Binary search trees of permuton samples

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AofA 2024, Bath

Full version at arXiv:2403.03151

Outline

I. BSTs and permutons

Binary Search Trees

A Binary Search Tree (BST) is a rooted labeled binary tree such that: for each vertex v, all labels of vertices in its left-subtree are smaller than that of v (resp. right-subtree, greater).

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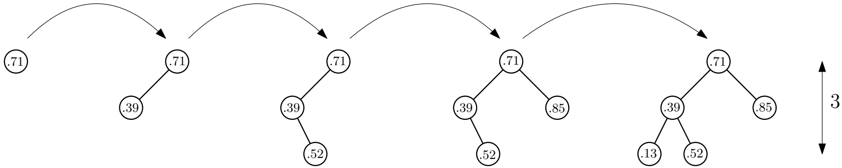
BST associated with a sequence of distinct numbers

Let $y = (y_1, \ldots, y_n)$ be a sequence of distinct numbers. We can construct a BST $\mathcal{T}\langle y \rangle$ by successively adding leaves with those labels.

Construction of $\mathcal{T}\langle y \rangle$ with $y = (.71, .39, .52, .85, .13)$:



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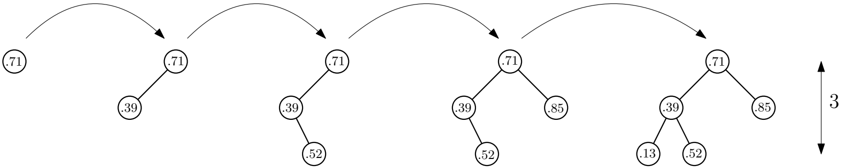


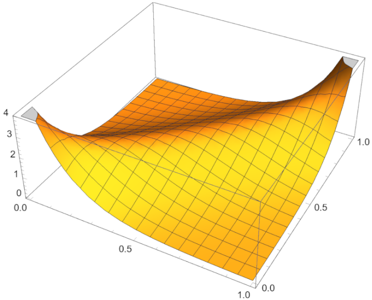
Rem: same shape as $\mathcal{T}\langle \sigma \rangle$ with $\sigma = (4, 2, 3, 5, 1)$.

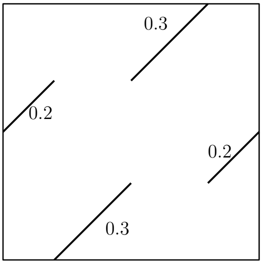
I. BSTs and permutons

Permutons

A *permuton* (or *copula*) is a probability measure μ on $[0,1]^2$ with uniform marginals:

$$\forall t \in [0,1], \quad \mu([0,t]\times[0,1]) = \mu([0,1]\times[0,t]) = t.$$





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Permuton samples

Let $\mathcal{P} = \{(x_1, y_1), \ldots, (x_n, y_n)\} \subset \mathbb{R}^2$ with no common x- or y-coordinate.
Define a reordering $(x_{(1)}, y_{(1)}), \ldots, (x_{(n)}, y_{(n)})$ by $x_{(1)} < \cdots < x_{(n)}$.
There exists a unique permutation $\sigma = \sigma\langle\mathcal{P}\rangle$ such that $(y_{(1)}, \ldots, y_{(n)})$ and $(\sigma(1), \ldots, \sigma(n))$ are in the same relative order.

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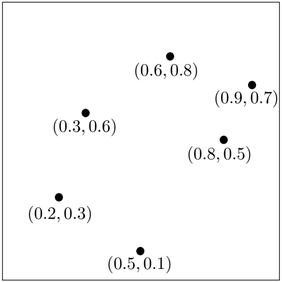

$\mathcal{P}$

$$(y_{(1)}, \dots, y_{(6)}) = (0.3, 0.6, 0.1, 0.8, 0.5, 0.7)$$

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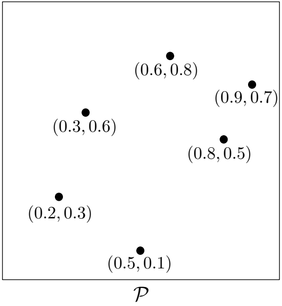

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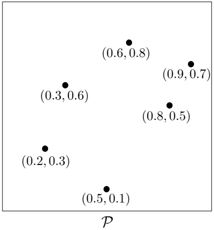

