

The Recurrence/Transience of Random Walks on a Bounded Grid in an Increasing Dimension

Shuma Kumamoto (Kyushu Univ.),

*Shuji Kijima (Shiga Univ.),

Tomoyuki Shirai (Kyushu Univ.)

Plan of talk

1. Introduction

- \mathbb{Z}^3
- RWoGG
- Tree

2. Related work

- Exploration

3. Previous work

- LHaGG
- $\{0,1\}^n$ proof
- Extension to $\{0,1, \dots, N\}^n$

4. Main result

- Weakly LHaGG
- Recurrence
- Transience
- pausing coupling

5. Concluding remarks

Plan of talk ≥ 49 min.

1. Introduction (≥ 9 min.)
 - \mathbb{Z}^3
 - RWoGG
 - Tree
2. Related work (≥ 6 min.)
 - Exploration
3. Previous work (≥ 8 min.)
 - LHaGG
 - $\{0,1\}^n$ proof
 - Extension to $\{0,1, \dots, N\}^n$
4. Main result (≥ 25 min.)
 - Weakly LHaGG
 - Recurrence
 - Transience
 - pausing coupling
5. Concluding remarks (1 min.)

Plan of talk ~~≥ 49 min.~~ 25 min.

1. Introduction (~~≥ 9 min.~~ 6 min.)

- \mathbb{Z}^3
- RWoGG
- ~~Tree~~

2. Related work (~~≥ 6 min.~~ 3 min.)

- ~~Exploration~~

3. Previous work (~~≥ 8 min.~~)

- LHaGG
- $\{0,1\}^n$ proof
- Extension to $\{0,1, \dots, N\}^n$

4. Main result (~~≥ 25 min.~~ 7 min.)

- Weakly LHaGG
- ~~Recurrence~~
- ~~Transience~~
- pausing coupling

5. Concluding remarks (1 min.)

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Shuji Kijima

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1. Introduction w/ examples

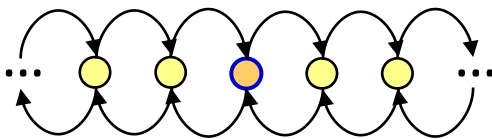
Recurrence/Transience of Random walks on *infinite* graphs

A random walk on an infinite graph is **recurrent** at vertex v if it visits v **infinitely many times**, i.e.,

$$\sum_{t=0}^{\infty} \Pr[X_t = v] = \infty$$

holds, otherwise it is said to be **transient**.

For instance,



RW on \mathbb{Z} is **recurrent** at 0,

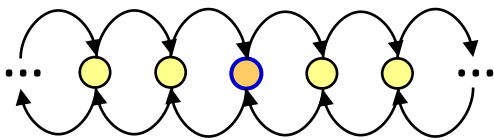
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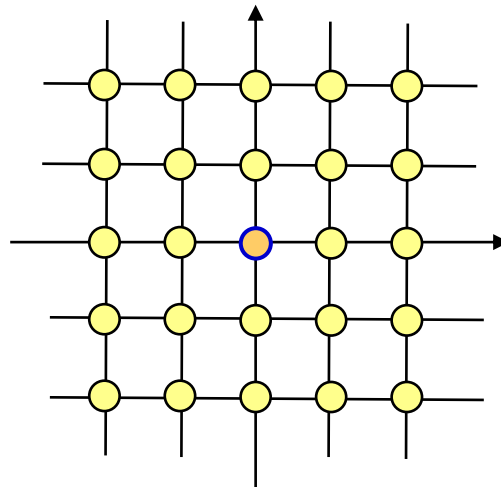
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RW on \mathbb{Z} is **recurrent** at o ,



RW on \mathbb{Z}^2 is **recurrent** at o ,

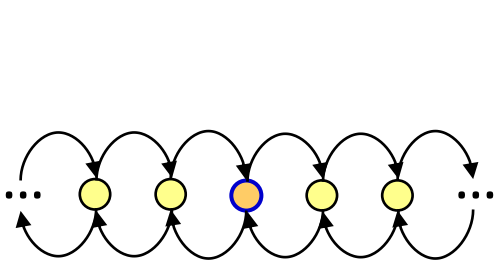
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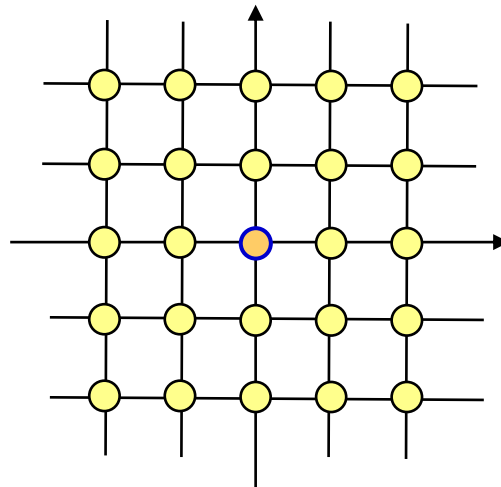
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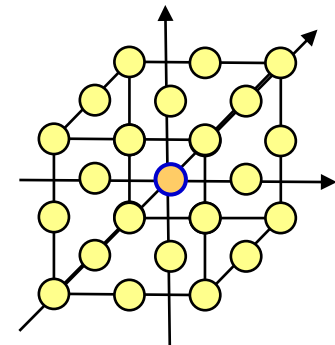
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RW on \mathbb{Z} is **recurrent** at o ,



RW on \mathbb{Z}^2 is **recurrent** at o ,



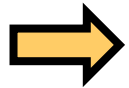
RW on \mathbb{Z}^3 is **transient** at o ,

Example 1. Random walk in a growing region of \mathbb{Z}^3

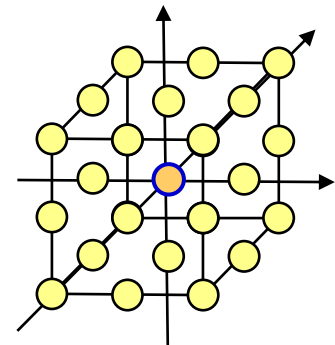
- ✓ Random walk on \mathbb{Z}^3 is **transient** at o .
- ✓ Random walk on $\{-n, \dots, n\}^3$ is **recurrent** at o .

Q. Is a random walk on $\{-n, \dots, n\}^3$ **recurrent** or **transient** if n **increases** as time go on?

A. It depends on the increasing speed.



Find the phase transition point regarding the growing speed.



RW on \mathbb{Z}^3 is **transient** at o ,

Model: Random Walk on a Growing Graph (RWoGG)

□ Growing graph is a sequence of static graphs

[K, Shimizu, Shiraga '21]

$$\mathcal{G} = \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \dots$$

where each $\mathcal{G}_t = (\mathcal{V}_t, \mathcal{E}_t)$ is a static simple graph.

We assume $\mathcal{V}_t \subseteq \mathcal{V}_{t+1}$, for convenience.

Furthermore, $\mathcal{E}_t \subseteq \mathcal{E}_{t+1}$ holds in this talk.

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□ RWoGG (\mathfrak{d}, G, P) is a specific model:

➤ $\mathfrak{d}(1), \mathfrak{d}(2), \mathfrak{d}(3), \dots \in \mathbb{Z}$ denote the **duration** time.

➤ **Growing graph** is given by

$$\mathcal{G}_t = G(n) \text{ for } t \in [T_{n-1}, T_{n-1} + \mathfrak{d}(n))$$

where $T_n = \sum_{i=1}^n \mathfrak{d}(i)$, i.e.,

$$\mathcal{G}_t = \begin{cases} G(1) & \text{for the first } \mathfrak{d}(1) \text{ steps} \\ G(2) & \text{for the next } \mathfrak{d}(2) \text{ steps} \\ G(3) & \text{for the next } \mathfrak{d}(3) \text{ steps} \\ \vdots & \vdots \end{cases}$$

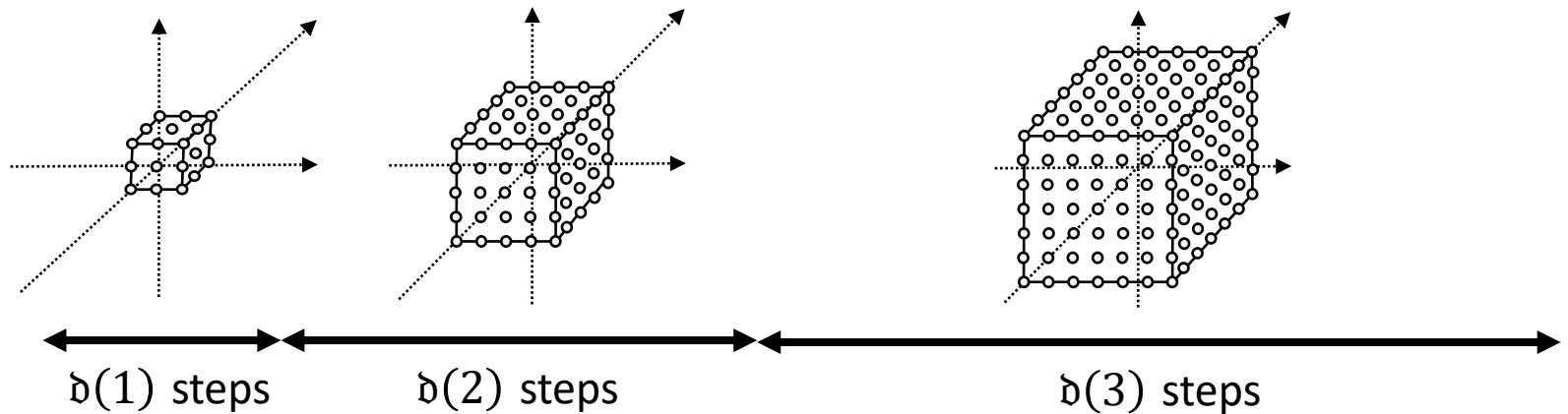
➤ $P(n)$ denotes the **transition matrix** on $G(n)$.

\mathfrak{d} represents
(inverse) growing speed

Example 1. Random walk in a growing region of \mathbb{Z}^d

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

- $\mathfrak{d}(n) = n^2$,
- $G(n)$ is a grid graph $\{-n, \dots, n\}^3$,
- $P(n)$ denotes the simple random walk w/ reflection bound,
i.e., move to a neighbor w.p. $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$ unless boundary,
for $n = 1, 2, \dots$



Thm. [Dembo et al. 2014, Kumamoto et al. 2024]

If $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{n^d} = \infty$ then recurrent, otherwise transient.

