Lattice walks in the Weyl chamber A_2

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Analysis of Algorithms
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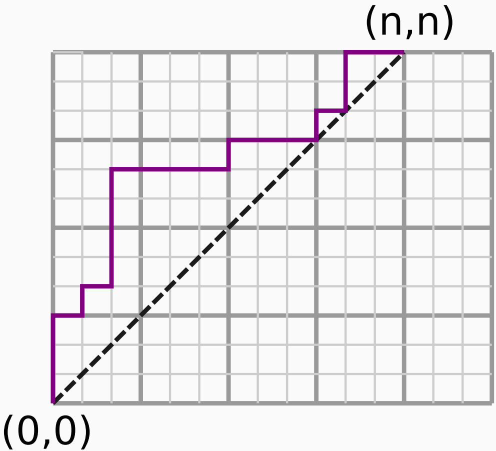
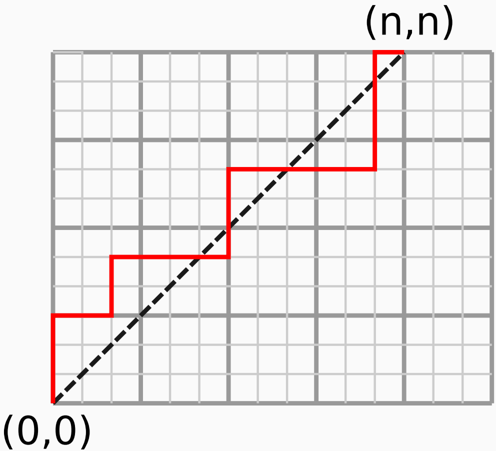
Dyck Paths

How many paths start at $(0, 0)$, end at (n, n), take steps in $\{\rightarrow, \uparrow\}$, and stay weakly above the line $y = x$?



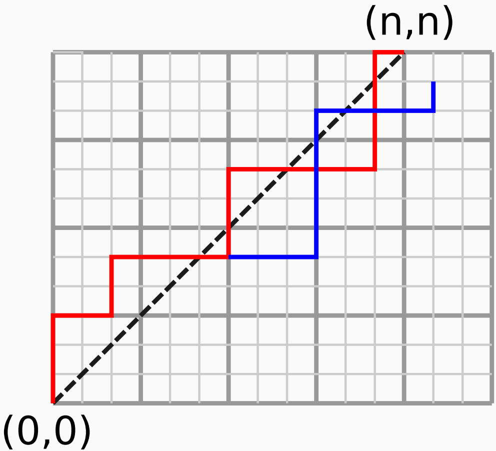
Bad Paths

How many paths start at $(0, 0)$, end at (n, n), take steps in $\{\rightarrow, \uparrow\}$, and stay weakly above the line $y = x$?



Bad Path Bijection

How many paths start at $(0, 0)$, end at (n, n), take steps in $\{\rightarrow, \uparrow\}$, and stay weakly above the line $y = x$?



Dyck Path formula

$$\begin{aligned} \text{All - Bad} &= \binom{2n}{n} - \binom{2n}{n-1} \\ &= \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$

Question:

How much can the reflection principle be generalized?

Why study lattice paths?

- Catalan objects are ubiquitous.
- Lattice paths in higher dimensions have bijections with trees, maps, permutations, lattice polygons, Young tableaux, queues, . . .
- Tracking the endpoint of a lattice path is the same as tracking the sum of steps, which could be viewed as a sum of independent discrete random variables.

Lattice models

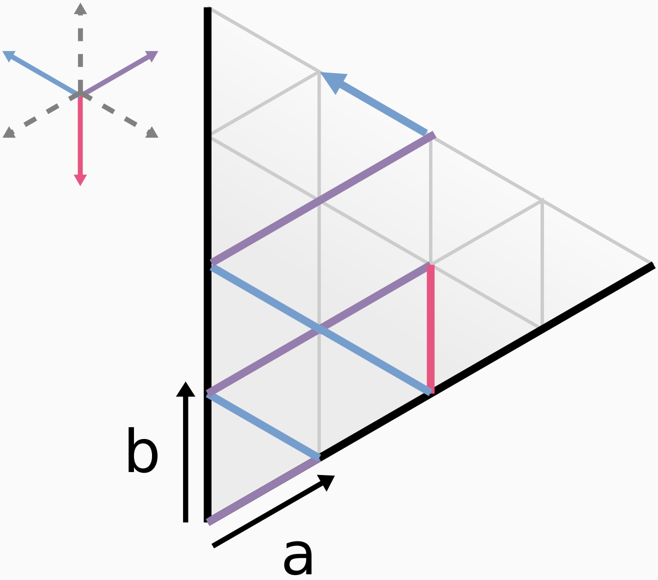
To generalize, we could change:

