(X_t^1, \dots, X_t^{n_t}\right) \in \{0,1,\dots,N\}^{n_t}$.

1. W.p. $\frac{1}{2}$, set $X_{t+1} = X_t$.
2. Else, choose $i \in \{1, \dots, n_t\}$ u.a.r.
3. If X_t^i is not 0 nor N then update as $X_{t+1}^i = X_t^i \pm 1$ w.p. $\frac{1}{2}$ resp.
4. Else if $X_t^i = 0$ then update as $X_{t+1}^i = X_t^i + 1$.
5. Else if $X_t^i = N$ then update as $X_{t+1}^i = X_t^i - 1$.

Lazy simple random walk on $\{0,1,\ldots,N\}^n$ w/ increasing n

Current state $X_t = \left(X_t^1, \ldots, X_t^{n_t}\right) \in \{0,1,\ldots,N\}^{n_t}$.

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5. Else if $X_t^i = N$ then update as $X_{t+1}^i = X_t^i - 1$.

If $\lambda = 0$ self-loop

Choose i u.a.r.

If $\rho = 0$ then -1

A transition $X_t \mapsto X_{t+1}$ is represented

by uniform r.v.s $(\lambda, i, \rho) \in \{0,1\} \times \{1,\ldots,n_t\} \times \{0,1\}$.