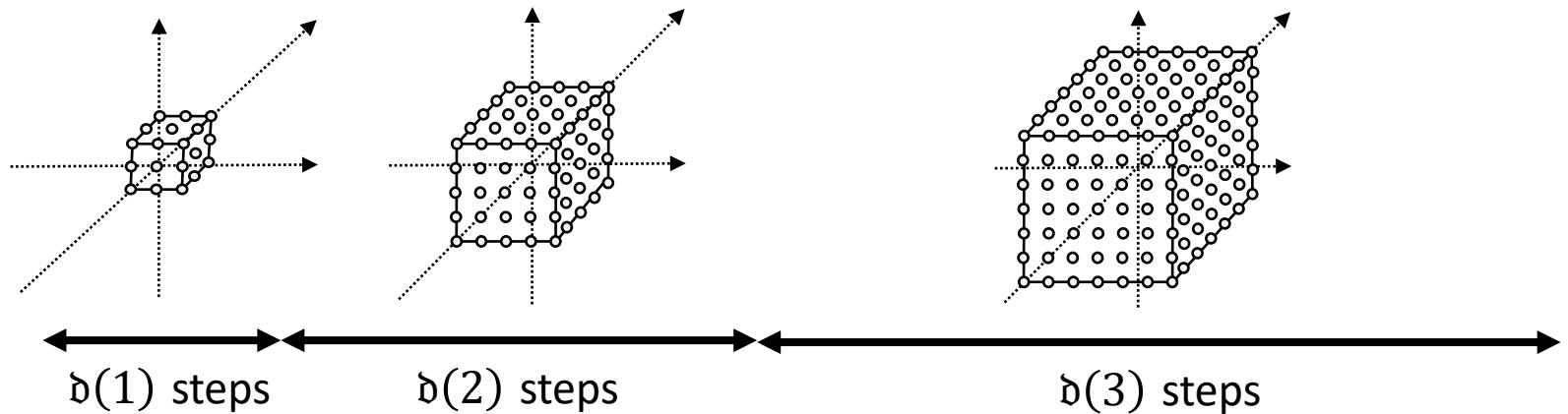
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Recurrent

$$\text{since } \sum_{n=1}^{\infty} \frac{n^2}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$



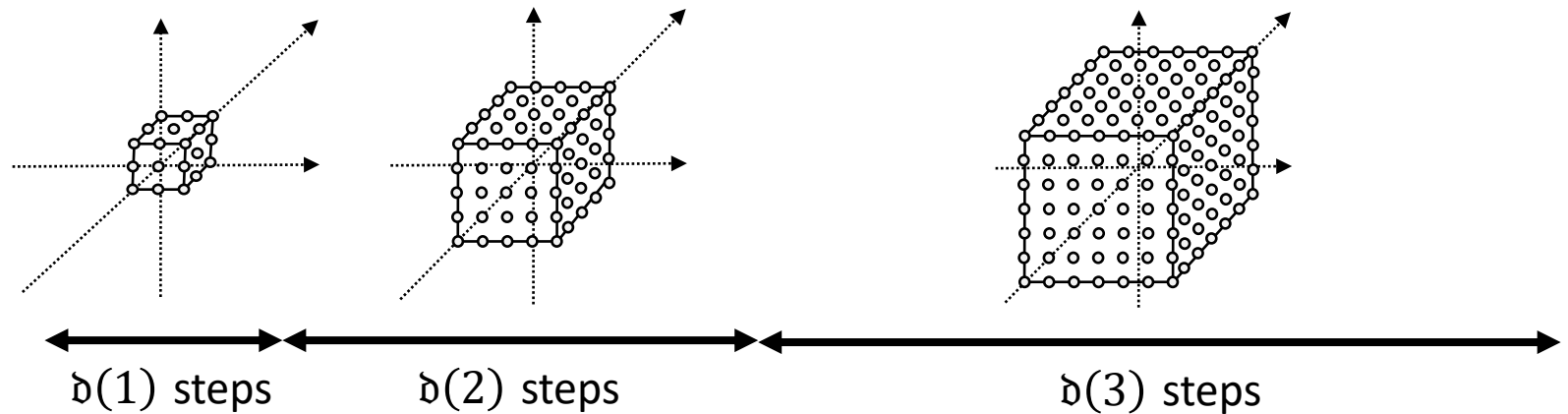
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Example 1. Random walk in a growing region of \mathbb{Z}^d

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

- $\mathfrak{d}(n) = n^{1.999}$,
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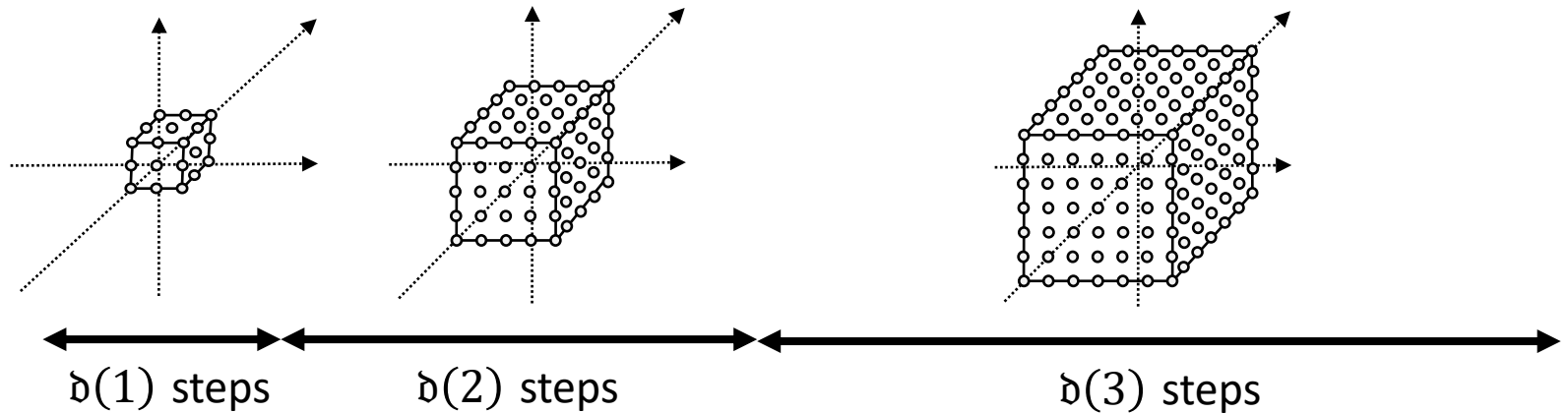
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Transient

$$\text{since } \sum_{n=1}^{\infty} \frac{n^{1.999}}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^{0.999}} < 1000.$$

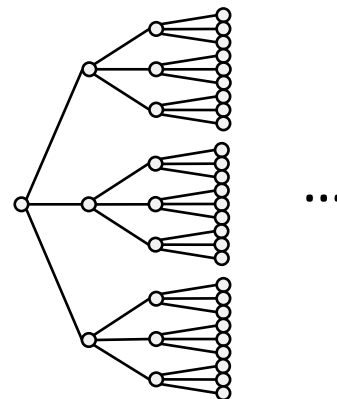


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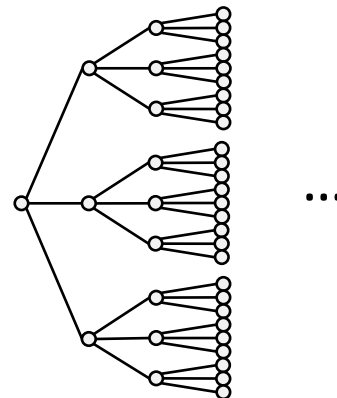
Example 2. RW on an infinite k -ary tree

- ✓ Random walk on an infinite k -ary tree is **transient** at r .



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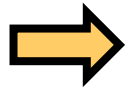


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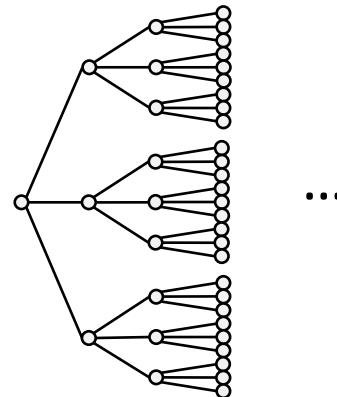
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A. It depends on the increasing speed.



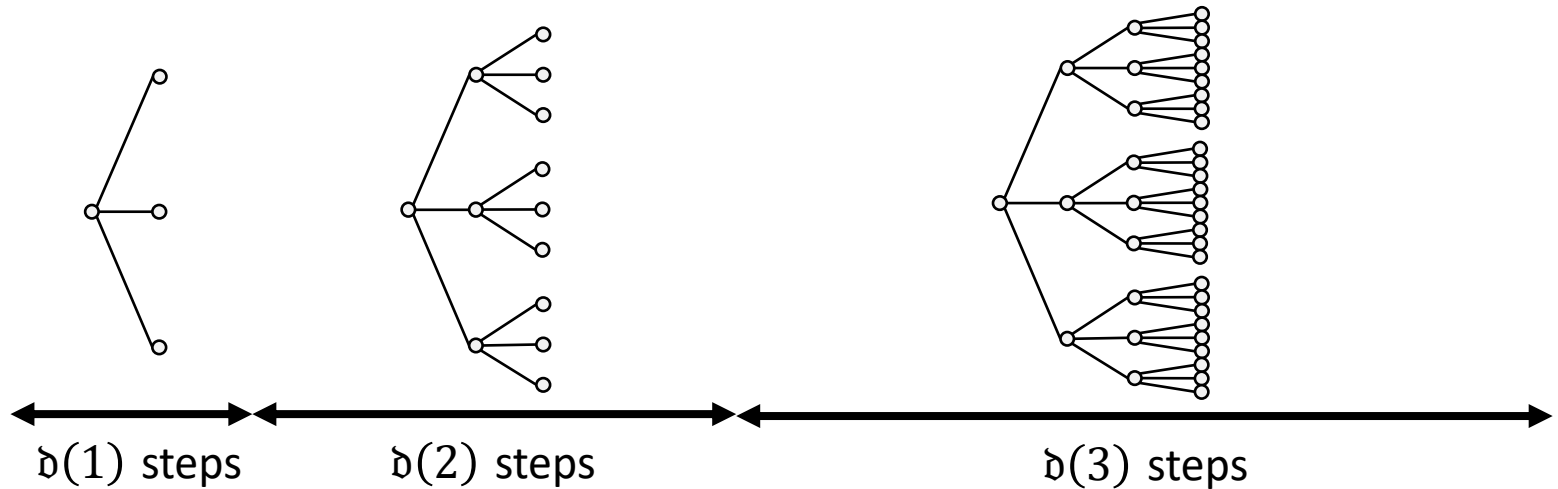
Find the phase transition point regarding the growing speed.



Example 2. Random walk on a growing k -ary tree

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

- $\mathfrak{d}(n) = 3^n$,
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i.e., move to a neighbor w.p. $1/4$ unless the root or a leaf,
for $n = 1, 2, \dots$



Thm. [Huang 2019, Kumamoto et al. 2024]

If $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{k^n} = \infty$ then recurrent, otherwise transient.

