Process Convergence of the QuickSelect Residual

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QuickSelect

- **Problem**: Given $n$ distinct elements $U_1, \ldots, U_n$ and $1 \leq k \leq n$, find the element at rank $k$, i.e. $U_{(k)}$ such that

$$\#\{i \mid U_i \leq U_{(k)}\} = k$$

- Sorting all values would take $O(n \log n)$ time
- QuickSelect, derived from QuickSort, needs expected $O(n)$ time
QuickSelect – Example

Find the fourth-biggest element

\[
6 \ 3 \ 1 \ 9 \ 4 \ 2 \ 7 \ 5
\]
QuickSelect – Example

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6 3 1 9 4 2 7 5

3 1 4 2 5 9 7

- Compare with 6 (the pivot) and sort into elements bigger and smaller
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1 4 2 5

9 7

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4 5

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Find the fourth-biggest element

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- Because 1, 2, 3 are smaller we look for the smaller of 4, 5
QuickSelect – Example

Find the fourth-biggest element

6 3 1 9 4 2 7 5

Compare with 6 (the pivot) and sort into elements bigger and smaller

New pivot 3

Because 1, 2, 3 are smaller we look for the smaller of 4, 5

We find 4
Analysis

- **Random model:** We assume $U_1, \ldots, U_n$ are independent and uniformly distributed on $[0, 1]$.
- Consider first the amount of comparisons.
- But how to chose the rank for $n \to \infty$?
  - Uniformly at random on $\{1, \ldots, n\}$ (grand average)
  - Rank $\lfloor tn \rfloor$ for some $t \in [0, 1]$ (**QuickQuant** process)
  - Let $S_{t,n}$ be the amount of comparisons for rank $\lfloor tn \rfloor$
  - $S_{t,n}/n$ converges for $n \to \infty$ a.s. to a limit $S_t$ (**Grübel** and **Rösler** 1996)
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**Theorem (I., Neininger 2024+)**

Let $F_n := \frac{1}{n} \sum_{i=1}^{n} 1[u_i, \infty)$ be the empirical distribution function of $U_1, \ldots, U_n$. The residual process converges

$$G_{t,n} := \frac{S_{t,n} - nS_n^{-1}(t)}{\sqrt{n}} \xrightarrow{d} G_{t,\infty} \quad \text{in } (D[0,1], d_{SK})$$

towards a mixed centred Gaussian process $G_{t,\infty}$.

What is $D[0,1]$? Why the empirical distribution function? And what are the covariances?
Functional Results

- We study $t \mapsto S_{t,n}$ resp. $G_{t,n}$ as process to be able to consider all choices of ranks simultaneously.
- Many properties, e.g. the amount of comparisons at fixed and random places or the maximum can be written as functions of the process $S_{t,n}$

**Theorem (Continuous mapping theorem)**

For a measurable function $\varphi$

$$X_n \xrightarrow{d} X \quad \Rightarrow \varphi(X_n) \xrightarrow{d} \varphi(X)$$

if $X$ is a.s. not at a discontinuity of $\varphi$.

- Process convergence of $S_{t,n}$ then implies convergence of these properties
- $S_{t,n}$ converges to $S_t$ as process (Grübel and Rösler 1996)
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The space of càdlàg functions

- But first, we have to describe the space the process $S_{t,n}$ lives on
- $S_{t,n}$ is a right-continuous step function
  - Functions that are right-continuous and have left limits are called càdlàg
  - Write $\mathcal{D}[0,1]$ for the space of càdlàg functions on $[0,1]$
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The space of càdlàg functions

- The Skorokhod metric uses a monotonously growing bijection $\lambda : [0, 1] \to [0, 1]$ to align jumps.
- For $f, g \in \mathcal{D}[0, 1]$ define

$$d_{SK}(f, g) := \inf_{\lambda} \| f - g \circ \lambda \|_\infty \lor \| \lambda - id \|_\infty$$

(id is the identity, $\lor$ the maximum, $\| \cdot \|_\infty$ the uniform norm)

- Example: For $a, b \in (0, 1)$: $d_{SK}(1_{[a, \infty)}, 1_{[b, \infty)}) = |b - a|$
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QuickVal

- Starting at the full interval $[0, 1]$, at step $0$ the first pivot divides the interval in left and right.
- The first value in each interval becomes the new pivot and splits the interval again.
- Write $l_{\alpha,k}$ for the length of the interval in step $k$ containing a value $\alpha$. 

\[ 0 \quad \quad \quad \quad \quad \quad \quad \quad \quad 1 \]
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![Diagram of interval splitting](image)
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- Starting at the full interval $[0, 1]$, at step 0 the first pivot divides the interval in left and right
- The first value in each interval becomes the new pivot and splits the interval again
- Write $I_{\alpha,k}$ for the length of the interval in step $k$ containing a value $\alpha$
We always go into the interval where the result of our algorithm is

The result is $F_n^{-1}(t)$, that is why the inverse empirical distribution function pops up

We should have indexed using the result $\alpha$ instead of using $t$!

