The Moment Method Revisited

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Average Case Analysis in Anal. Comb.

$\mathcal{C}$ ... class of combinatorial objects

$\boxed{c_n = \#\mathcal{C}_n}$ ... number of objects in $\mathcal{C}$ of size n, $c_n = [z^n]\, C(z)$

$$C(z) = \sum_{n\geq 0} c_n z^n = \sum_{\omega\in\mathcal{C}} z^{\mathsf{size}(\omega)} \quad \text{... GF of } \mathcal{C}$$

$\boxed{c_{n,k} = \#\mathcal{C}_{n,k}}$ ... number of objects in $\mathcal{C}_n$, where some **parameter of interest** has value k

X_n ... random variable with $\mathbb{P}[X_n = k] = \dfrac{c_{n,k}}{c_n}$

$$C(z,u) = \sum_{n,k\geq 0} c_{n,k} z^n u^k = \sum_{n\geq 0} \left(\mathbb{E}[u^{X_n}]\right) c_n z^n \quad \text{... bivariate GF}$$

$$\mathbb{E}[u^{X_n}] = \sum_{k\geq 0} \frac{c_{n,k}}{c_n} u^k = \frac{[z^n]\, C(z,u)}{[z^n]\, C(z,1)} \qquad \boxed{\mathbb{E}\, X_n = \frac{[z^n]\, C_u(z,u)|_{u=1}}{[z^n]\, C(z,1)}}$$

Limiting Distribution

Weak Limit

A **sequence of random variables** Y_n converges **weakly** to a random variable Y, if

$$\boxed{\mathbb{E}\, G(Y_n) \to \mathbb{E}\, G(Y)}$$

for all bounded functionals G. **Notation**: $\boxed{Y_n \to Y}$.

Equivalently we have

$$\boxed{\mathbb{E}\, e^{itY_n} \to \mathbb{E}\, e^{itY}} \qquad \text{(for all real } t\text{)}$$

or

$$\boxed{\mathbb{P}[Y_n \le t] \to \mathbb{P}[Y \le t]}$$

(for all continuity points of the distribution function $F(t) = \mathbb{P}[Y \le t]$).

Limiting Distribution

Weak Limit with Moments

Theorem (the Moment Method)

Suppose that all moments $\mathbb{E}[Y^r]$, $r \geq 1$, of a random variable exist and determine uniquely the distribution of Y. Furthermore let Y_n be a sequence of random variables. If for all integers $r \geq 1$

$$\boxed{\mathbb{E}[Y_n^r] \to \mathbb{E}[Y^r]}$$

then Y_n **converges to** Y **weakly**: $\boxed{Y_n \to Y}$

Examples

Height in binary trees (Flajolet and Odlyzko, 1982)

H_n ... height of a binary tree of size n

$Y_n = \dfrac{H_n}{2\sqrt{n}}$ ... normalized height

$$\boxed{\mathbb{E}[Y_n^r] \to \mu_r = r(r-1)\Gamma(r/2)\zeta(r)} \implies \boxed{\frac{H_n}{2\sqrt{n}} \to Y}.$$

μ_r are the moments of the **theta distribution** Y with distribution function

$$F(t) = \sum_{k\in\mathbb{Z}} (1 - k^2t^2)e^{-k^2t^2}$$

and density

$$f(t) = 4t \sum_{k\geq 1} k^2(2k^2t^2 - 3)e^{-k^2t^2}$$

Examples

Selected Problems

- Path length in binary trees (Takács, 1992, 1994)
- Cost of linear probing hashing (Flajolet, Plobete, Viola, 1998)
- Maximum degree in triangulations (Gao and Wormald, 2000)
- etc. (many many examples!!!)

Moment Method for Central Limit Theorems

Moments of the Standard Normal Distribution $N(0,1)$.

$$\mu_{2r}^{(N)} = (2r-1)!!, \qquad \mu_{2r+1}^{(N)} = 0$$

Moment Method for a sequence of random variables X_n:

$$\mathbb{E}\,(X_n - \mathbb{E}\,X_n)^r = \sum_{\ell=0}^{r} (-1)^\ell \binom{r}{\ell} \mathbb{E}[X_n^{r-\ell}](\mathbb{E}\,X_n)^\ell \sim \mu_r^{(N)}\,(\mathbb{V}\mathrm{ar}\,X_n)^{r/2}.$$

$$\Longrightarrow \boxed{\frac{X_n - \mathbb{E}\,X_n}{\sqrt{\mathbb{V}\mathrm{ar}\,X_n}} \to N(0,1)}$$

Several cancellations of asymptotic leading terms !!

Moment Method for Central Limit Theorems

Asymptotics for Centered Moments [Hwang et al.]

$\mathbb{E}\,(X_n - \mathbb{E}\,X_n)^r$

- Recursive random variable: e.g. $X_n \equiv X_{I_n} + X_{n-1-I_n} + t_n$, ($I_n$ u.d. on $\{0, 1, \ldots, n-1\}$
- Recurrence for scaled moment generating function $\mathbb{E}[e^{t(X_n - \mathbb{E}X_n)}]$
- Asymptotic transfer results
- Asymptotics for centered moments

This method is technically highly involved !!

