Process Convergence of the QuickSelect Residual

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QuickSelect

Problem: Given n distinct elements U_1, \ldots, U_n and $1 \le k \le n$, find the element at rank k, i.e. $U_{(k)}$ such that

$$\#\{i \mid U_i \leq U_{(k)}\} = k$$

- Sorting all values would take $O(n \log n)$ time
- QuickSelect, derived from QuickSort, needs expected O(n) time

Find the fourth-biggest element

$$\left(6\ 3\ 1\ 9\ 4\ 2\ 7\ 5\right)$$

Find the fourth-biggest element



 Compare with 6 (the *pivot*) and sort into elements bigger and smaller

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New pivot 3

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We find 4

- **•** Random model: We assume U_1, \ldots, U_n are independent and uniformly distributed on [0, 1].
- Consider first the amount of comparisons
- But how to chose the rank for $n \to \infty$?
- Uniformly at random on $\{1, \ldots, n\}$ (grand average)
- Rank $\lfloor tn \rfloor$ for some $t \in [0, 1]$ (QuickQuant process)
- Let $S_{t,n}$ be the amount of comparisons for rank $\lfloor tn \rfloor$
- $S_{t,n}/n$ converges for $n \to \infty$ a.s. to a limit S_t (Grübel and Rösler 1996)

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QuickSelect

The Limit Process



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Process Convergence of the QuickSelect Residual

The residual

What happens if we subtract the limit and rescale? We call this residual.

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Theorem (I., Neininger 2024+)

Let $F_n := \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{[U_i,\infty)}$ be the empirical distribution function of U_1, \ldots, U_n . The residual process converges

$$G_{t,n} := \frac{S_{t,n} - nS_{F_n^{-1}(t)}}{\sqrt{n}} \stackrel{d}{\longrightarrow} G_{t,\infty} \quad in \ (\mathcal{D}[0,1], d_{\mathrm{SK}})$$

towards a mixed centred Gaussian process $G_{t,\infty}$.

What is $\mathcal{D}[0,1]$? Why the empirical distribution function? And what are the covariances?

Process Convergence of the QuickSelect Residual

Functional Results

- We study $t \mapsto S_{t,n}$ resp. $G_{t,n}$ as process to be able to consider all choices of ranks simultaneously, .
- Many properties, e.g. the amount of comparisons at fixed and random places or the maximum can be written as functions of the process $S_{t,n}$

Theorem (Continuous mapping theorem)

For a measurable function arphi

$$X_n \xrightarrow{d} X \quad \Rightarrow \varphi(X_n) \xrightarrow{d} \varphi(X)$$

if X is a.s. not at a discontinuity of φ .

Process convergence of S_{t,n} then implies convergence of these properties
 S_{t,n} converges to S_t as process (Grübel and Rösler 1996)

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The space of càdlàg functions

- But first, we have to describe the space the process $S_{t,n}$ lives on
- $S_{t,n}$ is a right-continuous step function
- Functions that are right-continuous and have left limits are called càdlàg
- Write $\mathcal{D}[0,1]$ for the space of càdlàg functions on [0,1]
- For measurability we need a metric accommodating for jumps not aligning, the Skorokhod metric

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The space of càdlàg functions

- The Skorokhod metric uses a monotonously growing bijection $\lambda : [0, 1] \rightarrow [0, 1]$ to align jumps.
- For $f,g \in \mathcal{D}[0,1]$ define

$$d_{ ext{SK}}(f,g) := \inf_{oldsymbol{\lambda}} \|f - g \circ oldsymbol{\lambda}\|_\infty ee \|oldsymbol{\lambda} - \operatorname{\mathsf{id}}\|_\infty$$

(id is the identity, \lor the maximum, $\|\cdot\|_{\infty}$ the uniform norm) Example: For $a, b \in (0, 1)$: $d_{SK}(\mathbf{1}_{[a,\infty)}, \mathbf{1}_{[b,\infty)}) = |b - a|$
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- Starting at the full interval [0, 1], at step 0 the first pivot divides the interval in left and right
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• We always go into the interval where the result of our algorithm is