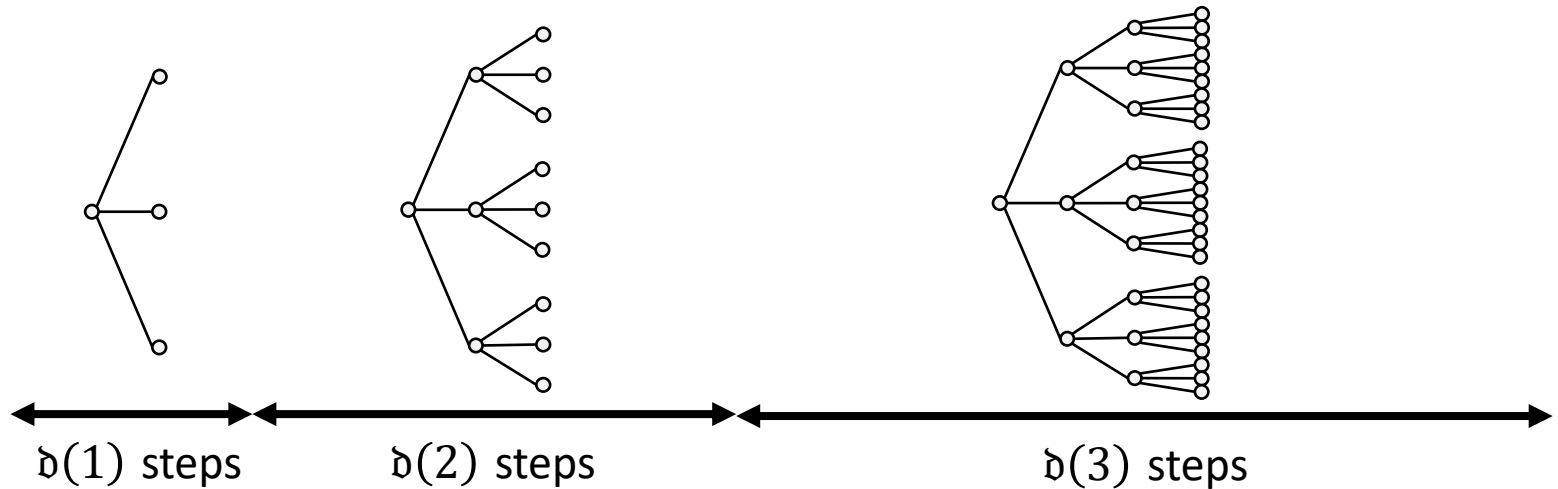
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Recurrent

since $\sum_{n=1}^{\infty} \frac{3^n}{3^n} = \sum_{n=1}^{\infty} 1 = \infty$.



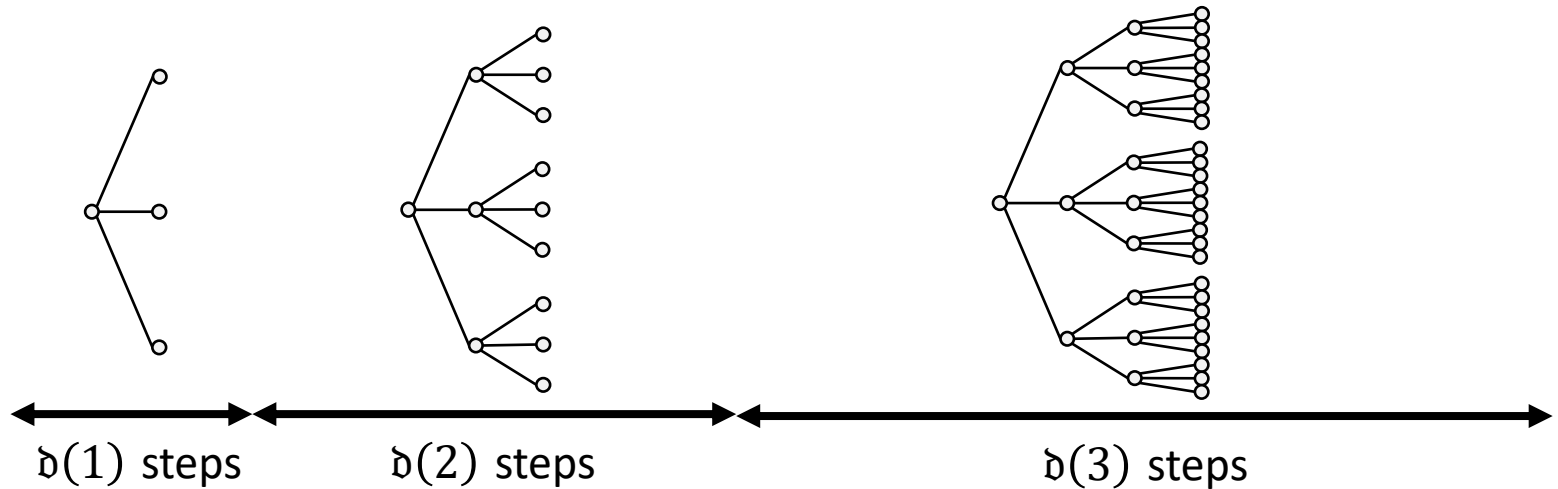
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Example 2. Random walk on a growing k -ary tree

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

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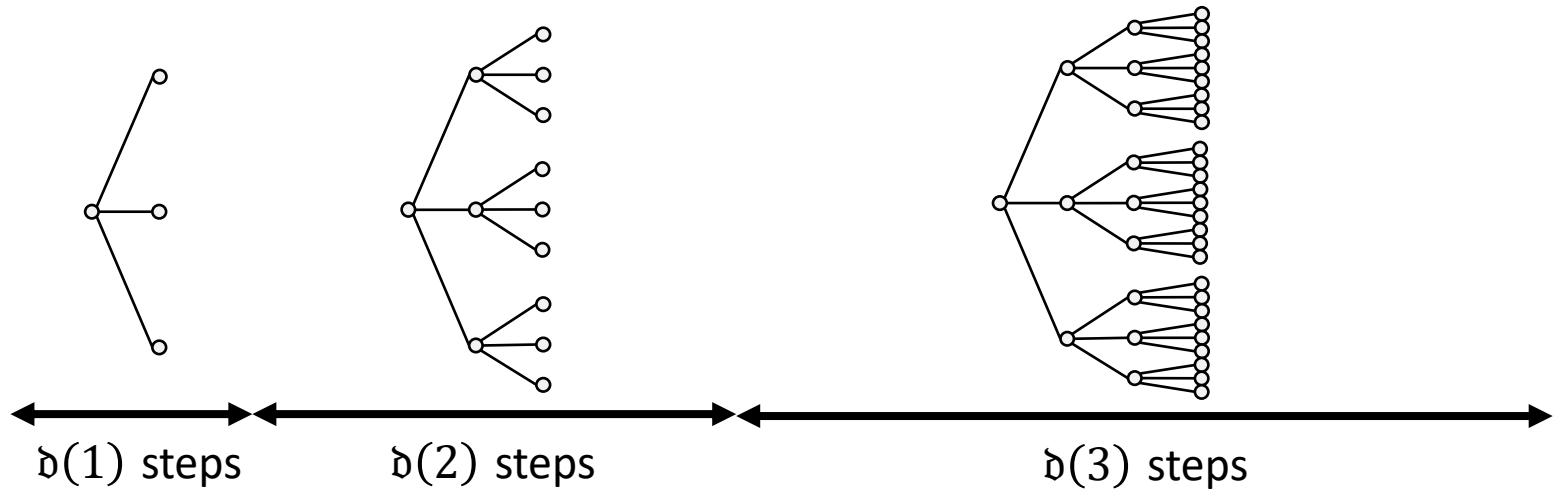
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Transient

since $\sum_{n=1}^{\infty} \frac{2.999999^n}{3^n} < 1,000,000$.



Thm. [Huang 2019, Kumamoto et al. 2024]

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2. Related work

About analysis of algorithms in dynamic environment

Related work (1/2): Random walks on dynamic graphs

□ Graph search by RW --- related to cover time

- Copper and Frieze (2003): Crawling on simple models of web graphs.
- Avin, Koucky and Lotker (2008): a bad example for hitting-time (cover time as well) w/ $\Omega(2^n)$ for the number of vertices n .
- Denysyuk and Rodrigues (2014): cover time under some fairness condition.
- Lamprou, Martin and Spirakis (2018): edge-uniform stochastically graphs.
- Sauerwald and Zanetti (2019): $O(n^2)$ cover time for d -regular graphs.
- K, Shimizu, Shiraga (2021): cover ratio of **RWoGG**

□ Mixing time

- Saloff-Coste and Zuniga (2009,2011): mixing time for time-inhomogeneous Markov chains w/ an invariant stationary distribution.
- Dembo, Huang and Zheng (2019) analyzed the conductance of growing subregion of \mathbb{Z}^d .
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□ Recurrence/transience

... Continued

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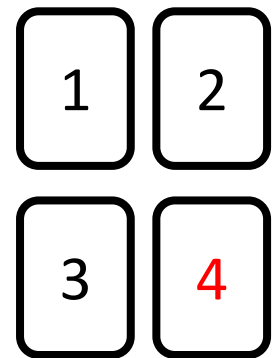
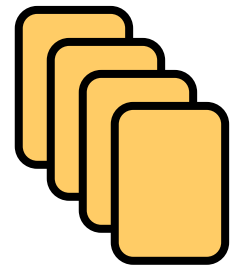
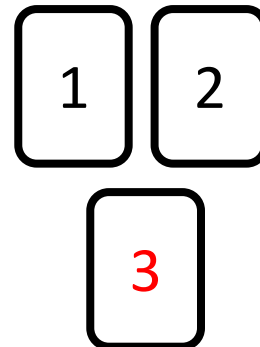
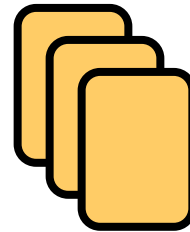
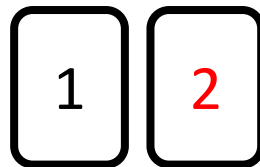
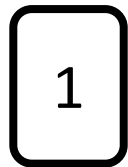
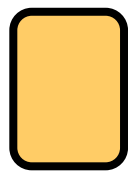
□ Recurrence/transience

... Continued

Collecting an increasing number of coupons [K, Shimizu, Shiraga '21]

Day	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7	Day 8	Day 9
# types	1	2	2	3	3	3	4	4	4
$\Pr[X_t = k]$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

1st period 2nd period 3rd period 4th period

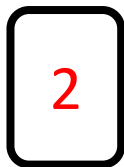
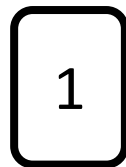
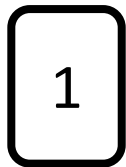
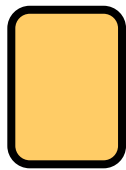


Q. How many types are collected in the end of n^{th} period?

Collecting an increasing number of coupons [K, Shimizu, Shiraga '21]

Day	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7	Day 8	Day 9
# types	1	2	2	3					
$\Pr[X_t = k]$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$					

1st period 2nd period 3rd



1. $O(\log n)$

2. $O(\sqrt{n})$

3. $\frac{n}{2}$

4. $.99n$

Q. How many types are collected in the end of n^{th} period?

Collecting an **increasing** number of coupons [K, Shimizu, Shiraga '21]

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1st period 2nd period 3rd period

1. $O(\log n)$
 2. $O(\sqrt{n})$
 3. $\frac{n}{2}$
 4. $.99n$
5. at least $n - 1$ in expectation

Q. How many types are collected in the end of n^{th} period?

Collecting an increasing number of coupon

Draw a coupon everyday
 $\mathfrak{d}(n)$: #days of the n^{th} period
 U_n : #items uncollected
in the end of n^{th} period

Prop.

$$\text{If } \mathfrak{d}(n) = n \text{ then } E[U_n] < \frac{1}{e-1}.$$

[K, Shimizu, Shiraga '21]

Proof.

$$\checkmark \quad \mathcal{E}_{i,n} := \begin{cases} 1 & \text{(item } i \text{ is uncollected in the end of the } n^{\text{th}} \text{ period)} \\ 0 & \text{(item } i \text{ is collected by the end of the } n^{\text{th}} \text{ period)} \end{cases}$$

for $i = 1, 2, \dots, n$.

$$\checkmark \quad U_n = \sum_{i=1}^n \mathcal{E}_{i,n}$$

✓ Prob. that item n is uncollected in the end of the n th period:

$$\Pr[\mathcal{E}_{n,n} = 1] = \left(1 - \frac{1}{n}\right)^n < e^{-1}$$

✓ Prob. that item i ($i \leq n$) is uncollected in the end of the n^{th} period:

$$\Pr[\mathcal{E}_{i,n} = 1] = \left(1 - \frac{1}{i}\right)^i \left(1 - \frac{1}{i+1}\right)^{i+1} \cdots \left(1 - \frac{1}{n}\right)^n < \left(\frac{1}{e}\right)^{n+1-i}$$

$$\checkmark \quad E[U_n] = \sum_{i=1}^n \Pr[\mathcal{E}_{i,n}] < \sum_{i=1}^n \left(\frac{1}{e}\right)^{n+1-i} = \frac{1}{e} + \frac{1}{e^2} + \cdots + \frac{1}{e^n} < \frac{\frac{1}{e}}{1 - \frac{1}{e}} = \frac{1}{e-1} < 0.582.$$

RWoGG (\mathfrak{d}, G, P)

Coupon collector is often regarded as a RW on the complete graph, and we can extend the arguments to RWoGG for general graphs.