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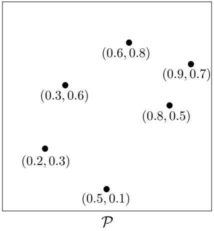

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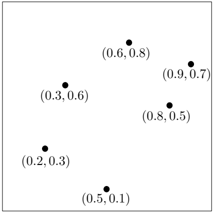

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- Insight on the links between random permutations and their limit permuton

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BST of a point process

Let $\mathcal{P} = \{(x_1, y_1), \ldots, (x_n, y_n)\} \subset \mathbb{R}^2$ with no common x- or y-coordinate.
Define a reordering $(x_{(1)}, y_{(1)}), \ldots, (x_{(n)}, y_{(n)})$ by $x_{(1)} < \cdots < x_{(n)}$.
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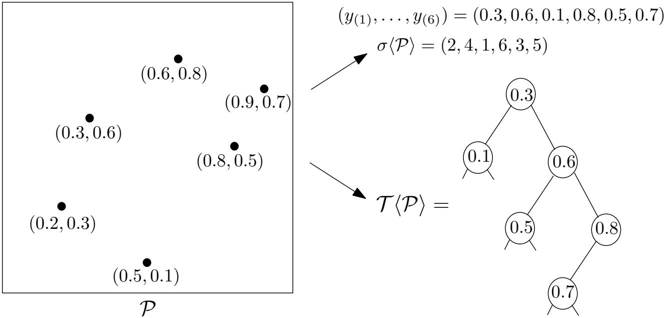

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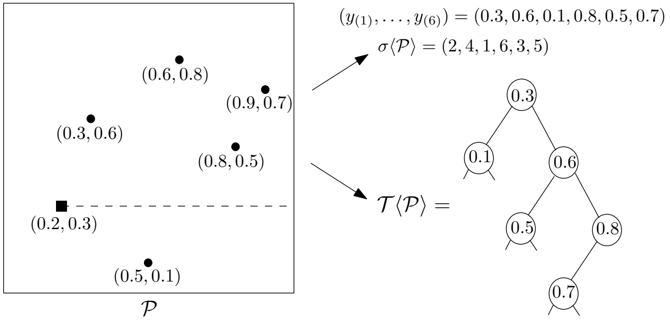

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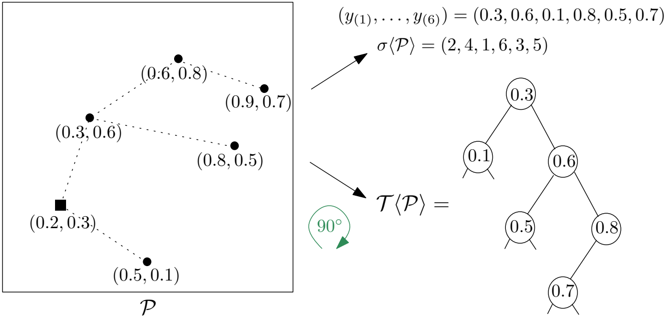

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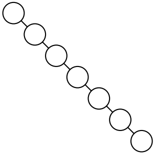
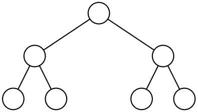
$$n \geq h(\mathcal{T}\langle\sigma(1),\ldots,\sigma(n)\rangle) \geq \log_2(n+1)$$





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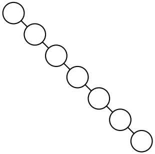

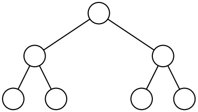

Depending on the law of σ, asymptotic results for $h(\mathcal{T}\langle\sigma\rangle)$?