- The result is F_n⁻¹(t), that is why the inverse empirical distribution function pops up
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- The result is $F_n^{-1}(t)$, that is why the inverse empirical distribution function pops up
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Let S_{α,k,n} the amount of comparisons in step k ∈ N₀
That is the amount of elements in the interval with length I_{α,k}
Conditional on the pivots, S_{α,k,n} is thus almost B(n, I_{α,k}) -distributed
The error are the k pivots, so we write S_{α,k,n} ~ B(n, I_{α,k}) + O(k).

- Let $S_{\alpha,k,n}$ the amount of comparisons in step $k \in \mathbb{N}_0$
- That is the amount of elements in the interval with length $I_{\alpha,k}$
- Conditional on the pivots, $S_{\alpha,k,n}$ is thus almost $B(n, I_{\alpha,k})$ -distributed
- The error are the k pivots, so we write $S_{\alpha,k,n} \sim B(n, I_{\alpha,k}) + O(k)$.

The limit process

Since $S_{\alpha,k,n} \sim B(n, I_{\alpha,k}) + O(k)$,

$$rac{S_{lpha,k,n}}{n}
ightarrow I_{lpha,k}$$
 a.s.

Indeed, our limit process S_{α} is given by

$$S_{\alpha} := \sum_{k \in \mathbb{N}} I_{\alpha,k}$$

and $n^{-1}S_{n,lpha} o S_{lpha}$ both a.s. (Fill and Nakama 2013) and in L^2 (Fill and Matterer 2014).

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Variances

What about the variation around this limit, the residual?

• Conditional on $I_{\alpha,k}$,

$$\frac{S_{\alpha,k,n}-nI_{\alpha,k}}{\sqrt{n}} \stackrel{d}{\longrightarrow} N(0,I_{\alpha,k}(1-I_{\alpha,k}))$$

As for the sum over all steps (Matterer 2015)

$$\frac{S_{\alpha,n}-nS_{\alpha}}{\sqrt{n}} \xrightarrow{d} N(0, \Sigma_{\alpha,\alpha})$$

If an element is in $I_{\alpha,k}$, then it is also in $I_{\alpha,l}$ for all $l \leq k$, so

$$\Sigma_{\alpha,\alpha} = \sum_{k,l \in \mathbb{N}_0} I_{\alpha,k \vee l} - I_{\alpha,k} I_{\alpha,l}$$

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Covariances

- For α, β ∈ [0, 1], let J(α, β) be the (random) last index where α and β are in the same interval
- An element is in $I_{\alpha,k}$ and $I_{\beta,l}$ for $k \leq l \in \mathbb{N}_0$ at the same time if and only if $k \leq J(\alpha, \beta)$ and the element is in $I_{\beta,l}$
- Conflating interval and interval length, let us write

 $|I_{\alpha,k} \cap I_{\beta,l}|$

- for the length of the intersection of $I_{\alpha,k}$ and $I_{\beta,l}$
- Then, the covariances can be written as

$$\Sigma_{\alpha,\beta} = \sum_{k,l \in \mathbb{N}_0} |I_{\alpha,k} \cap I_{\beta,l}| - |I_{\alpha,k}I_{\beta,l}|$$

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Main theorem

We can now restate our main theorem with QuickVal

Theorem (I., Neininger 2024+)

The residual process converges

$$rac{S_{lpha,n}-nS_{lpha}}{\sqrt{n}} \stackrel{d}{\longrightarrow} G_{lpha,\infty} \quad \textit{in} \ (\mathcal{D}[0,1],d_{
m SK})$$

towards a mixed centred Gaussian process G_∞ with covariances

$$\Sigma_{\alpha,\beta} = \sum_{k,l \in \mathbb{N}_0} |I_{\alpha,k} \cap I_{\beta,l}| - |I_{\alpha,k}|_{\beta,l}.$$