Thm. (general upper bound)

If $\mathfrak{d}(i) \geq ct_{\text{hit}}(i)$ ($c \geq 1$) then $E[U] = O(1)$.

Particularly, if $\frac{\mathfrak{d}(i)}{t_{\text{hit}}(i)} \xrightarrow{i \rightarrow \infty} \infty$ then $E[U_n] \xrightarrow{n \rightarrow \infty} 0$.

Thm. (upper bound for lazy and reversible walk)

Suppose $P^{(i)}$ is lazy and reversible.

If $\frac{t_{\text{hit}}(i)}{t_{\text{mix}}(i)} \geq \frac{i^\gamma}{c}$ and $\mathfrak{d}(i) \geq \frac{3ct_{\text{hit}}(i)}{i^\gamma}$ ($c > 0$) then $E[U_n] \leq \frac{8n^\gamma}{c} + 32$.

S. Kijima, N. Shimizu, T. Shiraga, How many vertices does a random walk miss in a network with moderately increasing the number of vertices?, in Proc. SODA 2021, 106—122.

Related work (2/2): recurrence/transience of RW

- Much work about the recurrence/transience on growing graphs exist in the context of self-interacting random walks including reinforced random walks, excited random walks, etc. since 1990s, or before.
- Dembo, Huang and Sidoravicius (2014× 2): recurrent $\Leftrightarrow \sum_{t=0}^{\infty} \pi_t(0) = \infty$ for growing subregion of \mathbb{Z}^d (fixed d), by conductance argument.
 - See also Huang and Kumagai (2016), Dembo, Huang, Morris and Peres (2017), Dembo, Huang and Zheng (2019), etc. about heat kernel, evolving set arguments.
- Amir, Benjamini, Gurel-Gurevich and Kozma (2015): random walk on growing tree. (random walk in changing environment).
- Huang (2017): growing graph w/ *uniformly bounded degrees*.
- Kumamoto, K. and Shirai (2024): k -ary tree, $\{0,1\}^n$ w/ an increasing n under **RWoGG** model by coupling.
- This work (2024): $\{0,1, \dots, N\}^n$ (fixed N , increasing n) by pausing coupling.



3. Our previous work [SAND '24]

About the recurrence/transience of RWoGG,
for an introduction of the basic technique and its issue.

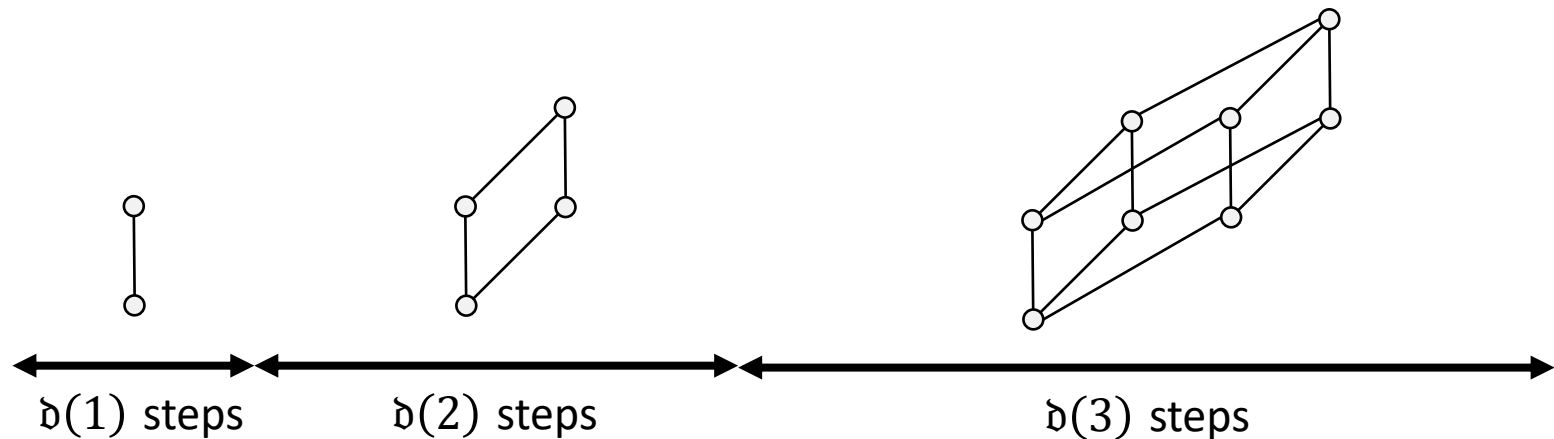
S. Kumamoto, S. Kijima and T. Shirai, An analysis of the recurrence/transience of random walks on growing trees and hypercubes, Proc. SAND 2024, 17:1-17:17

Example 3. Random walk on $\{0,1\}^n$ w/ an increasing n

[SAND '24]

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

- $\mathfrak{d}(n) = 2^n$,
- $G(n)$ is a $\{0,1\}^n$ skeleton,
- $P(n)$ denotes the simple random walk,
i.e., move to a neighbor w.p. $1/n$,
for $n = 1, 2, \dots$



Example 3. Random walk on $\{0,1\}^n$ w/ an increasing n

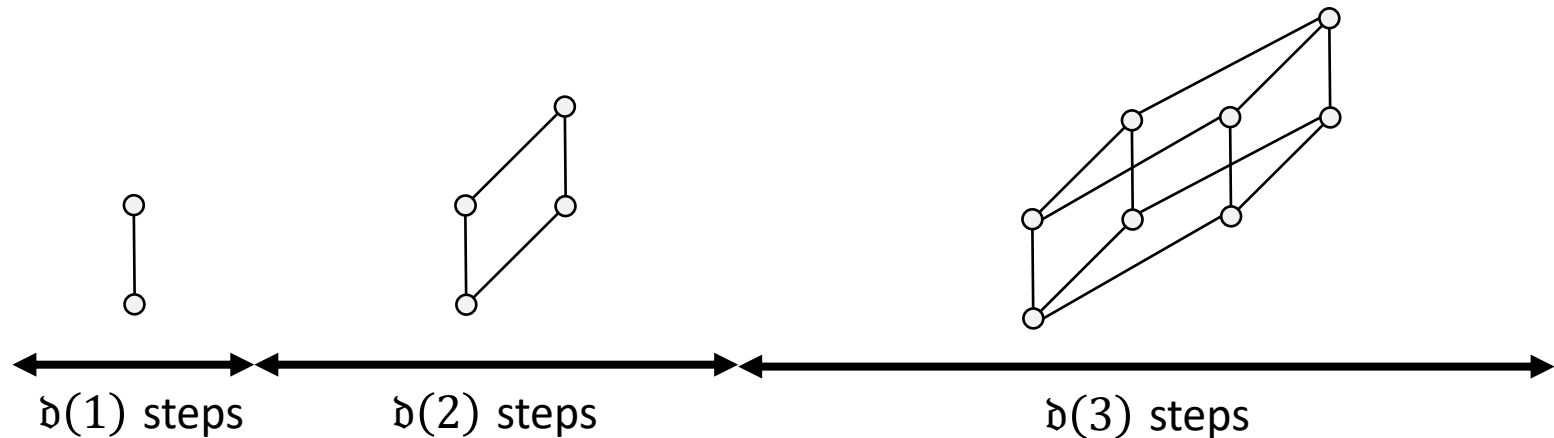
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for $n = 1, 2, \dots$

Recurrent

since $\sum_{n=1}^{\infty} \frac{2^n}{2^n} = \infty$.



Thm. [Kumamoto et al. 2024]

If $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{2^n} = \infty$ then recurrent, otherwise transient.

Example 3. Random walk on $\{0,1\}^n$ w/ an increasing n

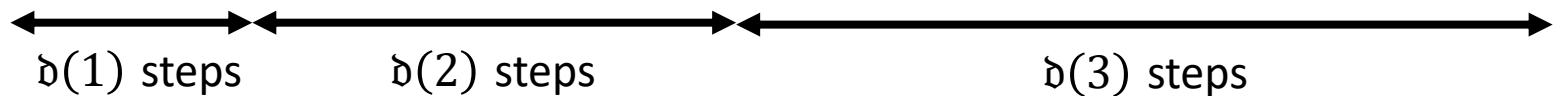
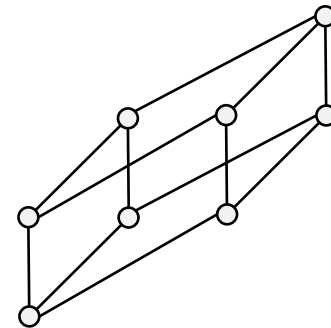
[SAND '24]

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- for $n = 1, 2, \dots$

Lem. [Kumamoto et al. 2024]

Random walk on $\{0,1\}^n$ is **LHaGG**.



Thm. [Kumamoto et al. 2024]

If $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{2^n} = \infty$ then recurrent, otherwise transient.

LHaGG [SAND '24]

Defs.

- $\mathcal{D}_1 = (f_1, G_1, P_1)$ is **less homesick** than $\mathcal{D}_2 = (f_2, G_2, P_2)$
if $R_1(t) \leq R_2(t)$ for any t where $R_1(t)$ and $R_2(t)$ respectively denote the return probabilities of \mathcal{D}_1 and \mathcal{D}_2 at time t .
- $\mathcal{D} = (f, G, P)$ is **less homesick as graph growing (LHaGG)**
if \mathcal{D} is less homesick than $\mathcal{D}' = (g, G, P)$ for any g satisfying that $\sum_{k=1}^n f(k) \leq \sum_{k=1}^n g(k)$ for any n ,
i.e., \mathcal{D} and \mathcal{D}' grows similarly, but \mathcal{D} grows *faster*.

The faster a graph grows,
the smaller the return probability.

Theorems by LHaGG

The faster a graph grows,
the smaller the return probability.

Under the condition of LHaGG, we can prove the following sufficient conditions of recurrence/transience, respectively.

Thm. [\[Kumamoto, K., Shirai '24\]](#)

Suppose $\mathcal{D} = (\mathfrak{d}, G, P)$ is LHaGG. If

$$\sum_{n=1}^{\infty} \mathfrak{d}(n)p(n) = \infty$$

then \mathcal{D} is **recurrent** at v , where $p(n) = \pi_n(v)$.