Hwang's Quasi-Power-Theorem

Theorem [Hwang]

Suppose that X_n is a sequence of random variables that satisfies

$$\boxed{\mathbb{E}[u^{X_n}] = e^{nf(u)+g(u)+O(1/n)}}$$

uniformly for complex u with $|u-1| < \eta$ and analytic functions $f(u)$ and $g(u)$ with $f(1) = g(1) = 0$, $f'(1) > 0$, and $f'(1) + f''(1) > 0$. Then we have

$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}(X_n)}} \to \mathcal{N}(0,1),$$

where

$$\mathbb{E}(X_n) = f'(1)n + O(1) \quad \text{and} \quad \mathbb{V}(X_n) = (f''(1) + f'(1))n + O(1).$$

Applications of Hwang's Quasi-Power-Theorem

Moving Singularities

$\boxed{C(z,u) \approx h(u)F(z/\rho(u))}$ with $F(x)$ singular at $x = 1$
(e.g. $F(x) = \sqrt{1-x}$)

$$\Longrightarrow \quad [z^n]\, C(z,u) \sim f_n h(u) \rho(u)^{-n}$$

$$\Longrightarrow \quad \boxed{\mathbb{E}[u^{X_n}] = \frac{[z^n]\, C(z,u)}{[z^n]\, C(z,1)} \sim \frac{h(u)}{h(1)} \left(\frac{\rho(1)}{\rho(u)}\right)^n}$$

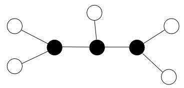
(System) of Functional Equations

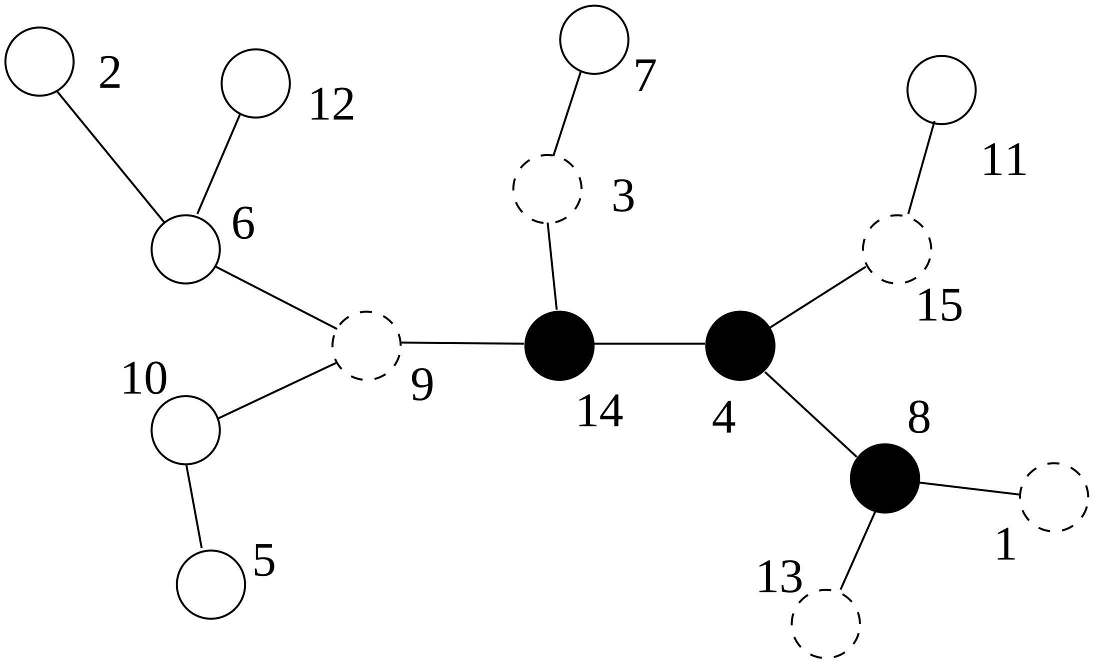
Unique combinatoral decompostions lead to recurrence relations that rewrite into a (system of) functional equation(s) for $C(z,u)$

$$C(z,u) = G(z,u,C(z,u))$$

and leads "automatically" to **moving singularities** (and to a CLT).