Split for some $K = K_n \in \mathbb{N}$

$$\frac{S_{\alpha,n} - nS_{\alpha}}{\sqrt{n}} = \sum_{k=0}^{K} \frac{S_{\alpha,k,n} - nI_{\alpha,k}}{\sqrt{n}} + \sum_{k=K+1}^{\lfloor 4.5 \log n \rfloor} \frac{S_{\alpha,k,n} - nI_{\alpha,k}}{\sqrt{n}} + \sum_{k=\lceil 4.5 \log n \rceil}^{\infty} \frac{S_{\alpha,k,n} - nI_{\alpha,k}}{\sqrt{n}}$$

- The first steps have only finitely many values, use the (Multivariate) Central Limit Theorem
- The following steps have few elements, and should be small, use Chernov bounds.
- In the last steps there are no elements, and $I_{\alpha,k}$ is falling geometrically.

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Swaps

- The step to partition into smaller and bigger elements is usually done by rearranging
- Depending on machine, swapping positions can be significantly slower than comparing
- We consider 2 schemes: Hoare and Lomuto

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Lomuto

			l			

Lomuto



Lomuto



Lomuto



Cost: Amount of smaller elements



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Cost: Amount of smaller elements



st: Amount of smaller element on the right side (hypergeometric)

Swap Residual

- Given the interval sizes, the amount $W_{\alpha,k,n}$ of swaps for either Hoare or Lomuto at a fixed step k are asymptotically normal around some mean $W_{\alpha,k}$, a function of interval sizes
- Using this limit, we again define a residual process

$$\sum_{k=0}^{\infty} \frac{W_{\alpha,k,n} - nW_{\alpha,k}}{\sqrt{n}}$$

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Swap Residual

Theorem (I., Neininger 2024+)

For the Hoare scheme the residual process for swaps converges

$$\sum_{k=0}^{\infty} \frac{\mathcal{W}_{\alpha,k,n} - n\mathcal{W}_{\alpha,k}}{\sqrt{n}} \stackrel{d}{\longrightarrow} G_{\alpha,\infty}^{\mathrm{swap}} \quad \textit{in} \; (\mathcal{D}[0,1], d_{\mathrm{SK}})$$

towards a mixed centred Gaussian process $G_{\alpha,\infty}^{\text{swap}}$. The random covariances are functions of the interval sizes. The same holds for Lomuto, with other covariances.

- Other model: Comparison times depend on the items compared
- E.g. the time needed to compare strings is proportional to the length of their common prefix
- The same holds for decimals (number of bit comparisons)
- Typically, closer elements take longer to compare

• Let $\beta(u, v)$ be cost to compare pivot u to item v

• Let V be uniformly distributed on [0, 1]

• We call $\beta \varepsilon$ -tame for $\varepsilon > 0$ if there exists a C > 0 so that for all $x \ge 0, u \in [0, 1]$

$$\mathbb{P}(eta(u,V) > x) \leq C x^{-arepsilon^{-1}}$$

Sufficient that $\beta(u, V)$ has a ε^{-1} -th moment, uniformly bounded in u.

This covers bit comparisons, where β(u, V) has exponential tails, so β is ε-tame for all ε > 0

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Residual

The residual can be defined as before, and then

Theorem (Matterer 2015)

When β is ε -tame for $\varepsilon < \frac{1}{2}$, the residual at a fixed point converges in distribution to a centred mixed normal.

Theorem (I., Neininger 2024+)

When β is ε -tame for $\varepsilon < \frac{1}{4}$, the residual process converges in distribution on $(\mathcal{D}[0,1], d_{SK})$ to a centred mixed Gaussian process with covariances given as functions of β and the interval sizes.

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Thank you for your attention