Thm. [\[Kumamoto, K., Shirai '24\]](#)

Suppose $\mathcal{D} = (\mathfrak{d}, G, P)$ is LHaGG. If

$$\sum_{n=1}^{\infty} \max\{\mathfrak{d}(n), \mathfrak{t}(n)\} p(n) < \infty$$

then \mathcal{D} is **transient** at v , where $\mathfrak{t}(n)$ represents the mixing time.

Example 3. Random walk on $\{0,1\}^n$ w/ an increasing n

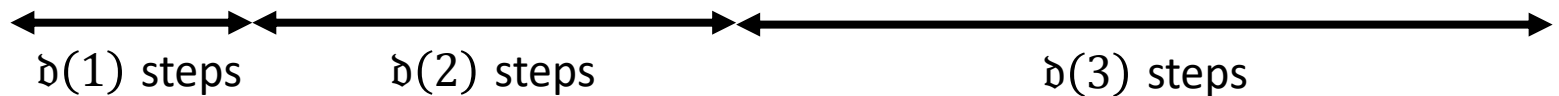
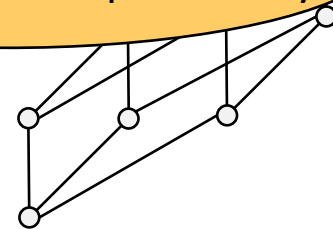
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i.e., move to a neighbor w.p. $1/n$,
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Lem. [Kumamoto et al. 2024]

Random walk on $\{0,1\}^n$ is **LHaGG**.

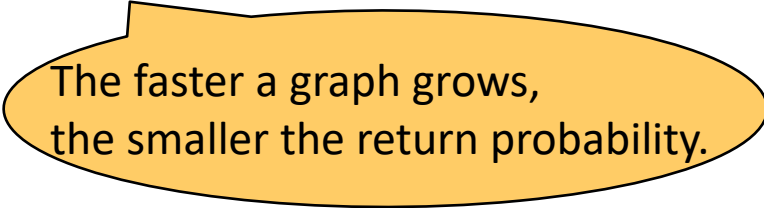
The faster a graph grows,
the smaller the return probability?



Thm. [Kumamoto et al. 2024]

If $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{2^n} = \infty$ then recurrent, otherwise transient.

FAQ: Any example for *not* LHaGG?



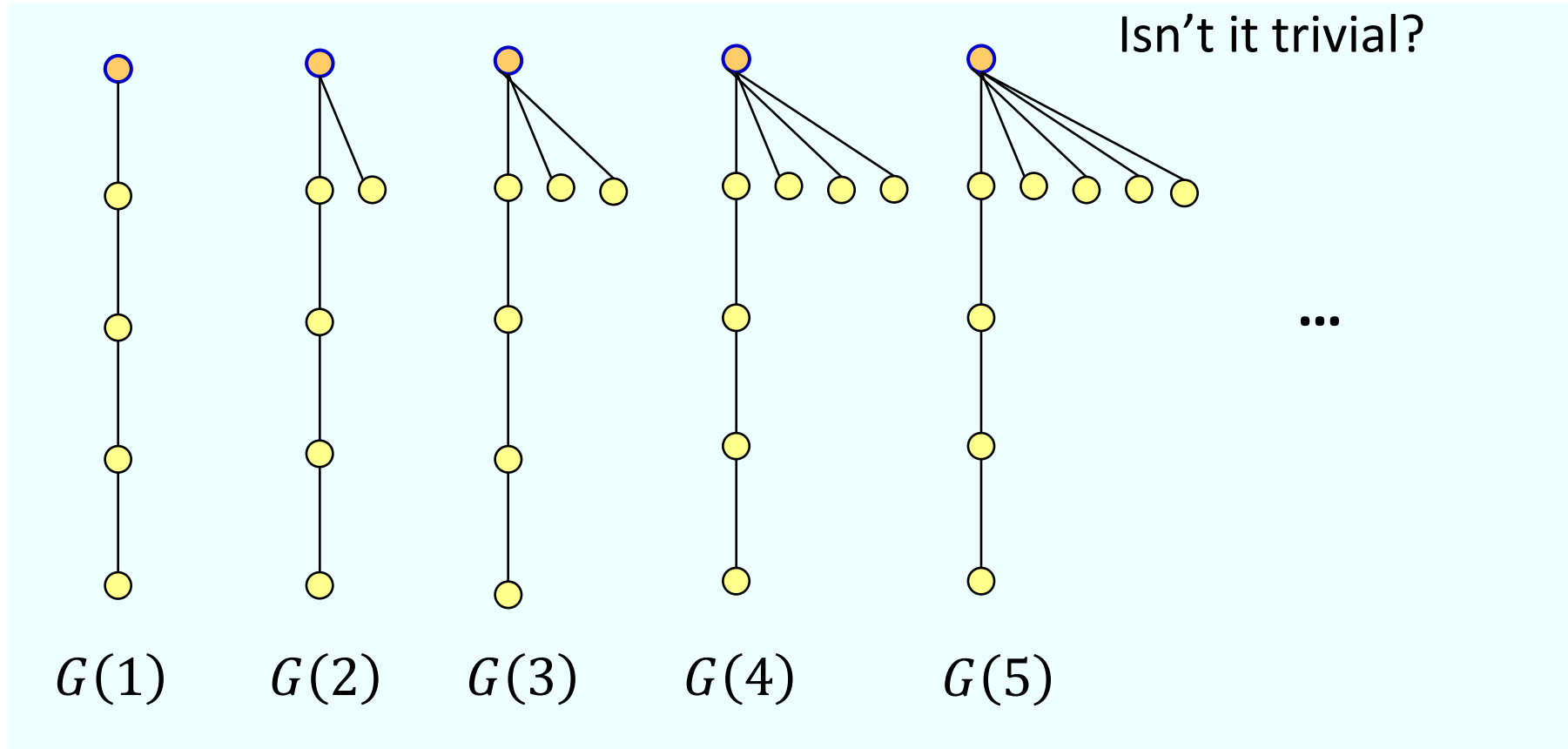
The faster a graph grows,
the smaller the return probability.

Isn't it trivial?

FAQ: Any example for *not* LHaGG?

A (lazy) simple random walk on

The faster a graph grows,
the smaller the return probability.



is *not* LHaGG.

Lazy RW on $\{0,1\}^n$ w/ increasing n is LHaGG

[SAND '24]

Proof.

The proof is a **monotone coupling**.

- Let $X_t \sim \mathcal{D}_f = (f, G, P)$ and $Y_t \sim \mathcal{D}_g = (g, G, P)$ where $\sum_{i=1}^n f(i) \geq \sum_{i=1}^n g(i)$,
 ➤ i.e., the graph of \mathcal{D}_g grows faster than that of \mathcal{D}_f .
- Let $|X_t|, |Y_t|$ denote the number of 1s in $X_t \in \{0,1\}^{n_t}, Y_t \in \{0,1\}^{m_t}$
 where notice that $n_t \leq m_t$. Then,

$$\Pr[|X_{t+1}| - 1 = |X_t|] = \frac{1}{2} \frac{|X_t|}{n_t}, \quad \Pr[|X_{t+1}| = |X_t|] = \frac{1}{2}, \quad \Pr[|X_{t+1}| + 1 = |X_t|] = \frac{1}{2} \left(1 - \frac{|X_t|}{n_t}\right)$$

$$\Pr[|Y_{t+1}| - 1 = |Y_t|] = \frac{1}{2} \frac{|Y_t|}{m_t}, \quad \Pr[|Y_{t+1}| = |Y_t|] = \frac{1}{2}, \quad \Pr[|Y_{t+1}| + 1 = |Y_t|] = \frac{1}{2} \left(1 - \frac{|Y_t|}{m_t}\right)$$

- if $|X_t| < |Y_t|$ then we can couple so that $|X_{t+1}| \leq |Y_{t+1}|$
 ➤ thanks to the self-loop w.p. $\frac{1}{2}$.
- If $|X_t| = |Y_t|$ then we can couple so that $|X_{t+1}| \leq |Y_{t+1}|$ since $n_t \leq m_t$.

Thus, $X_t = o$ if $Y_t = o$,

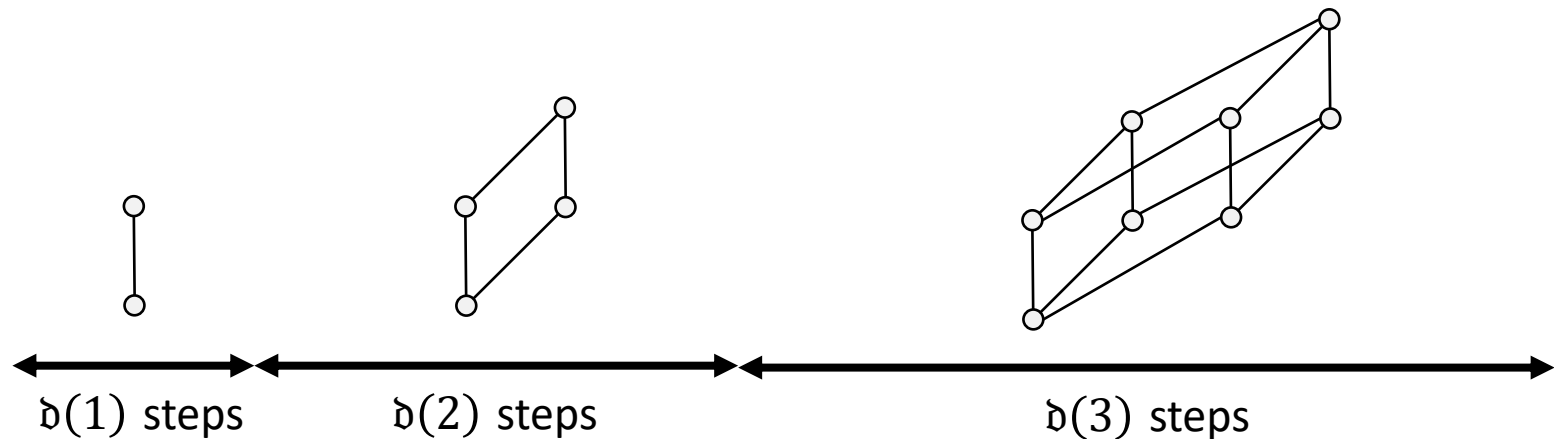
meaning that $\Pr[X_t = o] \geq \Pr[Y_t = o]$. □

It looks a very simple exercise if you are familiar with **coupling**, but $n_t \neq m_t$ makes some trouble, in general.

Example 3. Random walk on $\{0,1\}^n$ w/ an increasing n

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

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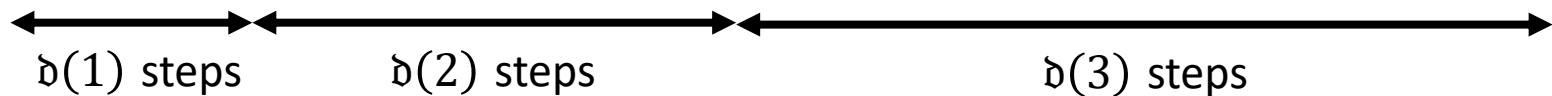
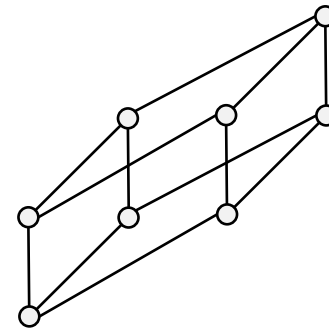
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since $\sum_{n=1}^{\infty} \frac{2^n}{2^n} = \infty$.

