The Recurrence/Transience of Random Walks on a Bounded Grid in an Increasing Dimension

Shuma Kumamoto (Kyushu Univ.),
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Tomoyuki Shirai (Kyushu Univ.)
Plan of talk

1. Introduction
   - $\mathbb{Z}^3$
   - RWoGG
   - Tree
2. Related work
   - Exploration
3. Previous work
   - LHaGG
   - $\{0,1\}^n$ proof
   - Extension to $\{0,1,\ldots,N\}^n$
4. Main result
   - Weakly LHaGG
   - Recurrence
   - Transience
   - pausing coupling
5. Concluding remarks
Plan of talk $\geq 49$ min.

1. Introduction ($\geq 9$ min.)
   - $\mathbb{Z}^3$
   - RWoGG
   - Tree

2. Related work ($\geq 6$ min.)
   - Exploration

3. Previous work ($\geq 8$ min.)
   - LHaGG
   - $\{0,1\}^n$ proof
   - Extension to $\{0,1,\ldots,N\}^n$

4. Main result ($\geq 25$ min.)
   - Weakly LHaGG
   - Recurrence
   - Transience
   - pausing coupling

5. Concluding remarks (1 min.)
Plan of talk ≥49 min. 25 min.

1. Introduction (≥9 min. 6 min.)
   ➢ ℤ³
   ➢ RWoGG
   ➢ Tree
2. Related work (≥6 min. 3 min.)
   ➢ Exploration
3. Previous work (≥8 min.)
   ➢ LHaGG
   ➢ {0,1}ⁿ proof
   ➢ Extension to {0,1,...,N}ⁿ
4. Main result (≥25 min. 7 min.)
   ➢ Weakly LHaGG
   ➢ Recurrence
   ➢ Transience
   ➢ pausing coupling
5. Concluding remarks (1 min.)

Find this slide in my HP
https://shuji-kijima.com/

Shuji Kijima
1. Introduction w/ examples
A random walk on an infinite graph is recurrent at vertex $\nu$ if it visits $\nu$ infinitely many times, i.e.,

$$\sum_{t=0}^{\infty} \Pr[X_t = \nu] = \infty$$

holds, otherwise it is said to be transient.

For instance,

RW on $\mathbb{Z}$ is recurrent at $0$. 

\[ \cdots \quad \circ \quad \cdots \]
Recurrence/Transience of Random walks on infinite graphs

A random walk on an infinite graph is **recurrent** at vertex \( v \) if it visits \( v \) infinitely many times, i.e.,

\[
\sum_{t=0}^{\infty} \Pr[X_t = v] = \infty
\]

holds, otherwise it is said to be **transient**.

For instance,

- RW on \( \mathbb{Z} \) is **recurrent** at \( o \),
- RW on \( \mathbb{Z}^2 \) is **recurrent** at \( o \),
Recurrence/Transience of Random walks on infinite graphs

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For instance,

- RW on $\mathbb{Z}$ is **recurrent** at $o$,
- RW on $\mathbb{Z}^2$ is **recurrent** at $o$,
- RW on $\mathbb{Z}^3$ is **transient** at $o$, 
Example 1. Random walk in a growing region of $\mathbb{Z}^3$

✓ Random walk on $\mathbb{Z}^3$ is transient at $o$.
✓ Random walk on $\{-n, ..., n\}^3$ is recurrent at $o$.

Q. Is a random walk on $\{-n, ..., n\}^3$ recurrent or transient if $n$ increases as time goes on?

A. It depends on the increasing speed.

Find the phase transition point regarding the growing speed.

RW on $\mathbb{Z}^3$ is transient at $o$,
**Model: Random Walk on a Growing Graph (RWoGG)**

- **Growing graph** is a sequence of static graphs
  \[ \mathcal{G} = \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \ldots \]
  where each \( \mathcal{G}_t = (\mathcal{V}_t, \mathcal{E}_t) \) is a static simple graph.
  We assume \( \mathcal{V}_t \subseteq \mathcal{V}_{t+1} \), for convenience.
  Furthermore, \( \mathcal{E}_t \subseteq \mathcal{E}_{t+1} \) holds in this talk.

[K, Shimizu, Shiraga ‘21]
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- **RWoGG** \((\mathcal{d}, \mathcal{G}, \mathcal{P})\) is a specific model:
  - \( \mathcal{d}(1), \mathcal{d}(2), \mathcal{d}(3), \ldots \in \mathbb{Z} \) denote the duration time.
  - Growing graph is given by
    \[ \mathcal{G}_t = \mathcal{G}(n) \text{ for } t \in [T_{n-1}, T_{n-1} + \mathcal{d}(n)) \]
    where \( T_n = \sum_{i=1}^{n} \mathcal{d}(n) \), i.e.,
    \[ \mathcal{G}_t = \begin{cases} 
    \mathcal{G}(1) & \text{for the first } \mathcal{d}(1) \text{steps} \\
    \mathcal{G}(2) & \text{for the next } \mathcal{d}(2) \text{steps} \\
    \mathcal{G}(3) & \text{for the next } \mathcal{d}(3) \text{steps} \\
    \vdots & \vdots \\
  \end{cases} \]
  - \( \mathcal{P}(n) \) denotes the transition matrix on \( \mathcal{G}(n) \).
  - \( \mathcal{d} \) represents (inverse) growing speed

[K, Shimizu, Shiraga ‘21]
Example 1. Random walk in a growing region of $\mathbb{Z}^d$

Let $\mathcal{D} = (\mathcal{G}, G, P)$ be a RWoGG where

- $\mathcal{G}(n) = n^2$,
- $G(n)$ is a grid graph $\{-n, ..., n\}^3$,
- $P(n)$ denotes the simple random walk w/ reflection bound, i.e., move to a neighbor w.p. $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$ unless boundary, for $n = 1, 2, ...$