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Devroye '86

If $\sigma_n \sim \mathrm{Unif}(\mathfrak{S}_n)$ then, as $n \to \infty$:

$$\frac{h(\mathcal{T}\langle\sigma_n\rangle)}{c^* \log n} \longrightarrow 1$$

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Let $q \in [0,1]$. The Mallows distribution $M_{n,q}$ is defined by:

$$\forall \sigma \in \mathfrak{S}_n, \quad M_{n,q}(\sigma) \propto q^{\mathrm{inv}(\sigma)}.$$

Addario-Berry – Corsini '21

If $\sigma_n \sim M_{n,q_n}$ where $n(1-q_n)/\log n \to 0$, then:

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Let $\theta \geq 0$. The record-biased distribution $R_{n,\theta}$ is defined by:

$$\forall \sigma \in \mathfrak{S}_n, \quad R_{n,\theta}(\sigma) \propto \theta^{\mathrm{rec}(\sigma)}.$$

Corsini '23

If $\sigma_n \sim R_{n,\theta}$ where $\theta \leq c^*$, then:

$$\frac{h\left(\mathcal{T}\langle\sigma_n\rangle\right)}{c^* \log n} \longrightarrow 1$$

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Corsini – D. – Féray '24

Let μ be a permuton with a bounded density on $[0,1]^2$, which is continuous and positive on a neighborhood of $\{0\} \times [0,1]$. Then:

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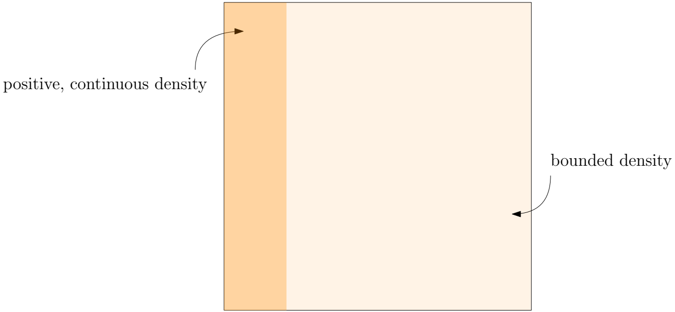

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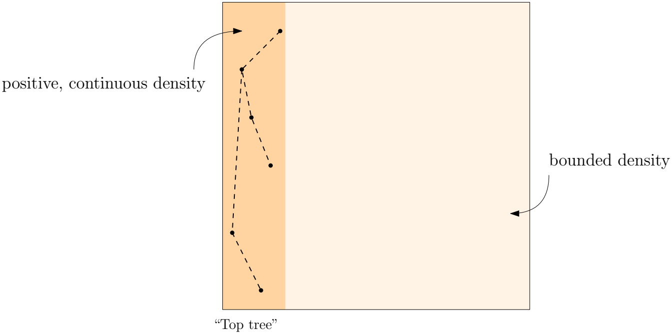

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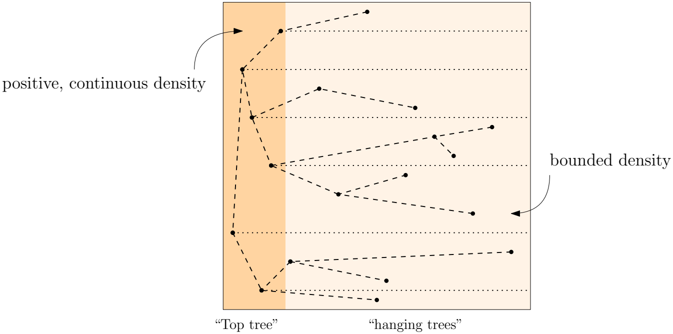

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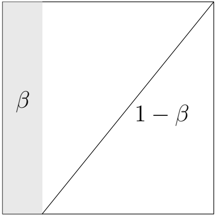
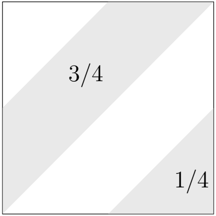
Necessity of assumptions:

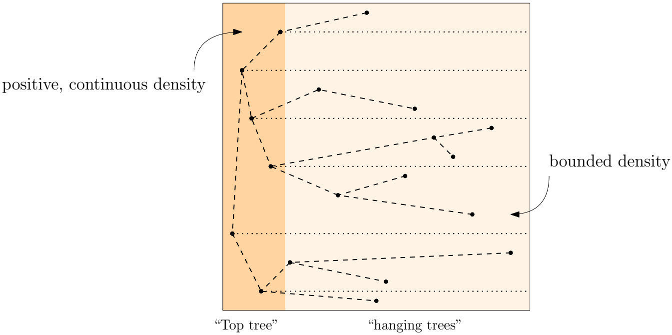


Height $\sim c_\beta \log n$ for some $c_\beta > c^*$



Height $\geq c\sqrt{n}$

III. Proof ideas



$$h\left(\mathcal{T}_{\mathrm{top}}\right) \;\leq\; h\left(\mathcal{T}\langle\sigma_\mu^n\rangle\right) \;\leq\; h\left(\mathcal{T}_{\mathrm{top}}\right) + 1 + \max_k h\left(\mathcal{T}^k_{\mathrm{hanging}}\right)$$