- the *stepset* $\mathcal{S}$ of the model (previously $\{\rightarrow, \uparrow\}$).
- the domain in which walks must stay (previously the upper half plane... or, really, $\mathbb{N}$).
- the weights of paths.

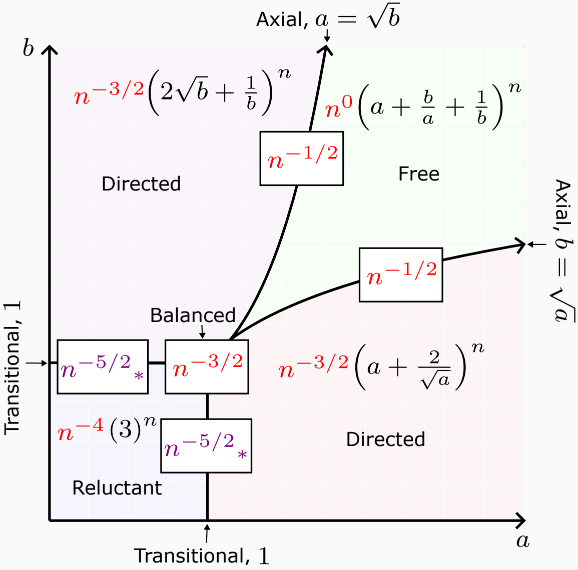
Interaction between lattice paths and ACSV

- Results on lattice paths have pushed forward results in Analytic Combinatorics in Several Variables (ACSV), and vice versa.
- There were regularly types of GF singularities that were first found in examples by studying lattice paths.

Our results: weighted walks in A_2



Graphical results



Increasing dimensions

- Banderier & Flajolet considered one-dimensional walks, bridges, meanders, and excursions.
- The quarter plane $\mathbb{N} \times \mathbb{N}$ proves to be much more difficult. Fayolle, Iasnogorodski, and Malyshev and Bousquet-Mélou and Mishna developed frameworks.
- The analysis depends heavily on the stepset.

Many additional authors have contributed: Bostan, Chyzak, Kauers, Krattenthaler, Kurkova, Raschel, Pech, van Hoeij.

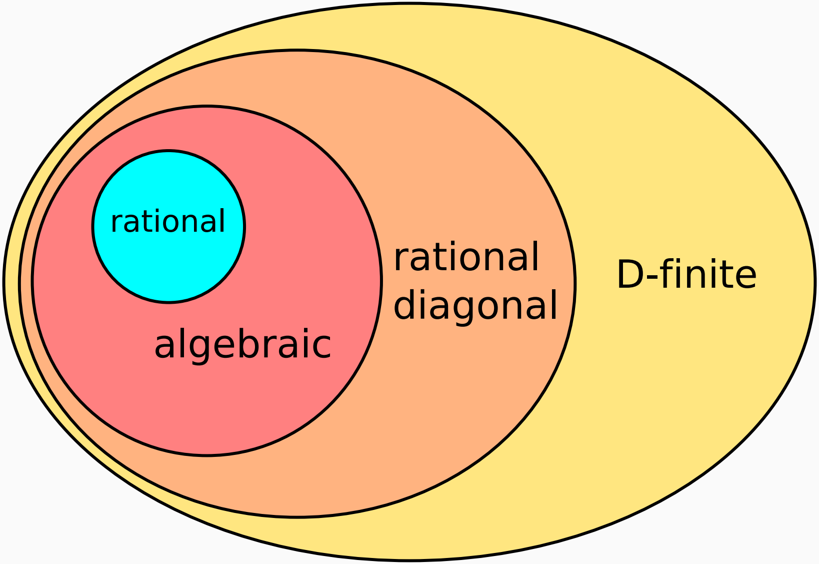
Analysis pipeline

- Figure out how to encode a_n, the number of walks with n total steps, in the generating function

$$F(t) = \sum a_n t^n.$$

- Use ACSV to extract asymptotics.