Thm. [Dembo et al. 2014, Kumamoto et al. 2024]

If $\sum_{n=1}^{\infty} \frac{d(n)}{n^d} = \infty$ then recurrent, otherwise transient.
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If $\sum_{n=1}^{\infty} \frac{\mathcal{D}(n)}{n^d} = \infty$ then recurrent, otherwise transient.
**Example 1. Random walk in a growing region of \( \mathbb{Z}^d \)**

Let \( \mathcal{D} = (\mathcal{d}, G, P) \) be a RWoGG where

- \( \mathcal{d}(n) = n^{1.999} \),
- \( G(n) \) is a grid graph \( \{-n, \ldots, n\}^3 \),
- \( P(n) \) denotes the simple random walk w/ reflection bound,
  i.e., move to a neighbor w.p. \( \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} \) unless boundary,

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Example 2. RW on an infinite $k$-ary tree

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Q. Is a random walk on a $k$-ary tree recurrent or transient if its height $n$ increases as time go on?

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Find the phase transition point regarding the growing speed.
Example 2. Random walk on a growing $k$-ary tree

Let $\mathcal{D} = (\delta, G, P)$ be a RWoGG where

- $\delta(n) = 3^n$,
- $G(n)$ is a $3$-ary tree of height $n$,
- $P(n)$ denotes the simple random walk w/ reflection bound, i.e., move to a neighbor w.p. $1/4$ unless the root or a leaf, for $n = 1, 2, ...$

Thm. [Huang 2019, Kumamoto et al. 2024]

If $\sum_{n=1}^{\infty} \frac{\delta(n)}{k^n} = \infty$ then recurrent, otherwise transient.
Example 2. Random walk on a growing $k$-ary tree

Let $\mathcal{D} = (\mathcal{D}, G, P)$ be a RWoGG where

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Recurrent since $\sum_{n=1}^{\infty} \frac{3^n}{3^n} = \sum_{n=1}^{\infty} 1 = \infty$.

Thm. [Huang 2019, Kumamoto et al. 2024]

If $\sum_{n=1}^{\infty} \frac{\mathcal{D}(n)}{k^n} = \infty$ then recurrent, otherwise transient.
Example 2. Random walk on a growing $k$-ary tree

Let $\mathcal{D} = (d, G, P)$ be a RWoGG where

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2. Related work

About analysis of algorithms in dynamic environment
Related work (1/2): Random walks on dynamic graphs

- Graph search by RW --- related to cover time
  - Avin, Koucky and Lotker (2008): a bad example for hitting-time (cover time as well) w/ $\Omega(2^n)$ for the number of vertices $n$.
  - Denysyuk and Rodrigues (2014): cover time under some fairness condition.
  - Sauerwald and Zanetti (2019): $O(n^2)$ cover time for $d$-regular graphs.
  - K, Shimizu, Shiraga (2021): cover ratio of RWoGG

- Mixing time
  - Dembo, Huang and Zheng (2019) analyzed the conductance of growing subregion of $\mathbb{Z}^d$.
  - Cai, Sauerwald and Zanetti (2020): mixing time for edge-Markovian graph.

- Recurrence/transience
  ... Continued
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- Recurrence/transience
  ... Continued
Collecting an **increasing number of coupons** [K, Shimizu, Shiraga ‘21]

<table>
<thead>
<tr>
<th>Day</th>
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<tbody>
<tr>
<td># types</td>
<td>1</td>
<td>2</td>
<td>2</td>
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<td>3</td>
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<td>4</td>
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<tr>
<td>Pr[$X_t = k$]</td>
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1$^{st}$ period  | 2$^{nd}$ period  | 3$^{rd}$ period  | 4$^{th}$ period

Q. How many types are collected in the end of $n^{th}$ period?
Collecting an *increasing number of coupons* \[K, Shimizu, Shiraga '21\]

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1. 0(\( \log n \))
2. 0(\( \sqrt{n} \))
3. \( \frac{n}{2} \)
4. .99\( n \)

Q. How many types are collected in the end of \( n \)th period?
Collecting an increasing number of coupons [K, Shimizu, Shiraga ‘21]

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Q. How many types are collected in the end of $n$th period?

1. $O(\log n)$
2. $O(\sqrt{n})$
3. $\frac{n}{2}$
4. $.99n$
5. at least $n - 1$

in expectation
Collecting an increasing number of coupons:

**Prop.**

If $d(n) = n$ then $E[U_n] < \frac{1}{e-1}$.

Proof.

1. $E_{i,n} := \begin{cases} 1 & \text{(item } i \text{ is uncollected in the end of the } n^{th} \text{ period)} \\ 0 & \text{(item } i \text{ is collected by the end of the } n^{th} \text{ period)} \end{cases}$ for $i = 1, 2, ..., n$.
2. $U_n = \Sigma_{i=1}^{n} E_{i,n}$
3. Prob. that item $n$ is uncollected in the end of the $n$th period:
   
   $\Pr[E_{n,n} = 1] = \left( 1 - \frac{1}{n} \right)^n < e^{-1}$

4. Prob. that item $i$ $(i \leq n)$ is uncollected in the end of the $n^{th}$ period:
   
   $\Pr[E_{i,n} = 1] = \left( 1 - \frac{1}{i} \right)^i \left( 1 - \frac{1}{i+1} \right)^i \cdots \left( 1 - \frac{1}{n} \right)^n < \left( \frac{1}{e} \right)^{n+1-i}$

5. $E[U_n] = \Sigma_{i=1}^{n} \Pr[E_{i,n}] < \Sigma_{i=1}^{n} \left( \frac{1}{e} \right)^{n+1-i} = \frac{1}{e} + \frac{1}{e^2} + \cdots + \frac{1}{e^n} < \frac{\frac{1}{e}}{1-\frac{1}{e}} = \frac{1}{e-1} < 0.582$. 