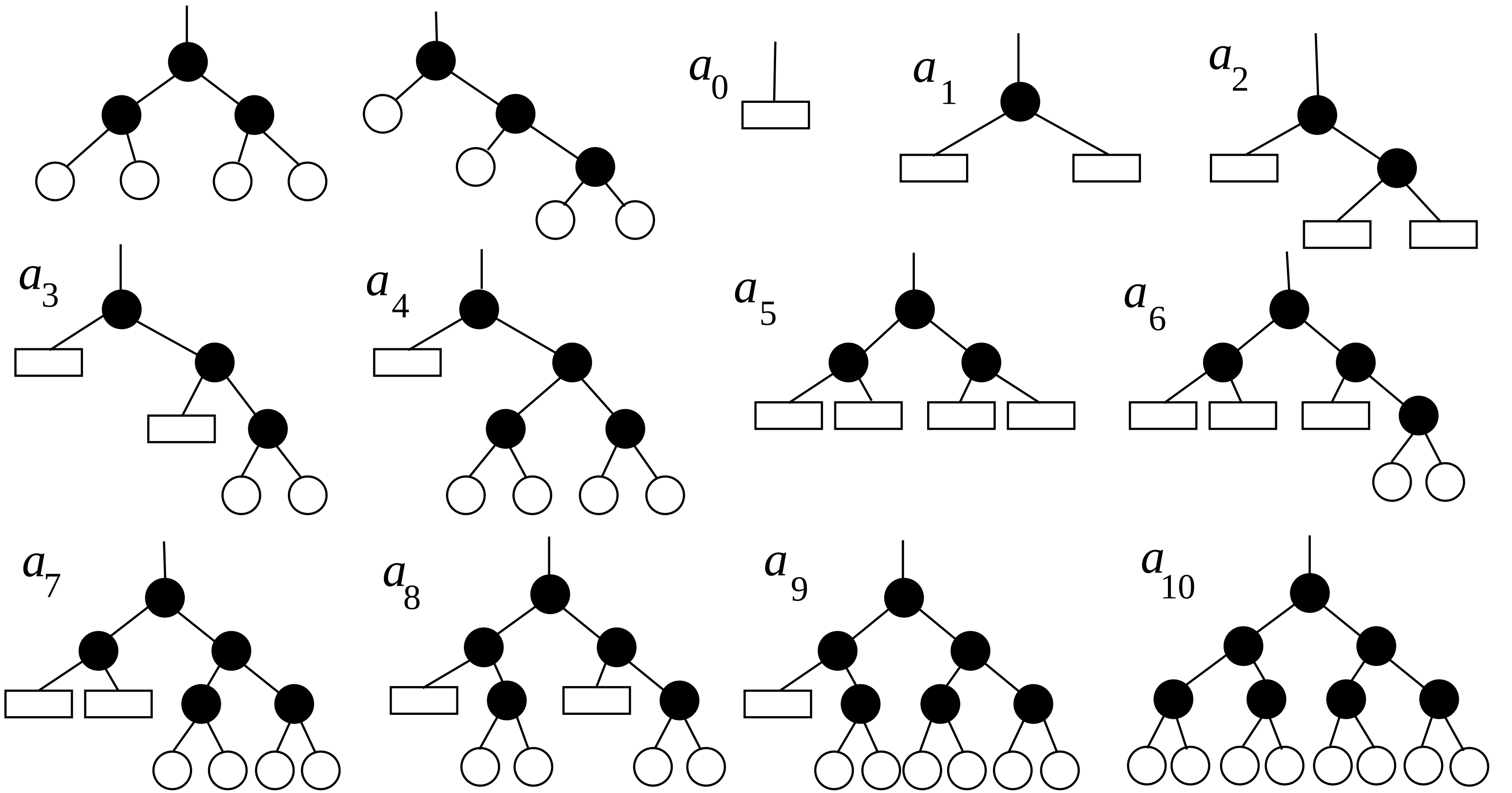
Pattern Occurences in Trees

Occurence of a pattern $\mathcal{M}$  in a labelled tree



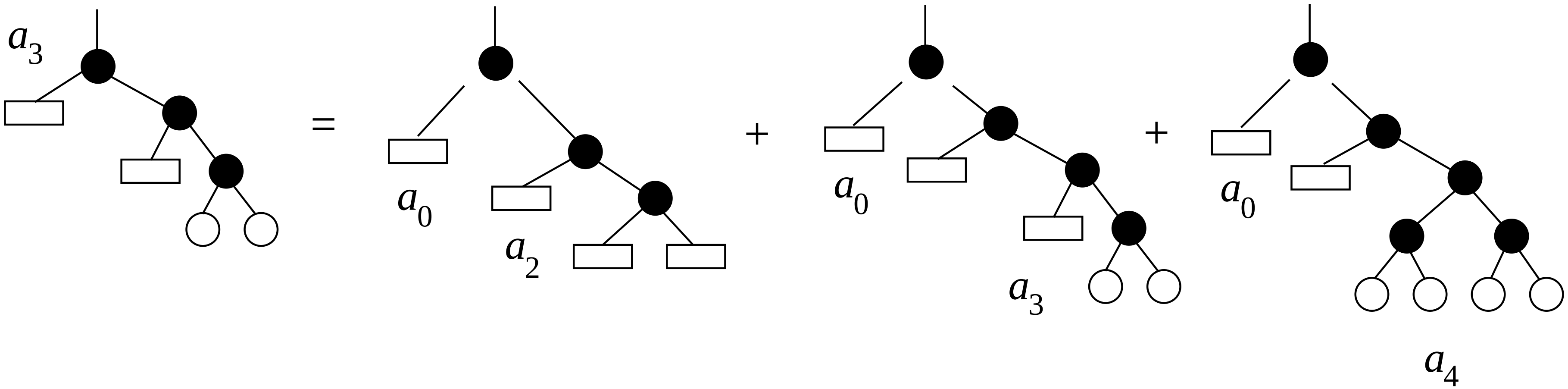
Pattern Occurences in Trees

Partition of trees in classes ( ... out-degree different from 2)