Classification of generating functions



Walks in $\mathbb{Z}^d$

Some classification results:

- Walks in all of $\mathbb{Z}^d$ have rational GFs.
- Walks in a half space of $\mathbb{Z}^d$ have algebraic GFs.
- Walks in the intersection of two half spaces could have algebraic GFs, or transcendental D-finite GFs, or neither.

More on walks in restricted domains

- Melczer, Mishna, and Wilson: d-dimensional orthant, (almost) highly symmetrical stepsets.
- Denisov and Wachtel examined walks in general cones, including Weyl chambers, from a probabilistic viewpoint.
- Bostan, Raschel, and Salvy: made explicit the results of Denisov and Wachtel in some settings.
- Courtiel, Melczer, Mishna, and Raschel: *weighted* Gouyou-Beauchamps model; found *universality classes* – our framework is modeled after theirs.

Reduced root systems

Definition (Reduced Root System)

For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, let $\sigma_{\mathbf{x}}(\mathbf{y})$ be the reflection of $\mathbf{y}$ through the hyperplane perpendicular to $\mathbf{x}$. A *reduced root system* is a finite set of vectors $\Phi \subset \mathbb{R}^d$ such that for any $\mathbf{x}, \mathbf{y} \in \Phi$,

- $\sigma_{\mathbf{x}}(\mathbf{y}) \in \Phi$,
- $\mathbf{y} - \sigma_{\mathbf{x}}(\mathbf{y})$ is an integer multiple of $\mathbf{x}$, and
- the only nontrivial scalar multiple of $\mathbf{x}$ in Φ is $-\mathbf{x}$.

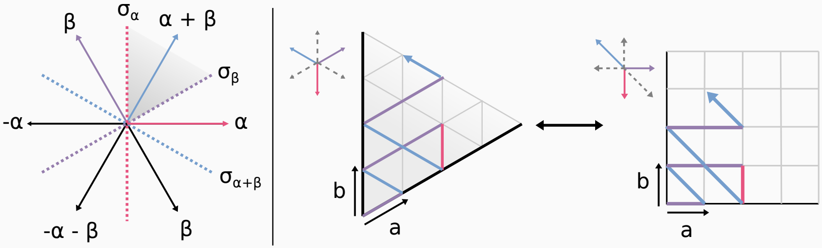
The group of reflections $\mathcal{G}$ determined by a root system is called a *Weyl group*.

Definition (Reflectable stepset)

Let $\mathcal{W}$ be a Weyl group acting on a real inner product space V with a distinguished basis $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_d)$. We say that a nonempty set of vectors $\mathcal{S}$ is a $(\mathcal{W}, \mathcal{B})$-*reflectable stepset* if

- for all $g \in \mathcal{W}$ and $s \in \mathcal{S}$, we have $g(s) \in \mathcal{S}$, and
- for all $s \in \mathcal{S}$ and $1 \leq i \leq d$, there is an integer c_i such that the dot product $\langle s, \mathbf{b}_i \rangle \in \{-c_i, 0, c_i\}$.

Walks in the Weyl chamber A_2



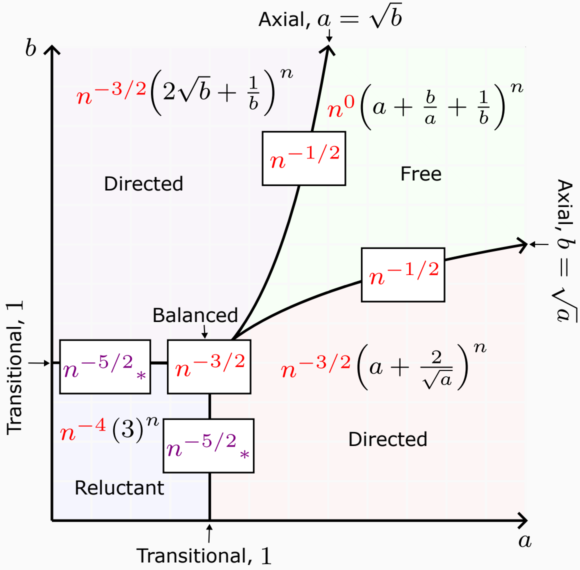
The three step version is called the Tandem model, and the six step version is called the Double Tandem model.