Draw a coupon everyday

$d(n)$: #days of the $n^{th}$ period

$U_n$: #items uncollected in the end of $n^{th}$ period

[K, Shimizu, Shiraga ’21]
**RWO GG (d, G P)**

Coupon collector is often regarded as a RW on the complete graph, and we can extend the arguments to RWO GG for general graphs.

**Thm. (general upper bound)**

If $d(i) \geq ct_{hit}(i)$ ($c \geq 1$) then $E[U] = O(1)$.

Particularly, if $\frac{d(i)}{t_{hit}(i)} \xrightarrow{i \to \infty} \infty$ then $E[U_n] \xrightarrow{n \to \infty} 0$.

**Thm. (upper bound for lazy and reversible walk)**

Suppose $P(i)$ is lazy and reversible.

If $\frac{t_{hit}(i)}{t_{mix}(i)} \geq \frac{i^\gamma}{c}$ and $d(i) \geq \frac{3ct_{hit}(i)}{i^\gamma}$ ($c > 0$) then $E[U_n] \leq \frac{8n^\gamma}{c} + 32$.

S. Kijima, N. Shimizu, T. Shiraga, How many vertices does a random walk miss in a network with moderately increasing the number of vertices?, in Proc. SODA 2021, 106—122.
Related work (2/2): recurrence/transience of RW

- Much work about the recurrence/transience on growing graphs exist in the context of self-interacting random walks including reinforced random walks, excited random walks, etc. since 1990s, or before.

- Dembo, Huang and Sidoravicius (2014× 2): recurrent ⇔ \( \sum_{t=0}^{\infty} \pi_t(0) = \infty \) for growing subregion of \( \mathbb{Z}^d \) (fixed \( d \)), by conductance argument.


- Huang (2017): growing graph w/ uniformly bounded degrees.

- Kumamoto, K. and Shirai (2024): \( k \)-ary tree, \( \{0,1\}^n \) w/ an increasing \( n \) under \textbf{RWoGG} model by coupling.

- This work (2024): \( \{0,1, \ldots, N\}^n \) (fixed \( N \), increasing \( n \)) by pausing coupling.
3. Our previous work [SAND ‘24]

About the recurrence/transience of RWoGG, for an introduction of the basic technique and its issue.

Example 3. Random walk on $\{0,1\}^n$ w/ an increasing $n$

Let $\mathcal{D} = (d, G, P)$ be a RWoGG where

- $d(n) = 2^n$,
- $G(n)$ is a $\{0,1\}^n$ skeleton,
- $P(n)$ denotes the simple random walk, i.e., move to a neighbor w.p. $1/n$,

for $n = 1, 2, ...$
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Thm. [Kumamoto et al. 2024]

If \( \sum_{n=1}^{\infty} \frac{\mathcal{D}(n)}{2^n} = \infty \) then recurrent, otherwise transient.
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Lem. [Kumamoto et al. 2024]
Random walk on \( \{0, 1\}^n \) is \textit{LHaGG}.

Thm. [Kumamoto et al. 2024]
If \( \sum_{n=1}^{\infty} \frac{\mathcal{d}(n)}{2^n} = \infty \) then recurrent, otherwise transient.
LHaGG \[\text{[SAND ‘24]}\]

Defs.

- $\mathcal{D}_1 = (f_1, G_1, P_1)$ is less homesick than $\mathcal{D}_2 = (f_2, G_2, P_2)$ if $R_1(t) \leq R_2(t)$ for any $t$ where $R_1(t)$ and $R_2(t)$ respectively denote the return probabilities of $\mathcal{D}_1$ and $\mathcal{D}_2$ at time $t$.

- $\mathcal{D} = (f, G, P)$ is less homesick as graph growing (LHaGG) if $\mathcal{D}$ is less homesick than $\mathcal{D}' = (g, G, P)$ for any $g$ satisfying that $\sum_{k=1}^{n} f(k) \leq \sum_{k=1}^{n} g(k)$ for any $n$, i.e., $\mathcal{D}$ and $\mathcal{D}'$ grows similarly, but $\mathcal{D}$ grows faster.

The faster a graph grows, the smaller the return probability.
Theorems by LHaGG

The faster a graph grows, the smaller the return probability.

Under the condition of LHaGG, we can prove the following sufficient conditions of recurrence/transience, respectively.

**Thm. [Kumamoto, K., Shirai ’24]**
Suppose $\mathcal{D} = (\mathcal{d}, G, P)$ is LHaGG. If
\[
\sum_{n=1}^{\infty} \mathcal{d}(n)p(n) = \infty
\]
then $\mathcal{D}$ is **recurrent** at $\nu$, where $p(n) = \pi_n(\nu)$.

**Thm. [Kumamoto, K., Shirai ’24]**
Suppose $\mathcal{D} = (\mathcal{d}, G, P)$ is LHaGG. If
\[
\sum_{n=1}^{\infty} \max\{\mathcal{d}(n), t(n)\} p(n) < \infty
\]
then $\mathcal{D}$ is **transient** at $\nu$, where $t(n)$ represents the mixing time.
Example 3. Random walk on \(\{0,1\}^n\) w/ an increasing \(n\)

Let \(\mathcal{D} = (\mathcal{G}, G, P)\) be a RWoGG where

- \(\mathcal{G}(n) = 2^n\),
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The faster a graph grows, the smaller the return probability?