Pattern Occurences in Trees

Recurrences $A_3 = xA_0A_2 + xA_0A_3 + xA_0A_4$

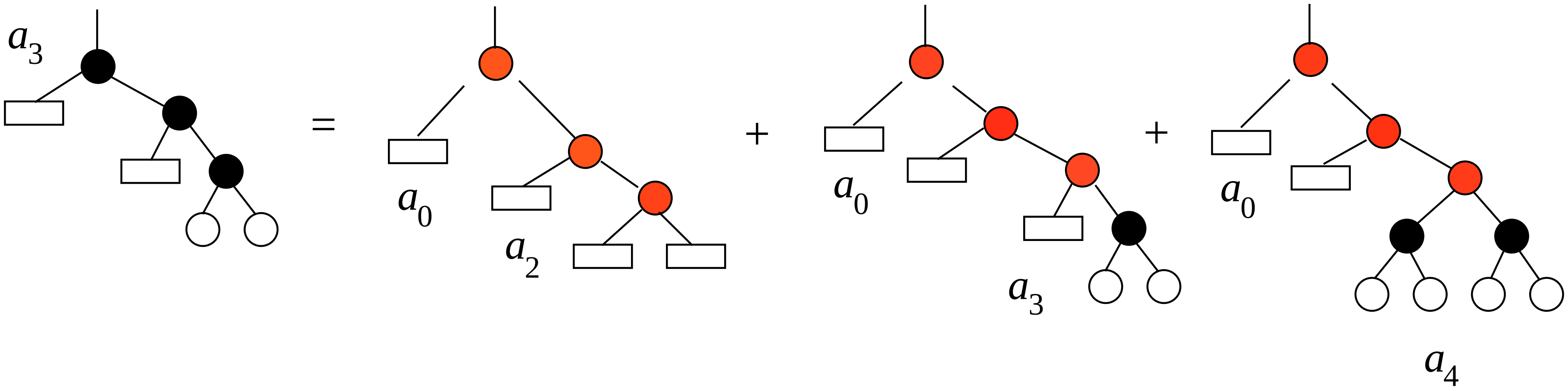


$$A_j(x) = \sum_{n,k} a_{j;n} \frac{x^n}{n!}$$

$a_{j;n}$ ... number of trees of size n in class j

Pattern Occurences in Trees

Recurrences $A_3 = xuA_0A_2 + xuA_0A_3 + xuA_0A_4$



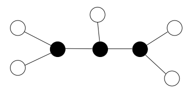
$$A_j(x,u) = \sum_{n,k} a_{j;n,m} \frac{x^n}{n!} u^m$$

$a_{j;n,m}$ ... number of trees of size n in class j with m occurences of $\mathcal{M}$

Pattern Occurences in Trees

$$
\begin{aligned}
A_0 &= A_0(x,u) = x + x\sum_{i=0}^{10} A_i + x\sum_{n=3}^{\infty}\frac{1}{n!}\left(\sum_{i=0}^{10} A_i\right)^n,\\
A_1 &= A_1(x,u) = \frac{1}{2}xA_0^2,\\
A_2 &= A_2(x,u) = xA_0A_1,\\
A_3 &= A_3(x,u) = xA_0(A_2+A_3+A_4)u,\\
A_4 &= A_4(x,u) = xA_0(A_5+A_6+A_7+A_8+A_9+A_{10})u^2,\\
A_5 &= A_5(x,u) = \frac{1}{2}xA_1^2u,\\
A_6 &= A_6(x,u) = xA_1(A_2+A_3+A_4)u^2,\\
A_7 &= A_7(x,u) = xA_1(A_5+A_6+A_7+A_8+A_9+A_{10})u^3,\\
A_8 &= A_8(x,u) = \frac{1}{2}x(A_2+A_3+A_4)^2u^3,\\
A_9 &= A_9(x,u) = x(A_2+A_3+A_4)(A_5+A_6+A_7+A_8+A_9+A_{10})u^4,\\
A_{10} &= A_{10}(x,u) = \frac{1}{2}x(A_5+A_6+A_7+A_8+A_9+A_{10})^2u^5.
\end{aligned}
$$

Pattern Occurences in Trees

Result for $\mathcal{M} =$ 

Central limit theorem for $(X_n - \mu n)/\sqrt{\sigma^2 n})$ with

$$\mu = \frac{5}{8e^3} = 0.0311169177\ldots$$

and

$$\sigma^2 = \frac{20e^3 + 72e^2 + 84e - 175}{32e^6} = 0.0764585401\ldots.$$

Theorem [Chyzak+D.+Klausner 2008]

For every tree pattern $\mathcal{M}$ the **number of it occurences** X_n in a random (labelled) tree of size n satisfies a **central limit theorem** with (asymptotically) linear mean and variance.

Subgraph Counts in Subcritical Graphs

Examples of subcritical graphs are **series-parallel graphs** or **outer-planar graphs**. They behave makroscopically like trees (they have the CRT as a scaling limit).

Theorem [D.+Ramos+Rue 2017]

Let $\mathcal{G}$ be a given subcritical class of (labelled) graphs. Then for every given graph H the **number of it occurences** X_n (as subgraphs) in a graph $\mathcal{G}$ of size n satisfies a **central limit theorem** with (asymptotically) linear mean and variance.

Here we need an **infinite system of equations** (that can be analyzed since the Jacobian of the system is a compact operator).

Factorial Moments

Problem.

What can we do if we expect a central limit theorem but the bivariate generating function $C(z, u)$ cannot be described in a proper way (explicitly or implicitly)?

Observation

Usually we can compute (factorial) moments.