The process $S_{\alpha,n}$ of comparisons for result $\alpha$ is called **QuickVal**
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QuickVal

- Let $S_{\alpha,k,n}$ the amount of comparisons in step $k \in \mathbb{N}_0$
- That is the amount of elements in the interval with length $l_{\alpha,k}$
- Conditional on the pivots, $S_{\alpha,k,n}$ is thus almost $B(n, l_{\alpha,k})$-distributed
- The error are the $k$ pivots, so we write $S_{\alpha,k,n} \sim B(n, l_{\alpha,k}) + O(k)$. 
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QuickVal
The limit process

- Since \( S_{\alpha,k,n} \sim B(n, l_{\alpha,k}) + O(k) \),

\[
\frac{S_{\alpha,k,n}}{n} \to l_{\alpha,k} \quad \text{a.s.}
\]

- Indeed, our limit process \( S_{\alpha} \) is given by

\[
S_{\alpha} := \sum_{k\in\mathbb{N}} l_{\alpha,k}
\]

and \( n^{-1}S_{n,\alpha} \to S_{\alpha} \) both a.s. (Fill and Nakama 2013) and in \( L^2 \) (Fill and Matterer 2014).
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Variations

What about the variation around this limit, the residual?
- Conditional on \( l_{\alpha,k} \),
  \[
  \frac{S_{\alpha,k,n} - n l_{\alpha,k}}{\sqrt{n}} \xrightarrow{d} N(0, l_{\alpha,k}(1 - l_{\alpha,k}))
  \]
- As for the sum over all steps (Matterer 2015)
  \[
  \frac{S_{\alpha,n} - n S_{\alpha}}{\sqrt{n}} \xrightarrow{d} N(0, \Sigma_{\alpha,\alpha})
  \]
- If an element is in \( l_{\alpha,k} \), then it is also in \( l_{\alpha,l} \) for all \( l \leq k \), so
  \[
  \Sigma_{\alpha,\alpha} = \sum_{k,l \in \mathbb{N}_0} l_{\alpha,k \vee l} - l_{\alpha,k} l_{\alpha,l}
  \]
Step-wise comparisons

Variances

What about the variation around this limit, the residual?

- Conditional on $I_{\alpha,k}$,

$$\frac{S_{\alpha,k,n} - nl_{\alpha,k}}{\sqrt{n}} \xrightarrow{d} N(0, l_{\alpha,k}(1 - l_{\alpha,k}))$$

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- If an element is in $I_{\alpha,k}$, then it is also in $I_{\alpha,l}$ for all $l \leq k$, so
  $$\Sigma_{\alpha,\alpha} = \sum_{k,l \in \mathbb{N}_0} I_{\alpha,k \lor l} - I_{\alpha,k}I_{\alpha,l}$$
Covariances

For $\alpha, \beta \in [0, 1]$, let $J(\alpha, \beta)$ be the (random) last index where $\alpha$ and $\beta$ are in the same interval.

An element is in $I_{\alpha,k}$ and $I_{\beta,l}$ for $k \leq l \in \mathbb{N}_0$ at the same time if and only if $k \leq J(\alpha, \beta)$ and the element is in $I_{\beta,l}$.

Conflating interval and interval length, let us write

$$|I_{\alpha,k} \cap I_{\beta,l}|$$

for the length of the intersection of $I_{\alpha,k}$ and $I_{\beta,l}$.

Then, the covariances can be written as

$$\Sigma_{\alpha,\beta} = \sum_{k,l \in \mathbb{N}_0} |I_{\alpha,k} \cap I_{\beta,l}| - I_{\alpha,k} I_{\beta,l}$$
We can now restate our main theorem with *QuickVal*

**Theorem (I., Neininger 2024+)**

The residual process converges

\[ \frac{S_{\alpha,n} - nS_\alpha}{\sqrt{n}} \overset{d}{\rightarrow} G_{\alpha,\infty} \quad \text{in} \ (D[0, 1], d_{SK}) \]

towards a mixed centred Gaussian process \( G_{\infty} \) with covariances

\[ \Sigma_{\alpha,\beta} = \sum_{k,l \in \mathbb{N}_0} |l_{\alpha,k} \cap l_{\beta,l}| - l_{\alpha,k} l_{\beta,l}. \]
Main Results