Lem. [Kumamoto et al. 2024]
Random walk on \(\{0,1\}^n\) is \textbf{LHaGG}.

Thm. [Kumamoto et al. 2024]
If \(\sum_{n=1}^{\infty} \frac{\mathcal{G}(n)}{2^n} = \infty\) then recurrent, otherwise transient.
FAQ: Any example for *not* LHaGG?

The faster a graph grows, the smaller the return probability.

Isn’t it trivial?
FAQ: Any example for not LHaGG?

A (lazy) simple random walk on $G(1)$, $G(2)$, $G(3)$, $G(4)$, $G(5)$ is not LHaGG.

The faster a graph grows, the smaller the return probability. Isn’t it trivial?
Lazy RW on $\{0,1\}^n$ w/ increasing $n$ is LHaGG

[SAND '24]

Proof.

The proof is a **monotone coupling**.

- Let $X_t \sim D_f = (f, G, P)$ and $Y_t \sim D_g = (g, G, P)$ where $\sum_{i=1}^n f(i) \geq \sum_{i=1}^n g(i)$, i.e., the graph of $D_g$ grows faster than that of $D_f$.

- Let $|X_t|, |Y_t|$ denote the number of 1s in $X_t \in \{0,1\}^{n_t}, Y_t \in \{0,1\}^{m_t}$, where notice that $n_t \leq m_t$. Then,

\[
\begin{align*}
\Pr[|X_{t+1}| - 1 = |X_t|] &= \frac{1}{2} |X_t| \cdot n_t, \\
\Pr[|X_{t+1}| = |X_t|] &= \frac{1}{2}, \\
\Pr[|X_{t+1}| + 1 = |X_t|] &= \frac{1}{2} \left( 1 - \frac{|X_t|}{n_t} \right) \\
\Pr[|Y_{t+1}| - 1 = |Y_t|] &= \frac{1}{2} |Y_t| \cdot m_t, \\
\Pr[|Y_{t+1}| = |Y_t|] &= \frac{1}{2}, \\
\Pr[|Y_{t+1}| + 1 = |Y_t|] &= \frac{1}{2} \left( 1 - \frac{|Y_t|}{m_t} \right)
\end{align*}
\]

- if $|X_t| < |Y_t|$ then we can couple so that $|X_{t+1}| \leq |Y_{t+1}|$ thanks to the self-loop w.p. $\frac{1}{2}$.
- If $|X_t| = |Y_t|$ then we can couple so that $|X_{t+1}| \leq |Y_{t+1}|$ since $n_t \leq m_t$.

Thus, $X_t = o$ if $Y_t = o$, meaning that $\Pr[X_t = o] \geq \Pr[Y_t = o]$. 

It looks a very simple exercise if you are familiar with **coupling**, but $n_t \neq m_t$ makes some trouble, in general.
Example 3. Random walk on $\{0,1\}^n$ w/ an increasing $n$

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

- $\mathfrak{d}(n) = 2^n$,
- $G(n)$ is a $\{0,1\}^n$ skeleton,
- $P(n)$ denotes the simple random walk, i.e., move to a neighbor w.p. $1/n$,

for $n = 1,2,\ldots$
Example 3. Random walk on \(\{0,1\}^n\) w/ an increasing \(n\)

Let \(\mathcal{D} = (\mathcal{d}, G, P)\) be a RWoGG where

- \(\mathcal{d}(n) = 2^n\),
- \(G(n)\) is a \(\{0,1\}^n\) skeleton,
- \(P(n)\) denotes the simple random walk, i.e., move to a neighbor w.p. \(1/n\), for \(n = 1,2,\ldots\)

\[\text{Lem. [Kumamoto et al. 2024]}\]

Random walk on \(\{0,1\}^n\) is **LHaGG**.

\[\text{Thm. [Kumamoto et al. 2024]}\]

If \(\sum_{n=1}^{\infty} \frac{\mathcal{d}(n)}{2^n} = \infty\) then recurrent, otherwise transient.

\[\text{Recurrent since } \sum_{n=1}^{\infty} \frac{2^n}{2^n} = \infty.\]
Three representations (or “applications”?) of \( \{0,1\}^n \)

- Random walk on \( \{0,1\}^n \) w/ an increasing dimensions
  
  \( \mathbb{d}(1) \) steps \( \rightarrow \) \( \mathbb{d}(2) \) steps \( \rightarrow \) \( \mathbb{d}(3) \) steps

- Random pick/drop items w/ an increasing number of items
  
  \{\} \rightarrow \{\text{🍎, 🍊}\} \rightarrow \{\text{🍎, 🍊, 🍒}\}

- Random bit flip of binary w/ an increasing bit length
  
  \( .1 \) \rightarrow \( .10 \) \rightarrow \( .101 \)
Three representations (or “applications”? ) of \( \{0,1\}^n \)

- Random walk on \( \{0,1\}^n \) w/ an increasing dimensions

- Random pick/drop items w/ an increasing number of items

- Random bit flip of binary w/ an increasing bit length
Extension from \(\{0,1\}^n\) to \(\{0,1, \ldots, 9\}^n\)

- Random walk on \(\{0,1, \ldots, 9\}^n\) w/ an increasing \(n\)
  
  \[
  \{0,1, \ldots, 9\}^1 \quad \{0,1, \ldots, 9\}^2 \quad \{0,1, \ldots, 9\}^3
  \]
  
  \[\text{\textbackslash{d}(1) steps} \quad \text{\textbackslash{d}(2) steps} \quad \text{\textbackslash{d}(3) steps}\]