Factorial Moments

$(x)_r = x(x-1)(x-2)\cdots(x-r+1)$ ... falling factorials

Factorial Moments

$$\boxed{\mathbb{E}\,(X)_r = \mathbb{E}[X(X-1)(X-2)\cdots(X-r+1)] = \frac{\partial^r}{\partial u^r}\mathbb{E}[u^X]\Big|_{u=1}}$$

They can be computed by the **Bivariate generating functions**

$$C(z,u) = \sum_{n,k\ge 0} c_{n,k} z^n u^k = \sum_{n\ge 0}\left(\mathbb{E}[u^{X_n}]\right) c_n z^n$$

$$\Longrightarrow \quad \boxed{\mathbb{E}[(X_n)_r] = \frac{[z^n]\,\frac{\partial^r}{\partial u^r}C(z,u)|_{u=1}}{[z^n]\,C(z,1)}}$$

Factorial Moments

... or by a

Combinatorial interpretation

Suppose that the parameter of interest is a **counting parameter**, e.g. the number of leaves in a tree or the number of triangles in a graph.

The factorial moment

$$\boxed{\mathbb{E}[(X_n)_r] = \frac{1}{c_n} \sum_{k \ge 0} k(k-1)\cdots(k-r+1) c_{n,k}}$$

is also the **number of objects of size n, where r different appearances of the parameter (that is considered) are marked** (and the order or marks is important) **divided by the number of objects of size n.**

Factoral Moments of the Binomial Distr.

$X_n \; \ldots \; Bi(n,p), \qquad \dfrac{X_n - np}{\sqrt{p(1-p)n}} \to N(0,1).$

$$\mathbb{E}\, u^{X_n} = (1-p+up)^n$$

$$\begin{aligned}\mathbb{E}[(X_n)_r] &= \frac{\partial^r}{\partial u^r}(1-p+up)^n\Big|_{u=1} \\ &= n(n-1)\cdots(n-r+1)p^r \sim (np)^r e^{-r^2/(2n)}\end{aligned}$$

for $\boxed{r = O(\sqrt{n})}$

QUESTION. Suppose that $\mathbb{E}\,(X_n)_r \sim (np)^r e^{-r^2/(2n)}$ for $\boxed{r = O(\sqrt{n})}$.

Does it follow that $\dfrac{X_n - np}{\sqrt{p(1-p)n}} \to N(0,1)$?

Factorial Moment Method by Gao and Wormald

$\mathbb{E}[(X)_r] = \mathbb{E}[X(X-1)\cdots(X-r+1)]$ ... r-th factorial moment

Lemma [Gao+Wormald 2004]

Suppose that $\mathbb{E}[X_n] = \mu_n \to \infty, \mathbb{V}\text{ar}[X_n] = \sigma_n^2 = o(\mu_n^2/\log^4 n), \mu_n = o(\sigma_n^3)$ and $(X_n)_{n\geq 1} \geq 0$ satisfies

$$\boxed{\mathbb{E}\left[(X_n)_r\right] \sim \mu_n^r \exp\left(\frac{r^2}{2}\frac{\sigma_n^2 - \mu_n}{\mu_n^2}\right)}$$

uniformly for all r in the range $\boxed{c\mu_n/\sigma_n \leq r \leq c'\mu_n/\sigma_n}$ for some constants $c' > c > 0$. Then

$$\boxed{\frac{X_n - \mu_n}{\sigma_n} \to \mathcal{N}(0,1)}.$$

Remark. $X_n \sim Bi(n,p)$, $\mu_n = np$, $\sigma_n^2 = p(1-p)n$, $r = \Theta(\sqrt{n})$.

Quasi-Powers and Factorial Moments

Lemma

Suppose that X_n is a sequence of random variables that satisfies

$$\boxed{\mathbb{E}[u^{X_n}] = e^{nf(u)+g(u)+O(1/n)}}$$

uniformly for complex u with $|u-1| < \eta$ and analytic functions $f(u)$ and $g(u)$ with $f(1) = g(1) = 0$ and $f'(1) > 0$. Then we have

$$\boxed{\mathbb{E}[(X_n)_r] \sim (nf'(1))^r \exp\left(\frac{r^2}{2n}\frac{f''(1)}{(f'(1))^2}\right)}$$

uniformly for $0 \le r \le C\sqrt{n}$, where $C > 0$ is an arbitrary constant.

Remark. This is consistent with Hwang's Quasi-Power-Theorem.

Quasi-Powers and Factorial Moments

Proof

$$\mathbb{E}[(X_n)_r] = r!\frac{1}{2\pi i}\int_\gamma \frac{\mathbb{E}[u^{X_n}]}{(u-1)^{r+1}}\,du,$$

γ is a cycle with center 1 and radius $\rho = r/(nf'(1))$: $u = 1 + \rho e^{i\varphi}$

$$\begin{aligned}\mathbb{E}[(X_n)_r] &= \frac{r!}{2\pi}\int_{-\pi}^{\pi} e^{nf'(1)\rho e^{i\varphi}+\frac{n}{2}f''(1)\rho e^{2i\varphi}+O(n\rho^3+\rho+1/n)}\rho^{-r}e^{-ir\varphi}\,d\varphi \\ &= (nf'(1))^r e^{\frac{r^2}{2n}\frac{f''(1)}{(f'(1))^2}}\frac{r!}{2\pi r^r e^{-r}} \\ &\quad\times\int_{-\pi}^{\pi} e^{r(e^{i\varphi}-1-i\varphi)+\frac{r^2}{2n}\frac{f''(1)}{(f'(1))^2}(e^{2i\varphi}-1)+O(\frac{r^3}{n^2}+\frac{r+1}{n})}\,d\varphi.\end{aligned}$$