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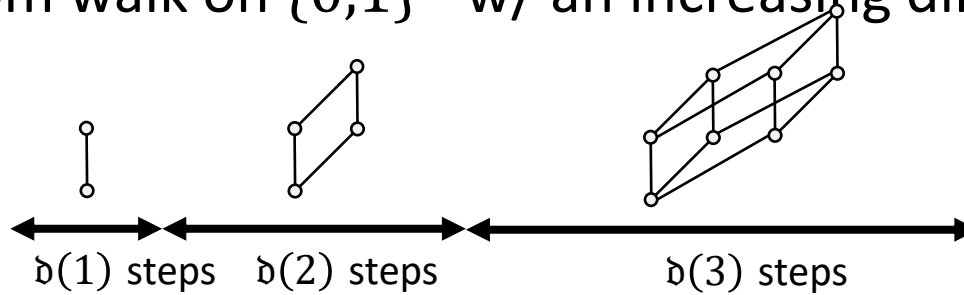


Thm. [Kumamoto et al. 2024]

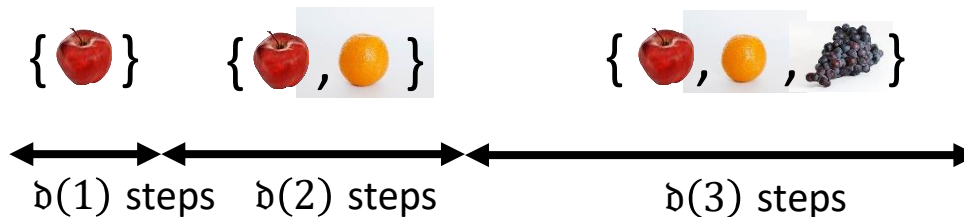
If $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{2^n} = \infty$ then recurrent, otherwise transient.

Three representations (or “applications”?) of $\{0,1\}^n$

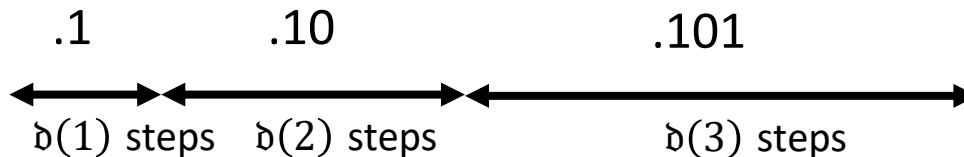
- Random walk on $\{0,1\}^n$ w/ an increasing dimensions



- Random pick/drop items w/ an increasing number of items

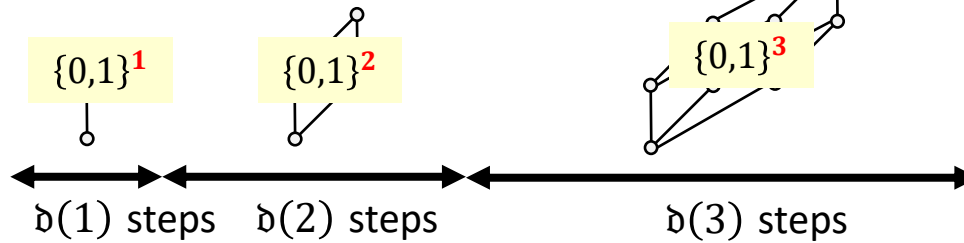


- Random bit flip of binary w/an increasing bit length

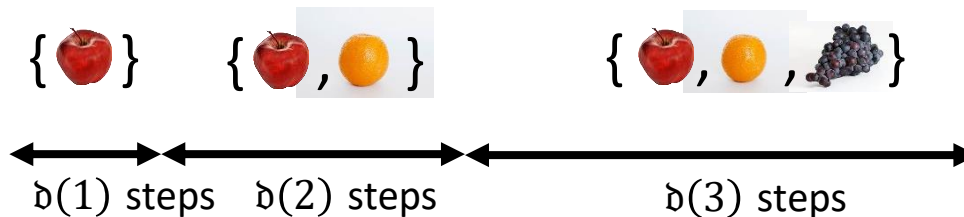


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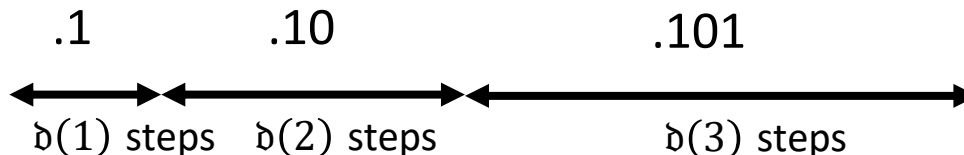
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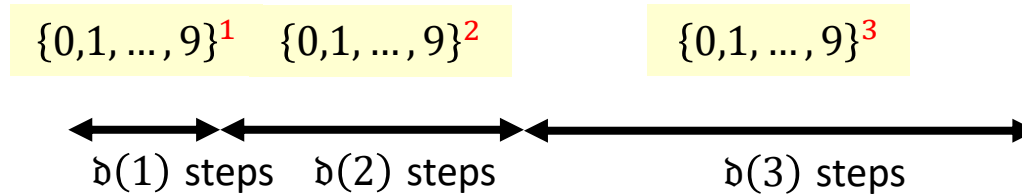


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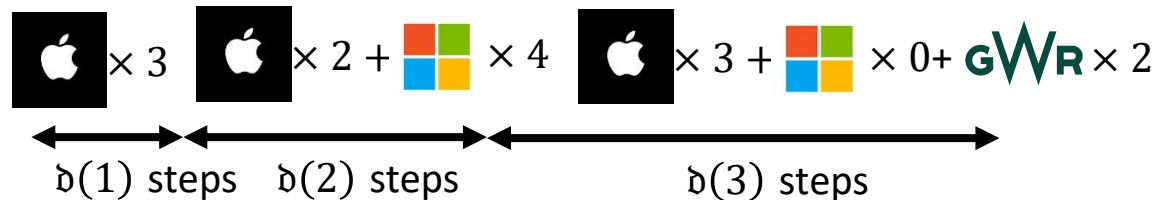


Extension from $\{0,1\}^n$ to $\{0,1,\dots,9\}^n$

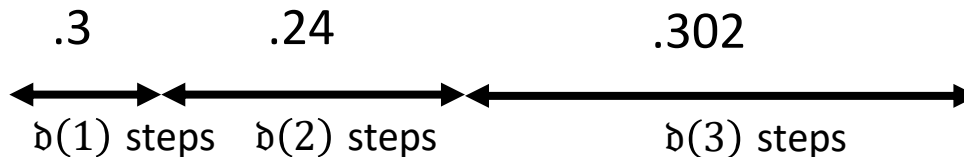
- Random walk on $\{0,1,\dots,9\}^n$ w/ an increasing n



- Random buy/sell stocks w/ an increasing #brands



- Random up/down digits w/ an increasing digit length

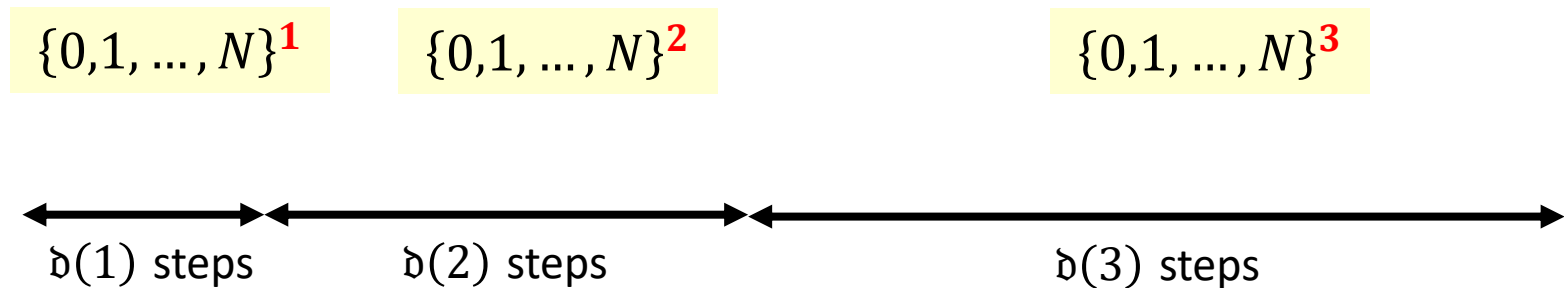


Target. Random walk on $\{0,1, \dots, N\}^n$ w/ an increasing n

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

w/ a fixed N .

- $\mathfrak{d}(n) = N^n$,
- $G(n)$ is a $\{0,1, \dots, N\}^n$ skeleton,
- $P(n)$ denotes the lazy simple random walk w/ reflection bound,
i.e., move to a neighbor w.p. $1/4n$, unless boundary
for $n = 1, 2, \dots$



Q.

Is random walk on $\{0,1, \dots, N\}^n$ LHaGG?

A. We can't prove it.



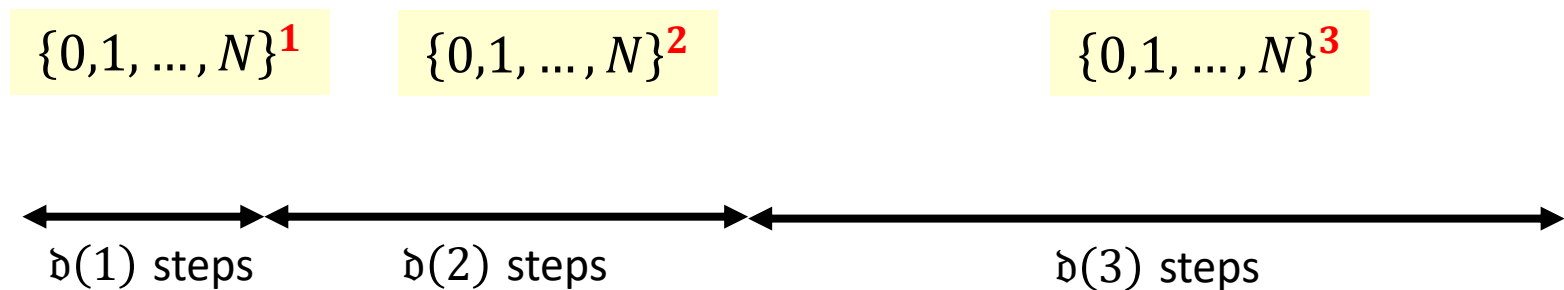
4. Main Result

Target. Random walk on $\{0,1, \dots, N\}^n$ w/ an increasing n

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

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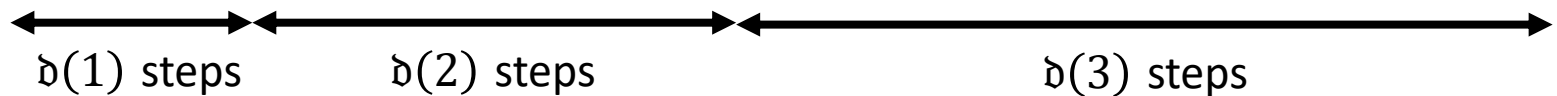
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Lem. 7.

Random walk on $\{0,1, \dots, N\}^n$ is **weakly** LHaGG.

$\{N\}^3$



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Random walk on $\{0,1, \dots, N\}^n$ is **weakly** LHaGG.