- Random buy/sell stocks w/ an increasing #brands
  
  \[
  \times 3 \quad \times 2 + \times 4 \quad \times 3 + \times 0 + \mathbf{g} \times 2
  \]
  
  \[\text{\textbackslash{d}(1) steps} \quad \text{\textbackslash{d}(2) steps} \quad \text{\textbackslash{d}(3) steps}\]

- Random up/down digits w/ an increasing digit length
  
  \[
  .3 \quad .24 \quad .302
  \]
  
  \[\text{\textbackslash{d}(1) steps} \quad \text{\textbackslash{d}(2) steps} \quad \text{\textbackslash{d}(3) steps}\]
Target. Random walk on \(\{0,1, \ldots, N\}^n\) w/ an increasing \(n\)

Let \(\mathcal{D} = (\mathfrak{d}, G, P)\) be a RWoGG where

- \(\mathfrak{d}(n) = N^n\),
- \(G(n)\) is a \(\{0,1, \ldots, N\}^n\) skeleton,
- \(P(n)\) denotes the lazy simple random walk w/ reflection bound,
  i.e., move to a neighbor w.p. \(1/4n\), unless boundary

for \(n = 1,2, \ldots\)

\[
\begin{align*}
\{0,1, \ldots, N\}^1 & \quad \{0,1, \ldots, N\}^2 & \quad \{0,1, \ldots, N\}^3 \\
\mathfrak{d}(1) \text{ steps} & \quad \mathfrak{d}(2) \text{ steps} & \quad \mathfrak{d}(3) \text{ steps}
\end{align*}
\]

Q.
Is random walk on \(\{0,1, \ldots, N\}^n\) LHaGG?

A. We can’t prove it.
4. Main Result
Target. Random walk on \( \{0, 1, ..., N\}^n \) w/ an increasing \( n \)

Let \( \mathcal{D} = (\mathcal{d}, G, P) \) be a RWoGG where

- \( \mathcal{d}(n) = N^n \),
- \( G(n) \) is a \( \{0, 1, ..., N\}^n \) skeleton,
- \( P(n) \) denotes the lazy simple random walk w/ reflection bound, i.e., move to a neighbor w.p. \( 1/4n \), unless boundary for \( n = 1, 2, ... \)

Q. Is random walk on \( \{0, 1, ..., N\}^n \) LHaGG?

A. We can’t prove it.
**Target. Random walk on \( \{0,1, \ldots, N\}^n \) w/ an increasing \( n \)**

Let \( \mathcal{D} = (\mathcal{A}, G, P) \) be a RWoGG where

- \( \mathcal{A}(n) = N^n \),
- \( G(n) \) is a \( \{0,1, \ldots, N\}^n \) skeleton,
- \( P(n) \) denotes the lazy simple random walk w/ reflection bound, i.e., move to a neighbor w.p. \( 1/4n \), unless boundary

for \( n = 1, 2, \ldots \)

**Lem. 7.**
Random walk on \( \{0,1, \ldots, N\}^n \) is *weakly* LHaGG.

**Q.**
Is random walk on \( \{0,1, \ldots, N\}^n \) LHaGG?
Target. Random walk on \( \{0,1,\ldots,N\}^n \) w/ an increasing \( n \)

Let \( \mathcal{D} = (\mathcal{d}, G, P) \) be a RWoGG where

- \( \mathcal{d}(n) = N^n \),
- \( G(n) \) is a \( \{0,1,\ldots,N\}^n \) skeleton,
- \( P(n) \) denotes the lazy simple random walk w/ reflection bound,
  i.e., move to a neighbor w.p. \( 1/4n \), unless boundary

for \( n = 1,2,\ldots \)

Lem. 7. Random walk on \( \{0,1,\ldots,N\}^n \) is **weakly** LHaGG.

Thm. 6. If \( \mathcal{D} = (\mathcal{d}, G, P) \) satisfies

\[
\sum_{n=1}^{\infty} \frac{d(n)}{(2N)^n} = \infty
\]

then \( o \) is recurrent, otherwise \( o \) is transient.
Recall: LHaGG [SAND ‘24]

Defs.

• $\mathcal{D}_1 = (f_1, G_1, P_1)$ is less homesick than $\mathcal{D}_2 = (f_2, G_2, P_2)$ if $R_1(t) \leq R_2(t)$ for any $t$ where $R_1(t)$ and $R_2(t)$ respectively denote the return probabilities of $\mathcal{D}_1$ and $\mathcal{D}_2$ at time $t$.

• $\mathcal{D} = (f, G, P)$ is less homesick as graph growing (LHaGG) if $\mathcal{D}$ is less homesick than $\mathcal{D}' = (g, G, P)$ for any $g$ satisfying that $\sum_{k=1}^{n} f(k) \leq \sum_{k=1}^{n} g(k)$ for any $n$, i.e., $\mathcal{D}$ and $\mathcal{D}'$ grows similarly, but $\mathcal{D}$ grows faster.

The faster a graph grows, the smaller the return probability.
Recall: LHaGG

Defs.

• $D_1 = (f_1, G_1, P_1)$ is less homesick than $D_2 = (f_2, G_2, P_2)$ if $R_1(t) \leq R_2(t)$ for any $t$ where $R_1(t)$ and $R_2(t)$ respectively denote the return probabilities of $D_1$ and $D_2$ at time $t$.

• $D = (f, G, P)$ is less homesick as graph growing (LHaGG) if $D$ is less homesick than $D' = (g, G, P)$ for any $g$ satisfying that $\sum_{k=1}^{n} f(k) \leq \sum_{k=1}^{n} g(k)$ for any $n$, i.e., $D$ and $D'$ grows similarly, but $D$ grows faster.

The faster a graph grows, the smaller the return probability.
wLHaGG

We replace the condition about the return prob. with a condition of the **sum of** return prob.

Defs.