The last integral evaluates (uniformly for $r \le C\sqrt{n}$) by Laplace's method

$$\int_{-\pi}^{\pi} e^{r(e^{i\varphi}-1-i\varphi)+\frac{r^2}{2n}\frac{f''(1)}{(f'(1))^2}(e^{2i\varphi}-1)+O(\frac{r^3}{n^2}+\frac{r+1}{n})}\,d\varphi \sim \int_{-\infty}^{\infty} e^{-\frac{r}{2}\varphi^2}\,d\varphi = \sqrt{\frac{2\pi}{r}}.$$

Applications of the Factorial Moment Method

- Gao+Wormald 2004: submap counts in random planar triangulations

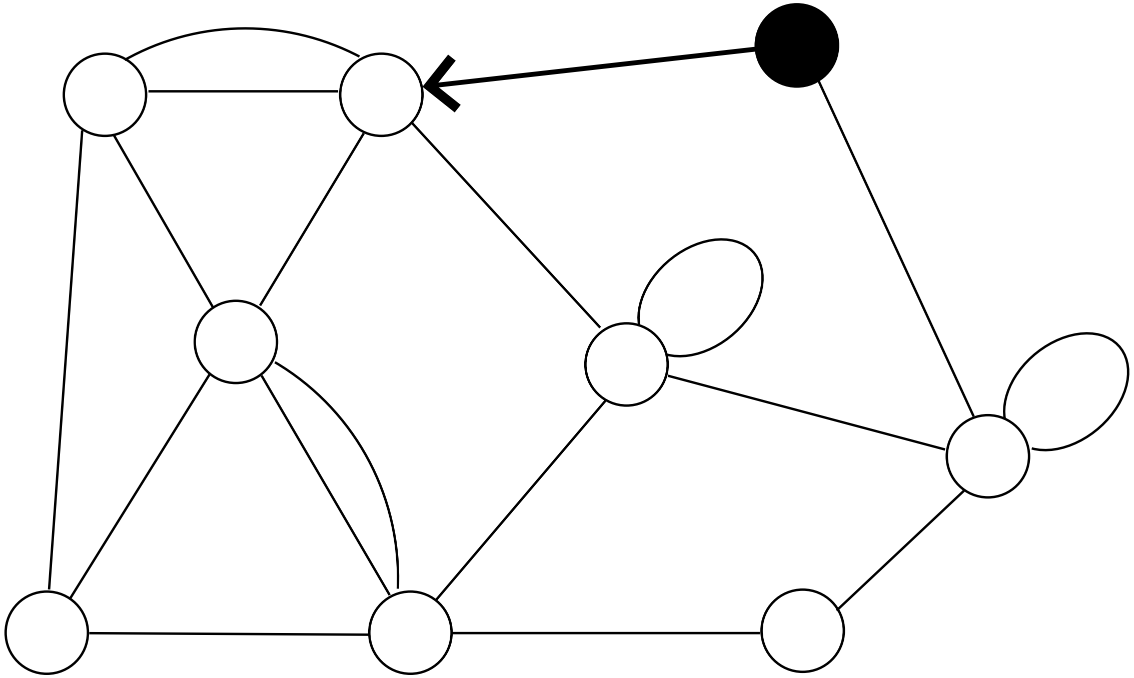
- Cai+Devroye 2017: subtrees in conditional Galton-Watson trees

- Hitczenko+Wormald 2023+: balls in bins in a classical allocation scheme (multivariate version)

- Ojeda+Holmgren+Janson 2023+: Fringe trees for random trees with given vertex degrees (multivariate version)

- **NEW:** D.+Hainzl+Wormald 2024+: pattern counts in random planar maps

Planar Maps



A **planar map** is a connected planar graph, possibly with loops and multiple edges, together with an embedding in the plane.
A map is **rooted** if a vertex v and an edge e incident with v are distinguished, and are called the root-vertex and root-edge, respectively. The face to the right of e is called the root-face and is usually taken as the **outer face**.

Planar Maps

M_n ... number of rooted maps with n edges [Tutte]

$$\boxed{M_n = \frac{2(2n)!}{(n+2)!n!}3^n}$$

The proof is given with the help of generating functions and the so-called **quadratic method**.

Asymptotics:

$$M_n \sim \frac{2}{\sqrt{\pi}} \cdot n^{-5/2}\, 12^n$$

Generating Function:

$$M(z) = \sum_{n\ge 0} M_n z^n = -\frac{1}{54z^2}\left(1 - 18z - \boxed{(1-12z)^{3/2}}\right)$$