We look at weighted walks where each step to the right has weight a and each step up has weight b.

Walks in Weyl chambers: previous results

- Grabiner and Magyar, Krattenthaler, and Feierl analyzed weighted walks in Weyl chambers using determinants or from a probabilistic viewpoint.
- Mishna and Simon analyzed walks in the Weyl chamber A_1^d, and found nice patterns and results for all dimensions.

Reminder of results



Finding the generating functions

The kernel method or the generalized reflection principle have already yielded the unweighted GFs (Melczer & Wilson), and the weighted GF can be interpreted as an evaluation.

$$F_T(x,y,t) = \frac{(b^2x - ay^2)(bx^2 - a^2y)(xy - ab)}{(1 - txy(\frac{a}{x} + \frac{bx}{ay} + \frac{y}{b}))a^3b^3(1-x)x(1-y)y}$$

and

$$F_{DT}(x,y,t) = \frac{(b^2x - ay^2)(bx^2 - a^2y)(xy - ab)}{(1 - txy(\frac{a}{x} + \frac{x}{a} + \frac{bx}{ay} + \frac{ay}{bx} + \frac{y}{b} + \frac{b}{y}))a^3b^3(1-x)x(1-y)y}$$

Recap: analytic combinatorics mantra

- The location of a GFs smallest singularities determine exponential growth rate. We call these *critical points*.
- The behavior of the GF near these singularities determines the subexponential growth.

ACSV pipeline

1. Classify what kinds of critical points are possible.
2. Identify the critical points.
3. Refine to *contributing* critical points.
4. Use pre-existing results to evaluate asymptotics.

Critical point classification

Our GFs have two types of critical points: smooth and transverse.

- A smooth critical point means that the singular set near the critical point is a smooth manifold that can be parameterized smoothly in $d - 1$ variables.
  - The implicit function theorem can help us identify these.
- A transverse critical point is one where the singular variety appears locally to be the intersection of hyperplanes whose gradients are linearly independent.

All of these conditions can be checked algebraically.

Additional criterion: *contributing* critical points

Definition

Let **w** be a transverse critical point, and let the denominator of the GF encoding the walks be $H(\mathbf{z}) = H_0(\mathbf{z})H_1(\mathbf{z}) \cdots H_m(\mathbf{z})$ where $H_0(\mathbf{w}) \neq 0$ but $H_i(\mathbf{w}) = 0$ for $i \geq 1$. The point **w** is called *contributing* if the vector $(1, 1, \ldots, 1)$ is in the normal cone

$$N(\mathbf{w}) := \left\{ \sum_{j=1}^{m} a_j \mathbf{v}_j : a_j > 0 \right\}$$

where $\mathbf{v}_j$ is the normalized logarithmic gradient of H_j at **w**.

Read-off results

Once the critical points are classified, identified, and then filtered, we can use a result like the following.

Theorem (Pemantle and Wilson)

If a rational GF $F(x, y, t)$ has a strictly minimal smooth critical point $\mathbf{w}$ then

$$[x^n y^n t^n]F(x, y, t) = \mathbf{w}^{-n} n^{-3/2} \frac{(2\pi)^{-3/2}}{\sqrt{(\det \mathcal{H})}} \left(\sum_{j=0}^{M} C_j n^{-j} + O\left(n^{-M-1}\right) \right)$$

for explicitly computable constants C_j and some explicit matrix $\mathcal{H}$.

Conjectured results

In the analysis of A_2, a type of critical point previously unseen in applications appeared:

- The critical point is transverse.
- The numerator of the GF vanishes at the critical point.
- The *direction* $(1, 1, 1)$ is at the *boundary of the normal cone.*

We verify numerically that this cuts the normal asymptotic contribution of such a critical point in half.

Future problems

- We are working with Melczer and Kroiter to verify the conjecture.
- A_2^d and other Weyl chambers should be accessible via the same framework.
- Products of Weyl chambers.

Thank you!