• \( \mathcal{D}_1 = (f_1, G_1, P_1) \) is **weakly less homesick** than \( \mathcal{D}_2 = (f_2, G_2, P_2) \) if
  \[ \sum_{t=1}^{T} R_1(t) \leq \sum_{t=1}^{T} R_2(t) \]
  for any \( T \) where \( R_1(t) \) and \( R_2(t) \) respectively denote the return probabilities of \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) at time \( t \).

• \( \mathcal{D} = (f, G, P) \) is **weakly less homesick as graph growing** (wLHaGG) if \( \mathcal{D} \) is weakly less homesick than \( \mathcal{D}' = (g, G, P) \) for any \( g \) satisfying that
  \[ \sum_{k=1}^{n} f(k) \leq \sum_{k=1}^{n} g(k) \]
  for any \( n \), i.e., \( \mathcal{D} \) and \( \mathcal{D}' \) grows similarly, but \( \mathcal{D} \) grows **faster**.

The faster a graph grows, the smaller the **expected number** of returns.
General theorems

Condition 0. (ergodic). In $\mathcal{D} = (\mathcal{V}, G, P)$, every transition matrix $P(n)$ is ergodic.

Condition 1. (mixing time). $\mathcal{D} = (\mathcal{V}, G, P)$ satisfies
\[
\sum_{k=1}^{\infty} \tau^*(k)p(k) < \infty
\]
where $p(k) = \pi_k(o)$ and $\tau^*(k) = t_{\text{mix}}^k \left( \frac{p(k)}{4} \right)$.

Mixing time is not very big. E.g., $O \left( \frac{1}{\pi_k(o)} \frac{1}{k \log k} \right)$

Thm. 2. (Recurrence).
Suppose $(\mathcal{V}, G, P)$ satisfies Conditions 0 and 1.
If $\sum_{k=1}^{\infty} d(k)p(k) = \infty$ then the initial vertex $v$ is recurrent.

Thm. 4. (Transience).
Suppose $(\mathcal{V}, G, P)$ satisfies Conditions 0 and 1, and it is wLHaGG.
If $\sum_{k=2}^{\infty} d(k)p(k - 1) < \infty$ then the initial vertex $v$ is transient.
Recurrence

Proof. Let $X_t$ follow $(b, G, P)$, and let $R(t) = \Pr[X_t = o]$. We claim

**Lem. 3.** $\sum_{t=1}^{T_n} R(t) \geq \frac{1}{2} \sum_{k=1}^{n} (b(k) - \tau^*(k))p(k)$

Proof of Lem. 3.

- Notice that $X_t$ follows $P_n$ for $t \in [T_{n-1}, T_{n-1} + b(n))$.
- If $b(n) > t_{\text{mix}}(\epsilon)$ then $R(t) \geq \pi_n(o) - \epsilon$ for $t \geq T_{n-1} + t_{\text{mix}}(\epsilon)$ where $\pi_n$ is the stationary distribution of $P_n$.
- Thus, $R(t) \geq \pi_n(o) - \frac{1}{2} p(n) = \frac{1}{2} p(n)$ since $\tau^*(k) = t_{\text{mix}}\left(\frac{1}{2} p(n)\right)$ and $p(n) = \pi_n(o)$.
- $\sum_{t=1}^{T_n} R(t) = \sum_{k=1}^{n} \sum_{s=1}^{b(k)} R(T_{n-1} + s) \geq \sum_{k=1}^{n} \sum_{s=\tau^*(n)}^{b(k)} R(T_{n-1} + s) \geq \sum_{k=1}^{n} \sum_{s=\tau^*(n)}^{b(k)} \frac{1}{2} p(n) = \frac{1}{2} \sum_{k=1}^{n} (b(k) - \tau^*(k))p(k)$

Once we obtain Lem. 3, Thm. 2 is easy: $\sum_{t=1}^{\infty} R(t) = \infty$ holds if $\sum_{k=1}^{\infty} b(k)p(k) = \infty$ and $\sum_{k=1}^{\infty} \tau^*(k)p(k) < \infty$.  

Mixing time condition
Transience

Thm. 4. (Transience).
Suppose \((d, G, P)\) satisfies Conditions 0 and 1, and it is wLHaGG.
If \(\sum_{k=2}^{\infty} d(k)p(k - 1) < \infty\) then the initial vertex \(v\) is transient.

Proof. Let \(f(k) = \max\{d, \tau^*(k)\}\).
By wLHaGG, \(\sum_{t=1}^{T_n} R_b(t) \leq \sum_{t=1}^{T_n} R_g(t)\).

Lem. 5. \(\sum_{t=1}^{T_n} R_g(t) \leq g(1) + \frac{3}{2} \sum_{k=2}^{n} g(k)p(k - 1)\)

Proof of Lem. 5.
Let \(f(k) = \begin{cases} g(k) & k \leq n - 1 \\ \infty & k = n \end{cases}\)
Then, \(\sum_{k=1}^{m} g(k) \leq \sum_{k=1}^{m} g(k)\) for any \(m\).