Planar Maps

Quadratic Method

$M_{n,k}$ ... number of maps with n edges and outer-face-valency k

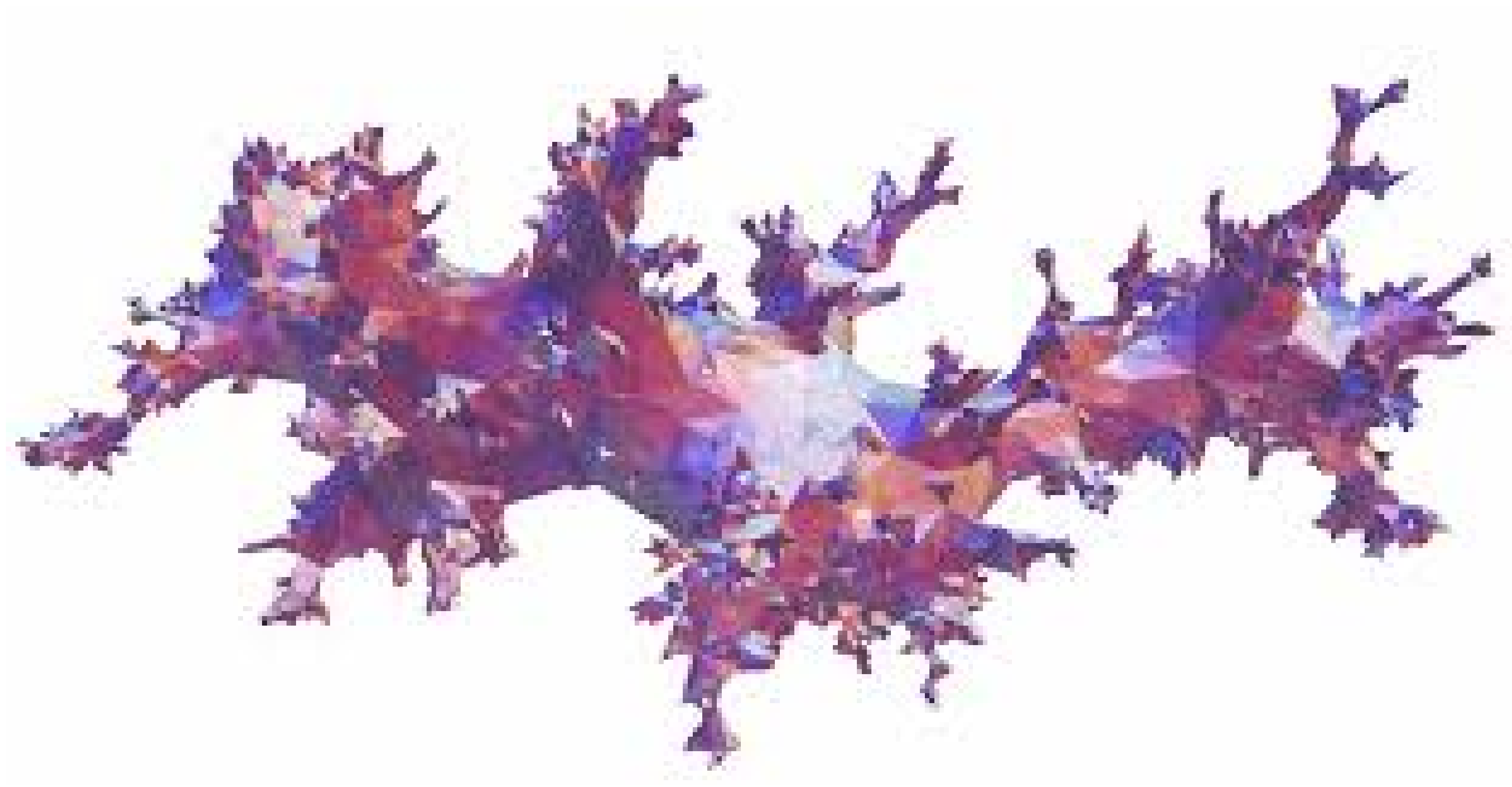
$$M(z,v) = \sum_{n,k} M_{n,k} v^k z^n$$

$$\boxed{M(z,v) = 1 + zv^2 M(z,v)^2 + vz\frac{vM(z,v) - M(z,1)}{v-1}.}$$

By binding z and v by a proper function $\boxed{v = v(z)}$ this equation can be solved and we get

$$M(z) = M(z,1) - \frac{1}{54z^2}\left(1 - 18z - (1-12z)^{3/2}\right).$$

Random Planar Maps



Picture by Nicolas Curien

Classes of Planar Maps

- Bipartite / Eulerian planar maps
- Quadrangulations / 4-regular planar maps
- Triangulations / 3-regular planar maps
- 2-connected planar maps
- 3-connected planar maps
- ...

Scaling Limit of Planar Maps

$\mathcal{M}_n$ ... random planar map with n edges
$\mathcal{Q}_n$ ... random quadrangulation with n edges

Theorem [Miermont 13, Le Gall 13, Bettinelli+Jacob+Miermont 14]

We have in distribution for the Gromov-Hausdorff topology

$$\left(\frac{9}{8n}\right)^{1/4} \mathcal{M}_n \to \mathcal{S}, \qquad \left(\frac{9}{8n}\right)^{1/4} \mathcal{Q}_n \to \mathcal{S},$$

where $\mathcal{S}$ denotes the **Brownian Map**.

In particular, the typical distance of two vertices is of order $n^{1/4}$

Local Limit of Planar Maps

$U_r(M)$ ... rooted map induced by those vertices of the rooted map M with **distance** $< r$ **from the root vertex of** M.