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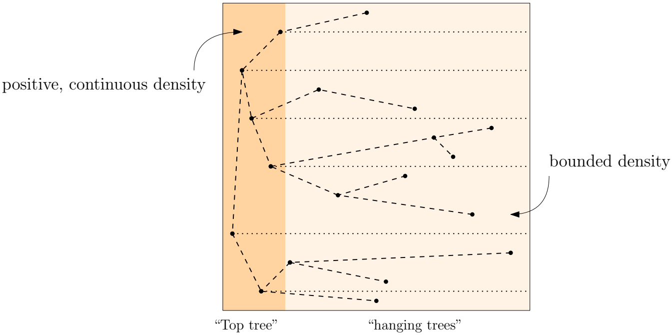

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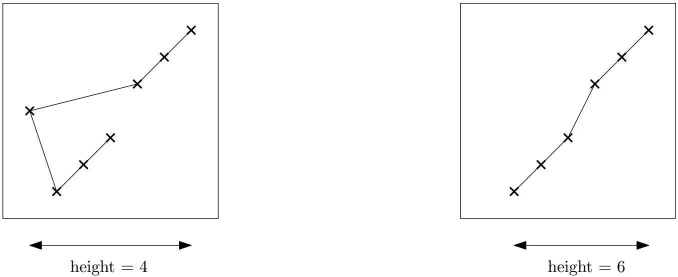
Problem: taking out a point might double the height...



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Let $\mathcal{P}_- \subseteq \mathcal{P}_+$ be two point sets.
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If $\mathcal{C}$ is a chain of maximal size in $\mathcal{T}\langle\mathcal{P}_+\rangle$ then:

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If $\mathcal{P}_- \subseteq \mathcal{P} \subseteq \mathcal{P}_+$, we can apply this lemma twice: with $\mathcal{P}_- \subseteq \mathcal{P}$, and with $\mathcal{P} \subseteq \mathcal{P}_+$. Thus:

$$h(\mathcal{T}\langle\mathcal{P}_+\rangle) - \underbrace{|\mathcal{C}_+ \cap (\mathcal{P}_+ \setminus \mathcal{P})|}_{\text{negligible?}} \leq h(\mathcal{T}\langle\mathcal{P}\rangle) \leq h(\mathcal{T}\langle\mathcal{P}_-\rangle) + \underbrace{|\mathcal{C} \cap (\mathcal{P} \setminus \mathcal{P}_-)|}_{\text{negligible?}}$$

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   $\longrightarrow$ (good?) estimate on $h(\mathcal{T}\langle\mathcal{P}\rangle)$;
3. "dePoissonization" techniques.

III. Proof ideas

Using Devroye's result:

If $\rho(x, y) = 1$, then σ_ρ^n is uniform and $h\left(\mathcal{T}\langle\sigma_\rho^n\rangle\right) \sim c^* \log n$.

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If $\rho(\mathsf{x}, \mathsf{y}) = \mathsf{f}(\mathsf{x}) \cdot \mathsf{g}(\mathsf{y})$, then σ_ρ^n is uniform and $h\left(\mathcal{T}\langle\sigma_\rho^n\rangle\right) \sim c^* \log n$.

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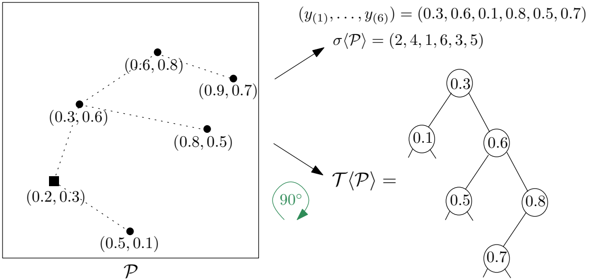
- Another result: the *subtree size convergence*. The BST of a permuton sample is not "balanced" in the same way as the BST of a uniform permutation.



Thank you for your attention!