Let \(X_t \sim D_g = (g, G, P)\) and \(Y_t \sim D_f = (f, G, P)\).
• Notice that \(Y_t\) follows \(P_{n-1}\) for \(t \geq T_{n-2}\).
• By wLHaGG, \(\sum_{t=1}^{T} \Pr[X_t = o] \leq \sum_{t=1}^{T} \Pr[Y_t = o]\) for any \(T\).
• \(R_f(t) \leq \pi_n(o) + \frac{1}{2} p(n - 1) = \frac{3}{2} p(n - 1)\) for \(t \geq T_{n-1}\)
• \(\sum_{t=1}^{T_n} R_g(t) = \sum_{k=1}^{n} \sum_{s=1}^{g(k)} R_g(T_{k-1} + s) \leq g(1) + \sum_{k=2}^{n} \sum_{s=1}^{g(k)} R_f(T_{k-1} + s) \leq g(1) + \sum_{k=2}^{n} \sum_{s=1}^{g(k)} \frac{3}{2} p(k - 1) = g(1) + \frac{3}{2} \sum_{k=2}^{n} g(k)p(k - 1)\)

Particularly, remark \(X_t \sim P_n\) but \(Y_t \sim P_{n-1}\) for \(t \in [T_{n-1}, T_n)\)

Once we obtain Lem. 5, Thm. 4 is clear.
Target. Random walk on $\{0, 1, \ldots, N\}^n$ w/ an increasing $n$

Let $\mathcal{D} = (\mathfrak{d}, G, P)$ be a RWoGG where

- $\mathfrak{d}(n) = N^n$,
- $G(n)$ is a $\{0, 1, \ldots, N\}^n$ skeleton,
- $P(n)$ denotes the lazy simple random walk w/ reflection bound, i.e., move to a neighbor w.p. $1/4n$, unless boundary for $n = 1, 2, \ldots$

Lem. 7.
Random walk on $\{0, 1, \ldots, N\}^n$ is *weakly* LHaGG.

Thm. 6. If $\mathcal{D} = (\mathfrak{d}, G, P)$ satisfies
\[ \sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{(2N)^n} = \infty \]
then $o$ is recurrent, otherwise $o$ is transient.
Target. Random walk on $\{0,1, \ldots, N\}^n$ w/ an increasing $n$

Let $\mathcal{D} = (\mathcal{D}, G, P)$ be a RWoGG where

- $\mathcal{D}(n) = N^n$,
- $G(n)$ is a $\{0,1, \ldots, N\}^n$ skeleton,
- $P(n)$ denotes the lazy simple random walk w/ reflection bound, i.e., move to a neighbor w.p. $1/4n$, unless boundary

for $n = 1, 2, \ldots$

Lem. 7.
Random walk on $\{0,1, \ldots, N\}^n$ is weakly LHaGG.

It looks a very simple exercise if you are familiar with coupling, but $n_t \neq m_t$ makes some trouble, in general.
Target. Random walk on \(\{0, 1, \ldots, N\}^n\) w/ an increasing \(n\)

Let \(D = (\mathcal{D}, G, P)\) be a RWoGG where

- \(\mathcal{D}(n) = N^n\),
- \(G(n)\) is a \(\{0, 1, \ldots, N\}^n\) skeleton,
- \(P(n)\) denotes the lazy simple random walk w/ reflection bound,
  i.e., move to a neighbor w.p. \(1/4n\), unless boundary
for \(n = 1, 2, \ldots\)

Lem. 7.
Random walk on \(\{0, 1, \ldots, N\}^n\) is **weakly** LHaGG.

We develop “pausing coupling”

It looks a very simple exercise if you are familiar with coupling, but \(n_t \neq m_t\) makes some trouble, in general.
We define time correspondence $t \mapsto S(t)$ depending on $Y$ so that

1. if $Y_t$ does self-loop then so does $X_{S^{-1}(t)}$,
2. if $Y_t$ updates $Y_t^i$ for $i \leq \dim(X_{S^{-1}(t)})$ then $X$ updates $X_{S^{-1}(t)}^i$,
3. if $Y_t$ updates $Y_t^i$ for $i > \dim(X_{S^{-1}(t)})$ then $X$ pauses.

We need to check “measure conservation” of the coupling.
Figure of pausing coupling

- Let \( X = X_0, X_1, X_2, \ldots \sim D_f \) and \( Y = Y_0, Y_1, Y_2, \ldots \sim D_g \) where let \( D_g \) grow faster than \( D_f \).
- We couple \( X \) and \( Y \), instead of \( X_t \) and \( Y_t \).

We define time correspondence \( t \mapsto S(t) \) depending on \( Y \) so that

1. if \( Y_t \) does self-loop then so does \( X_{S^{-1}(t)} \),
2. if \( Y_t \) updates \( Y^i_t \) for \( i \leq \dim(X_{S^{-1}(t)}) \) then \( X \) updates \( X^i_{S^{-1}(t)} \),
3. if \( Y_t \) updates \( Y^i_t \) for \( i > \dim(X_{S^{-1}(t)}) \) then \( X \) pauses.

We need to check “measure conservation” of the coupling.
We define time correspondence $t \mapsto S(t)$ depending on $Y$ so that

1. if $Y_t$ does self-loop then so does $X_{S^{-1}(t)}$,
2. if $Y_t$ updates $Y^i_t$ for $i \leq \dim(X_{S^{-1}(t)})$ then $X$ updates $X^i_{S^{-1}(t)}$,
3. if $Y_t$ updates $Y^i_t$ for $i > \dim(X_{S^{-1}(t)})$ then $X$ pauses.

We need to check “measure conservation” of the coupling.
Outline of the proof

Let $\eta: Y \mapsto X = \eta(Y)$ denote the coupling described in the previous slide. We prove two things:

- The coupling $\eta$ preserves the measure, i.e.,
  \[ \Pr[Y = y] = \Pr[X = \eta(y)] \]

- The coupling $\eta$ preserves $|X_t| \leq |Y_s|$ (meaning $|\eta(y_s)| \leq |y_s|$) for any $s$ satisfying $S(t) \leq s < S(t + 1)$.

  - This implies $\#\{t \leq T \mid X_t = o\} \geq \#\{t \leq T \mid Y_t = o\}$ for any $T$. 

Random walk on $\{0, 1, \ldots, N\}^n$ w/ increasing $n$ is wLHaGG.
**Def. S(t)**

Proof.