Proof Sketch

- Split for some $K = K_n \in \mathbb{N}$

$$\frac{S_{\alpha,n} - nS_{\alpha}}{\sqrt{n}} = \sum_{k=0}^{K} \frac{S_{\alpha,k,n} - nl_{\alpha,k}}{\sqrt{n}} + \sum_{k=K+1}^{\left\lfloor 4.5 \log n \right\rfloor} \frac{S_{\alpha,k,n} - nl_{\alpha,k}}{\sqrt{n}} + \sum_{k=\left\lceil 4.5 \log n \right\rceil}^{\infty} \frac{S_{\alpha,k,n} - nl_{\alpha,k}}{\sqrt{n}}$$

- The first steps have only finitely many values, use the (Multivariate) Central Limit Theorem

- The following steps have few elements, and should be small, use Chernov bounds.

- In the last steps there are no elements, and $l_{\alpha,k}$ is falling geometrically.
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- The first steps have only finitely many values, use the (Multivariate) Central Limit Theorem
- The following steps have few elements, and should be small, use Chernov bounds.
- In the last steps there are no elements, and $I_{\alpha,k}$ is falling geometrically.
The step to partition into smaller and bigger elements is usually done by rearranging. Depending on machine, swapping positions can be significantly slower than comparing.

We consider 2 schemes: Hoare and Lomuto.
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Lomuto and Hoare – Visualisation

Lomuto

Cost: Amount of smaller elements on the right side (hypergeometric)
Lomuto and Hoare – Visualisation

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**Lomuto**

**Hoare**

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Lomuto and Hoare – Visualisation

**Lomuto**

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**Hoare**

Cost: Amount of smaller elements on the right side (hypergeometric)
Given the interval sizes, the amount $W_{\alpha,k,n}$ of swaps for either Hoare or Lomuto at a fixed step $k$ are asymptotically normal around some mean $W_{\alpha,k}$, a function of interval sizes.

Using this limit, we again define a residual process

$$\sum_{k=0}^{\infty} \frac{W_{\alpha,k,n} - nW_{\alpha,k}}{\sqrt{n}}$$
For the Hoare scheme the residual process for swaps converges

\[ \sum_{k=0}^{\infty} \frac{W_{\alpha,k,n} - nW_{\alpha,k}}{\sqrt{n}} \xrightarrow{d} G_{\alpha,\infty}^{\text{swap}} \quad \text{in} \ (D[0,1], d_{\text{SK}}) \]

towards a mixed centred Gaussian process \( G_{\alpha,\infty}^{\text{swap}} \). The random covariances are functions of the interval sizes. The same holds for Lomuto, with other covariances.
Variable comparison costs

- Other model: Comparison times depend on the items compared
- E.g. the time needed to compare strings is proportional to the length of their common prefix
- The same holds for decimals (number of bit comparisons)
- Typically, closer elements take longer to compare
Variable comparison costs

- Let $\beta(u, v)$ be cost to compare pivot $u$ to item $v$
- Let $V$ be uniformly distributed on $[0, 1]$
- We call $\beta \varepsilon$-tame for $\varepsilon > 0$ if there exists a $C > 0$ so that for all $x \geq 0, u \in [0, 1]$
  \[ P(\beta(u, V) > x) \leq Cx^{-\varepsilon^{-1}} \]
- Sufficient that $\beta(u, V)$ has a $\varepsilon^{-1}$-th moment, uniformly bounded in $u$.
- This covers bit comparisons, where $\beta(u, V)$ has exponential tails, so $\beta$ is $\varepsilon$-tame for all $\varepsilon > 0$
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The residual can be defined as before, and then

Theorem (Matterer 2015)

*When $\beta$ is $\varepsilon$-tame for $\varepsilon < \frac{1}{2}$, the residual at a fixed point converges in distribution to a centred mixed normal.*

Theorem (I., Neininger 2024+)

*When $\beta$ is $\varepsilon$-tame for $\varepsilon < \frac{1}{4}$, the residual process converges in distribution on $(\mathcal{D}[0, 1], d_{SK})$ to a centred mixed Gaussian process with covariances given as functions of $\beta$ and the interval sizes.*
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**Theorem (Matterer 2015)**

When $\beta$ is $\varepsilon$-tame for $\varepsilon < \frac{1}{2}$, the residual at a fixed point converges in distribution to a centred mixed normal.

**Theorem (I., Neininger 2024+)**

When $\beta$ is $\varepsilon$-tame for $\varepsilon < \frac{1}{4}$, the residual process converges in distribution on $(\mathcal{D}[0, 1], d_{SK})$ to a centred mixed Gaussian process with covariances given as functions of $\beta$ and the interval sizes.
Thank you for your attention