$\{N\}^3$

Thm. 6. If $\mathcal{D} = (\mathfrak{d}, G, P)$ satisfies

$$\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{(2N)^n} = \infty$$

then o is recurrent, otherwise o is transient.

) steps

Recall: LHaGG [SAND '24]

Defs.

- $\mathcal{D}_1 = (f_1, G_1, P_1)$ is **less homesick** than $\mathcal{D}_2 = (f_2, G_2, P_2)$
if $R_1(t) \leq R_2(t)$ for any t where $R_1(t)$ and $R_2(t)$ respectively denote the return probabilities of \mathcal{D}_1 and \mathcal{D}_2 at time t .
- $\mathcal{D} = (f, G, P)$ is **less homesick as graph growing (LHaGG)**
if \mathcal{D} is less homesick than $\mathcal{D}' = (g, G, P)$ for any g satisfying that $\sum_{k=1}^n f(k) \leq \sum_{k=1}^n g(k)$ for any n ,
i.e., \mathcal{D} and \mathcal{D}' grows similarly, but \mathcal{D} grows *faster*.

The faster a graph grows,
the smaller the **return probability**.

Recall: LHaGG

We replace the condition about the return prob. with

Defs.

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 $\sum_{k=1}^n f(k) \leq \sum_{k=1}^n g(k)$ for any n ,
 i.e., \mathcal{D} and \mathcal{D}' grows similarly, but \mathcal{D} grows *faster*.

The faster a graph grows,
 the smaller the **return probability**.

wLHaGG

We replace the condition about the return prob. with a condition of the **sum of** return prob.

Defs.

- $\mathcal{D}_1 = (f_1, G_1, P_1)$ is **weakly less homesick** than $\mathcal{D}_2 = (f_2, G_2, P_2)$ if $\sum_{t=1}^T R_1(t) \leq \sum_{t=1}^T R_2(t)$ for any T where $R_1(t)$ and $R_2(t)$ respectively denote the return probabilities of \mathcal{D}_1 and \mathcal{D}_2 at time t .
- $\mathcal{D} = (f, G, P)$ is **weakly less homesick as graph growing (wLHaGG)** if \mathcal{D} is weakly less homesick than $\mathcal{D}' = (g, G, P)$ for any g satisfying that $\sum_{k=1}^n f(k) \leq \sum_{k=1}^n g(k)$ for any n , i.e., \mathcal{D} and \mathcal{D}' grows similarly, but \mathcal{D} grows *faster*.

The faster a graph grows,
the smaller the **expected number of returns**.

= sum of return prob.

General theorems

Condition 0. (**ergodic**). In $\mathcal{D} = (\mathfrak{d}, G, P)$, every transition matrix $P(n)$ is ergodic.

Condition 1. (**mixing time**). $\mathcal{D} = (\mathfrak{d}, G, P)$ satisfies

$$\sum_{k=1}^{\infty} \tau^*(k) p(k) < \infty$$

where $p(k) = \pi_k(o)$ and $\tau^*(k) = t_{\text{mix}}^k \left(\frac{p(k)}{4} \right)$.

Mixing time is not very big.

E.g., $O\left(\frac{1}{\pi_k(o)} \frac{1}{k \log k}\right)$

Thm. 2. (Recurrence).

Suppose (\mathfrak{d}, G, P) satisfies Conditions 0 and 1.

If $\sum_{k=1}^{\infty} \mathfrak{d}(k) p(k) = \infty$ then the initial vertex v is **recurrent**.

Thm. 4. (Transience).

Suppose (\mathfrak{d}, G, P) satisfies Conditions 0 and 1, and it is **wLHaGG**.

If $\sum_{k=2}^{\infty} \mathfrak{d}(k) p(k-1) < \infty$ then the initial vertex v is **transient**.

Suppose (\mathfrak{d}, G, P) satisfies Conditions 0 and 1.

If $\sum_{k=1}^{\infty} \mathfrak{d}(k)p(k) = \infty$ then the initial vertex v is **recurrent**.

Recurrence

Proof. Let X_t follow (\mathfrak{d}, G, P) , and

let $R(t) = \Pr[X_t = o]$. We claim

$$\text{Lem. 3. } \sum_{t=1}^{T_n} R(t) \geq \frac{1}{2} \sum_{k=1}^n (\mathfrak{d}(k) - \tau^*(k))p(k)$$

Proof of Lem. 3.

- Notice that X_t follows P_n for $t \in [T_{n-1}, T_{n-1} + \mathfrak{d}(n))$.
- If $\mathfrak{d}(n) > t_{\text{mix}}(\epsilon)$ then $R(t) \geq \pi_n(o) - \epsilon$ for $t \geq T_{n-1} + t_{\text{mix}}(\epsilon)$
where π_n is the stationary distribution of P_n .
- Thus, $R(t) \geq \pi_n(o) - \frac{1}{2}p(n) = \frac{1}{2}p(n)$
since $\tau^*(k) = t_{\text{mix}}\left(\frac{1}{2}p(n)\right)$ and $p(n) = \pi_n(o)$.
- $\sum_{t=1}^{T_n} R(t) = \sum_{k=1}^n \sum_{s=1}^{\mathfrak{d}(k)} R(T_{n-1} + s) \geq \sum_{k=1}^n \sum_{s=\tau^*(n)}^{\mathfrak{d}(k)} R(T_{n-1} + s) \geq$
 $\sum_{k=1}^n \sum_{s=\tau^*(n)}^{\mathfrak{d}(k)} \frac{1}{2}p(n) = \frac{1}{2} \sum_{k=1}^n (\mathfrak{d}(k) - \tau^*(k))p(k)$

Once we obtain Lem. 3, Thm. 2 is easy: $\sum_{t=1}^{\infty} R(t) = \infty$ holds

if $\sum_{k=1}^{\infty} \mathfrak{d}(k)p(k) = \infty$ and $\sum_{k=1}^{\infty} \tau^*(k)p(k) < \infty$.

← Mixing time condition

Thm. 4. (Transience).

Transience

Suppose (\mathfrak{d}, G, P) satisfies Conditions 0 and 1, and it is **wLHaGG**.

If $\sum_{k=2}^{\infty} \mathfrak{d}(k)p(k-1) < \infty$ then the initial vertex v is **transient**.

Proof. Let $f(k) = \max\{\mathfrak{d}, \tau^*(k)\}$.

By wLHaGG, $\sum_{t=1}^{T_n} R_{\mathfrak{d}}(t) \leq \sum_{t=1}^{T_n} R_g(t)$.

Lem. 5. $\sum_{t=1}^{T_n} R_g(t) \leq g(1) + \frac{3}{2} \sum_{k=2}^n g(k)p(\textcolor{red}{k} - 1)$

Proof of Lem. 5.

Let $f(k) = \begin{cases} g(k) & k \leq n-1 \\ \infty & k = n. \end{cases}$ Then, $\sum_{k=1}^m g(k) \leq \sum_{k=1}^m g(k)$ for any m .

Let $X_t \sim \mathcal{D}_g = (g, G, P)$ and $Y_t \sim \mathcal{D}_f = (f, G, P)$.

- Notice that Y_t follows P_{n-1} for $t \geq T_{n-2}$. ←
- By wLHaGG, $\sum_{t=1}^T \Pr[X_t = o] \leq \sum_{t=1}^T \Pr[Y_t = o]$ for any T .
- $R_f(t) \leq \pi_n(o) + \frac{1}{2}p(n-1) = \frac{3}{2}p(n-1)$ for $t \geq T_{n-1}$
- $\sum_{t=1}^{T_n} R_g(t) = \sum_{k=1}^n \sum_{s=1}^{g(k)} R_g(T_{k-1} + s) \leq g(1) + \sum_{k=2}^n \sum_{s=1}^{g(k)} R_{\textcolor{red}{f}}(T_{k-1} + s) \leq g(1) + \sum_{k=2}^n \sum_{s=1}^{g(k)} \frac{3}{2}p(k-1) = g(1) + \frac{3}{2} \sum_{k=2}^n g(k)p(k-1)$

Particularly, remark $X_t \sim P_{\textcolor{red}{n}}$ but $Y_t \sim P_{\textcolor{red}{n-1}}$ for $t \in [\textcolor{red}{T}_{n-1}, \textcolor{red}{T}_n)$

Once we obtain Lem. 5, Thm. 4 is clear.

Target. Random walk on $\{0,1, \dots, N\}^n$ w/ an increasing n

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

w/ a fixed N .

- $\mathfrak{d}(n) = N^n$,
- $G(n)$ is a $\{0,1, \dots, N\}^n$ skeleton,
- $P(n)$ denotes the lazy simple random walk w/ reflection bound,
i.e., move to a neighbor w.p. $1/4n$, unless boundary
for $n = 1, 2, \dots$

Lem. 7.

Random walk on $\{0,1, \dots, N\}^n$ is **weakly** LHaGG.

$V\}^3$

Thm. 6. If $\mathcal{D} = (\mathfrak{d}, G, P)$ satisfies

$$\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{(2N)^n} = \infty$$

then o is recurrent, otherwise o is transient.

) steps

Target. Random walk on $\{0,1, \dots, N\}^n$ w/ an increasing n

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

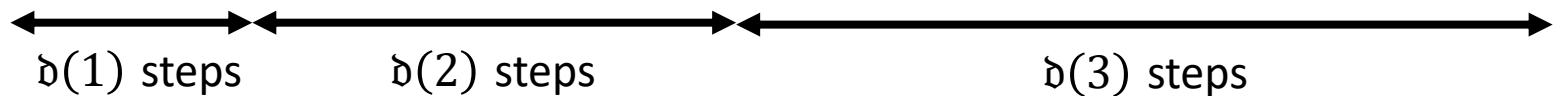
w/ a fixed N .

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$N\}^3$



It looks a very simple exercise if you are familiar with **coupling**, but $n_t \neq m_t$ makes some trouble, in general.

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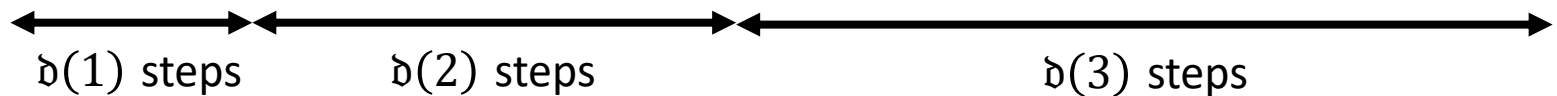
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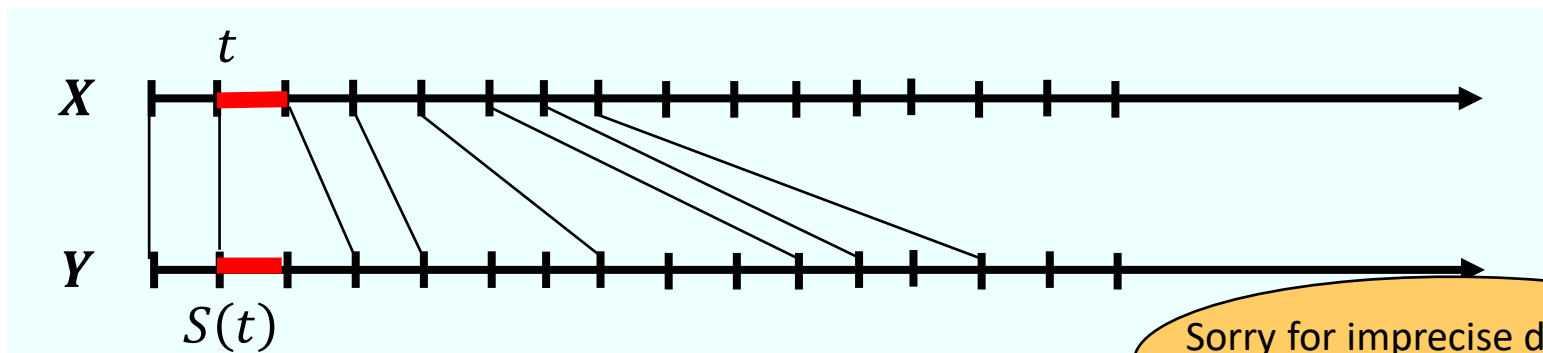


We develop “pausing coupling”

It looks a very simple exercise if you are familiar with **coupling**, but $n_t \neq m_t$ makes some trouble, in general.