Suppose $Y = Y_0, Y_1, Y_2, Y_3, ...$ is represented by
\[
\theta_Y = (\lambda_1, j_1, \rho_1), (\lambda_2, j_2, \rho_2), (\lambda_3, j_3, \rho_3), ...
\]

We define $S: \mathbb{Z} \to \mathbb{Z}$ according to $\theta$.

Let $S(1) = \min\{\min\{t > 0| \lambda_t = 0\}, \min\{t > 0| j_t \in n_0\}\}$.

Recursively, let
\[
S(k) = \min\{\min\{t > S(k - 1)| \lambda_t = 0\}, \min\{t > S(k - 1)| j_t \in n_{k-1}\}\}
\]
where let $\min\{\emptyset\} = \infty$.

If $S(k) = \infty$ then let $S(k + 1) = \infty$.

For convenience, let $S^{-1}(t) = k$ for $t = S(k) < \infty$ ($k = 1, 2, ...$).

Then, we define $X = X_0, X_1, X_2, ...$ by
\[
\theta_X = \left(\left(\lambda_{S^{-1}(k)}, j_{S^{-1}(k)}, \rho_{S^{-1}(k)}\right)\right)_{k=1,2,...}
= (\lambda_{S^{-1}(1)}, j_{S^{-1}(1)}, \rho_{S^{-1}(1)}), (\lambda_{S^{-1}(2)}, j_{S^{-1}(2)}, \rho_{S^{-1}(2)}), ...
\]
as far as $S(k) < \infty$.

If $S(k) = \infty$ then generate $(\lambda'_k, j'_k, \rho'_k)$ and transit to $X_{k+1}$ according to it.
**Def. $S(t)$**

Proof.

Suppose $Y = Y_0, Y_1, Y_2, Y_3, \ldots$ is represented by

$$\theta_Y = (\lambda_1, j_1, \rho_1), (\lambda_2, j_2, \rho_2), (\lambda_3, j_3, \rho_3), \ldots$$

We define $S: \mathbb{Z} \rightarrow \mathbb{Z}$ according to $\theta$.

Let $S(1) = \min\{\min\{t > 0 | \lambda_t = 0\}, \min\{t > 0 | j_t \in n_0\}\}$.

Recursively, let

$$S(k) = \min\{\min\{t > S(k-1) | \lambda_t = 0\}, \min\{t > S(k-1) | j_t \in n_{k-1}\}\}$$

where let $\min\{\emptyset\} = \infty$.

If $S(k) = \infty$ then let $S(k+1) = \infty$.

For convenience, let $S^{-1}(t) = k$ for $t = S(k) < \infty \ (k = 1, 2, \ldots)$.

Then, we define $X = X_0, X_1, X_2, \ldots$

$$\begin{align*}
\theta_X &= (\lambda_{S^{-1}(1)}, j_{S^{-1}(1)}, \rho_{S^{-1}(1)}) \\
&\quad (2), \ldots \\
\end{align*}$$

as far as $S(k) < \infty$.

If $S(k) = \infty$ then generate $(\lambda_k, j_k, \rho_k)$ and transit to $X_{k+1}$ according to it.

Time up...
5. Concluding remarks
Final slide

Result

- Recurrence/transience of \( \text{wLHaGG RWoGG} \).
- Random walk on \( \{0,1, \ldots, N\}^n \) w/ increasing \( n \) is \( \text{wLHaGG} \).
  - Proof by pausing coupling.

Future work

- Simplify the proof
  - Extension to other \( \text{RWoGGs} \)
    - E.g., GW tree, PA graph, and more general graphs,
    - Edge dynamics, e.g., growing + edge Markovian.
- Analysis of \( \text{RWoGG} \) beyond recurrence/transience
  - Hitting time, meeting time, gathering time, etc.
  - Find a new limit, undefined for an infinite graph.
The end

Thank you for the attention.
Lazy simple random walk on $\{0,1,...,N\}^n$ w/ increasing $n$

Current state $X_t = (X_t^1,...,X_t^{n_t}) \in \{0,1,...,N\}^{n_t}$.

1. W.p. $\frac{1}{2}$, set $X_{t+1} = X_t$.
2. Else, choose $i \in \{1,...,n_t\}$ u.a.r.
3. If $X_t^i$ is not 0 nor $N$ then update as $X_{t+1}^i = X_t^i \pm 1$ w.p. $\frac{1}{2}$ resp.
4. Else if $X_t^i = 0$ then update as $X_{t+1}^i = X_t^i + 1$.
5. Else if $X_t^i = N$ then update as $X_{t+1}^i = X_t^i - 1$. 
Lazy simple random walk on \(\{0,1, \ldots, N\}^n\) w/ increasing \(n\)

Current state \(X_t = (X_t^1, \ldots, X_t^{n_t}) \in \{0,1, \ldots, N\}^{n_t}\).

1. W.p. \(\frac{1}{2}\), set \(X_{t+1} = X_t\).

2. Else, choose \(i \in \{1, \ldots, n_t\}\) u.a.r.

3. If \(X_t^i\) is not 0 nor \(N\) then update as \(X_{t+1}^i = X_t^i \pm 1\) w.p. \(\frac{1}{2}\) resp.

4. Else if \(X_t^i = 0\) then update as \(X_{t+1}^i = X_t^i + 1\).

5. Else if \(X_t^i = N\) then update as \(X_{t+1}^i = X_t^i - 1\).

A transition \(X_t \mapsto X_{t+1}\) is represented by uniform r.v.s \((\lambda, i, \rho) \in \{0,1\} \times \{1, \ldots, n_t\} \times \{0,1\}\).