Figure of pausing coupling

- Let $\mathbf{X} = X_0, X_1, X_2, \dots \sim \mathcal{D}_f$ and $\mathbf{Y} = Y_0, Y_1, Y_2, \dots \sim \mathcal{D}_g$
where let \mathcal{D}_g grow faster than \mathcal{D}_f .
- We couple \mathbf{X} and \mathbf{Y} , instead of X_t and Y_t .



Sorry for imprecise description
to avoid bothering notation.

We define time correspondence $t \mapsto S(t)$ depending on \mathbf{Y} so that

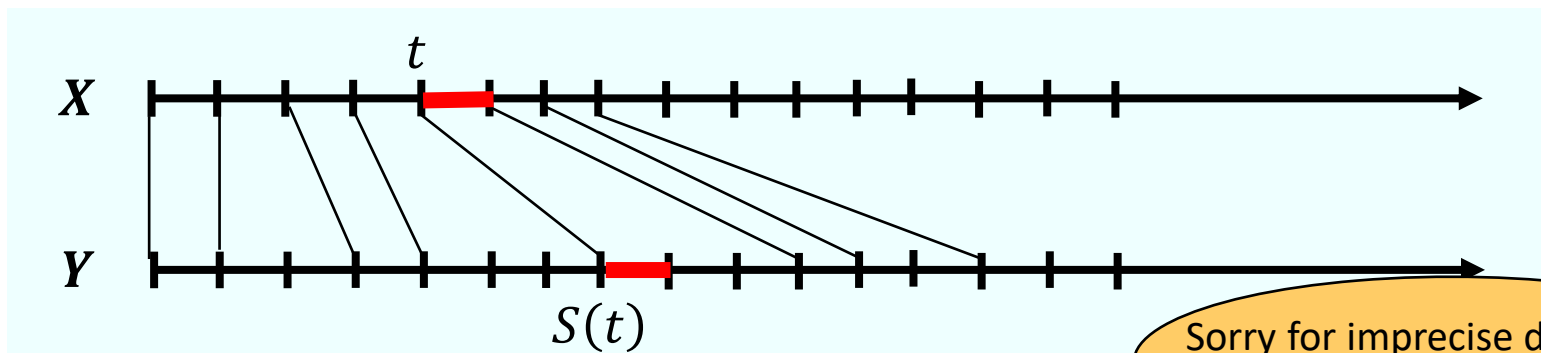
1. if Y_t does self-loop then so does $X_{S^{-1}(t)}$,
2. if Y_t updates Y_t^i for $i \leq \dim(X_{S^{-1}(t)})$ then \mathbf{X} updates $X_{S^{-1}(t)}^i$,
3. if Y_t updates Y_t^i for $i > \dim(X_{S^{-1}(t)})$ then \mathbf{X} **pauses**.



We need to check “measure conservation” of the coupling.

Figure of pausing coupling

- Let $\mathbf{X} = X_0, X_1, X_2, \dots \sim \mathcal{D}_f$ and $\mathbf{Y} = Y_0, Y_1, Y_2, \dots \sim \mathcal{D}_g$
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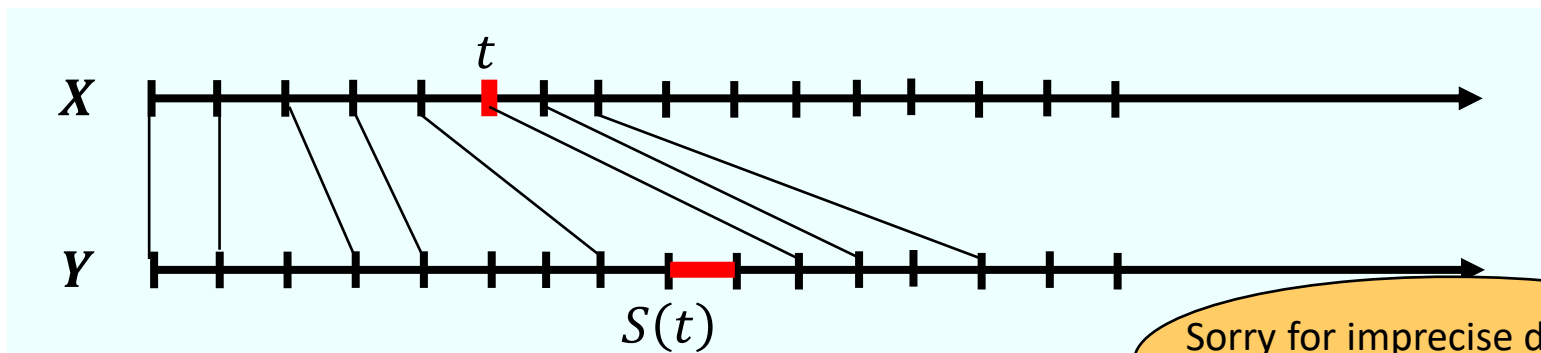
1. if Y_t does self-loop then so does $X_{S^{-1}(t)}$,
2. if Y_t updates Y_t^i for $i \leq \dim(X_{S^{-1}(t)})$ then X updates $X_{S^{-1}(t)}^i$,
3. if Y_t updates Y_t^i for $i > \dim(X_{S^{-1}(t)})$ then X **pauses**.



We need to check “measure conservation” of the coupling.

Figure of pausing coupling

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We need to check “measure conservation” of the coupling.

Outline of the proof

Random walk on $\{0, 1, \dots, N\}^n$ w/ increasing n is wLHaGG.

Let $\eta: Y \mapsto X = \eta(Y)$ denote the coupling described in the previous slide.

We prove two things:

□ The coupling η preserves the measure, i.e.,

$$\Pr[Y = \mathbf{y}] = \Pr[X = \eta(\mathbf{y})]$$

□ The coupling η preserves $|X_t| \leq |Y_s|$ (meaning “ $|\eta(y_s)| \leq |y_s|$ ”)

for any s satisfying $S(t) \leq s < S(t + 1)$.

➤ This implies $\#\{t \leq T \mid X_t = o\} \geq \#\{t \leq T \mid Y_t = o\}$ for any T .

Def. $S(t)$

Proof.

Suppose $\mathbf{Y} = Y_0, Y_1, Y_2, Y_3, \dots$ is represented by

$$\boldsymbol{\theta}_Y = (\lambda_1, j_1, \rho_1), (\lambda_2, j_2, \rho_2), (\lambda_3, j_3, \rho_3), \dots$$

We define $S: \mathbb{Z} \rightarrow \mathbb{Z}$ according to $\boldsymbol{\theta}$.

Let $S(1) = \min\{\min\{t > 0 \mid \lambda_t = 0\}, \min\{t > 0 \mid j_t \in n_0\}\}$.

Recursively, let

$$S(k) = \min\{\min\{t > S(k-1) \mid \lambda_t = 0\}, \min\{t > S(k-1) \mid j_t \in n_{k-1}\}\}$$

where let $\min\{\emptyset\} = \infty$.

If $S(k) = \infty$ then let $S(k+1) = \infty$.

For convenience, let $S^{-1}(t) = k$ for $t = S(k) < \infty$ ($k = 1, 2, \dots$).

Then, we define $\mathbf{X} = X_0, X_1, X_2, \dots$ by

$$\begin{aligned} \boldsymbol{\theta}_X &= \left((\lambda_{S^{-1}(k)}, j_{S^{-1}(k)}, \rho_{S^{-1}(k)}) \right)_{k=1,2,\dots} \\ &= (\lambda_{S^{-1}(1)}, j_{S^{-1}(1)}, \rho_{S^{-1}(1)}), (\lambda_{S^{-1}(2)}, j_{S^{-1}(2)}, \rho_{S^{-1}(2)}), \dots \end{aligned}$$

as far as $S(k) < \infty$.

If $S(k) = \infty$ then generate $(\lambda'_k, j'_k, \rho'_k)$ and transit to X_{k+1} according to it.

Def. $S(t)$

Proof.

Suppose $\mathbf{Y} = Y_0, Y_1, Y_2, Y_3, \dots$ is represented by

$$\boldsymbol{\theta}_Y = (\lambda_1, j_1, \rho_1), (\lambda_2, j_2, \rho_2), (\lambda_3, j_3, \rho_3), \dots$$

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Then, we define $\mathbf{X} = X_0, Y_1, Y_2, \dots$



as far as $S(k)$

If $S(k) = \infty$ then generate $(\lambda'_k, j'_k, \rho'_k)$ and transit to X_{k+1} according to it.



5. Concluding remarks

Final slide

Result

- Recurrence/transience of **wLHaGG** RWoGG.
- Random walk on $\{0,1, \dots, N\}^n$ w/ increasing n is wLHaGG.
 - Proof by **pausing coupling**.

Future work

- Simplify the proof
 - Extension to other RWoGGs
 - E.g., GW tree, PA graph, and more general graphs,
 - Edge dynamics, e.g., growing + edge Markovian.
- Analysis of RWoGG beyond recurrence/transience
 - **Hitting time, meeting time, gathering time, etc.**
 - **Find a new limit, undefined for an infinite graph.**



The end

Thank you for the attention.

Lazy simple random walk on $\{0, 1, \dots, N\}^n$ w/ increasing n

Current state $X_t = (X_t^1, \dots, X_t^{n_t}) \in \{0, 1, \dots, N\}^{n_t}$.

1. W.p. $\frac{1}{2}$, set $X_{t+1} = X_t$.
2. Else, choose $i \in \{1, \dots, n_t\}$ u.a.r.
3. If X_t^i is not 0 nor N then update as $X_{t+1}^i = X_t^i \pm 1$ w.p. $\frac{1}{2}$ resp.
4. Else if $X_t^i = 0$ then update as $X_{t+1}^i = X_t^i + 1$.
5. Else if $X_t^i = N$ then update as $X_{t+1}^i = X_t^i - 1$.

Lazy simple random walk on $\{0, 1, \dots, N\}^n$ w/ increasing n

Current state $X_t = (X_t^1, \dots, X_t^{n_t}) \in \{0, 1, \dots, N\}^{n_t}$.

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4. Else if $X_t^i = 0$ then update as $X_{t+1}^i = X_t^i + 1$.
5. Else if $X_t^i = N$ then update as $X_{t+1}^i = X_t^i - 1$.

If $\lambda = 0$ self-loop

Choose i u.a.r.

If $\rho = 0$ then -1

A transition $X_t \mapsto X_{t+1}$ is represented

by uniform r.v.s $(\lambda, i, \rho) \in \{0, 1\} \times \{1, \dots, n_t\} \times \{0, 1\}$.