Theorem [Ménard+Nolin 2014, Stephenson 2016]

We have for any rooted map M *and for all* $r > 0$

$$\lim_{n\to\infty} \mathbb{P}[U_r(\mathcal{M}_n) = M] = \mathbb{P}[U_r(\mathrm{s}) = M],$$

where s *denotes the* **Uniform Infinite Planar Map**.

Remark. There are similar results for triangulations by [Angel+Schramm 2003], for quadrangulations by Krikum 2005+] and by [Curien+Ménard+Miermont 2013], and for bipartite maps by [Björnberg+Stefánsson 2014].

Local Limit of Planar Maps

$\mathcal{M}_n^\bullet$ ... random planar map with n edges and a **randomly chosen distinguished vertex** v

$U_r^v(M)$ ... rooted map induced by those vertices of M the **with distance** $< r$ **from the vertex** v (of M)

Corollary [D.+Stufler 2019]

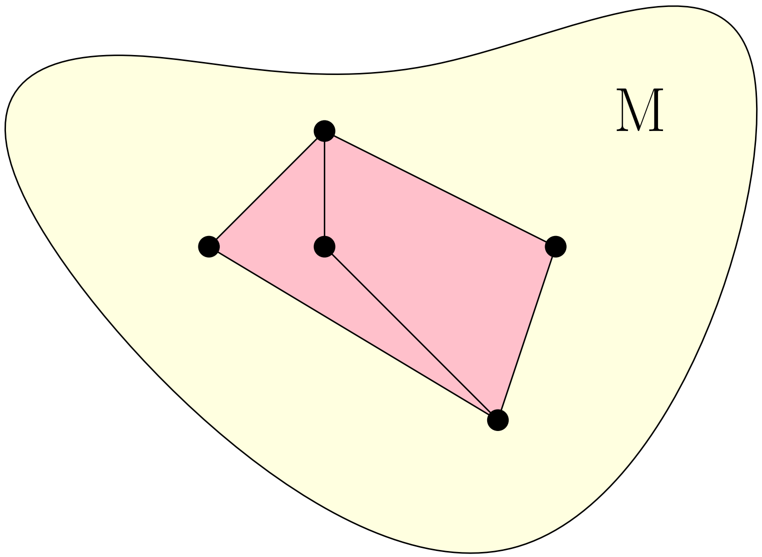
We have for every vertex rooted map $\tilde{M}$ *and for all* $r > 0$

$$\lim_{n\to\infty} \mathbb{P}[U_r^v(\mathcal{M}_n^\bullet) = \tilde{M}] = \mathbb{P}[U_r^v(\mathbf{s}^*) = \tilde{M}],$$

where $\mathbf{s}^*$ *denotes the corresponding* **Benjamini-Schramm limit**.

Pattern in a Planar Map

A **pattern** P in a planar map M is a planar map, if M can be constructed by adding successively faces to the outer face of P and to the outer faces of the appearing maps.



Simplest pattern: face of valency r

Coloured pictures by Eva-Maria Hainzl

Local Limit of Planar Maps

P ... **planar pattern**

$X_n^{(P)}$ ... number of occurrences of P in M_n

Theorem [D.+Stufler 2019]

There exists $c(P) > 0$ *with*

$$\boxed{\mathbb{E}\, X_n^{(P)} \sim c(P)\, n.}$$

Problem. What can be said about the difference $X_n^{(P)} - \mathbf{E}\, X_n^{(P)}$?

Is there always a Central Limit Theorem ?

Results

- Submaps counts of a 3-connected map N in 2-connected triangulations satisfy a CLT. [Gao+Wormald 2004]

- The number of faces of degree r in (2-connected) random planar maps satisfy a CLT. [D+Panagiotou 2013, Collet+D.+Klausner+Kok 2019]

- Double-triangles in random planar maps [D.+Yu 2018]

- **NEW**: pattern with simply boundary [D.+Hainzl+Wormald]

Methods

Proof Methods for CLT

- **Bijective method** with mobiles (restricted to face valencies, no other pattern, no connectivity assumptions): **Quasi-Power-Th.**

- **Quadratic method** (face valencies, pattern without self-intersections, several map classes): **Quasi-Power-Th.**

- **NEW**; **Gao-Wormald-Moment-Method with "Quasi-Power-Preprocessing"** (pattern with simple boundary)

Bijective Method

Theorem 1 [Collet+D.+Klausner 2019]

Ω ... an arbitrary set of positive integers, not a subset of $\{1,2\}$

$\mathcal{M}_\Omega$ ... planar rooted maps such that all face valencies are in Ω

$X_n^{(r)}$ ... **number of faces of valency** r in a random planar map in $\mathcal{M}_\Omega$. $(r \in \Omega)$

Then we have

$$\mathbb{E}[X_n^{(r)}] = \mu_r n + O(1), \quad \mathbb{V}ar[X_n^{(k)}] = \sigma_r^2 n + O(1)$$

for certain constants $\mu_r > 0$, $\sigma_r^2 \geq 0$, and

$$\boxed{\frac{X_n^{(r)} - \mathbb{E}[X_n^{(r)}]}{\sqrt{n}} \to N(0, \sigma_r^2)}.$$

Bijective Method

Examples.

$\Omega = \{3\}$ ... triangulations

$\Omega = \{4\}$ ... quadrangulations

$\Omega = 2\mathbb{N}$ ... bipartiite maps

$\Omega = \mathbb{N}$ ... all maps

$\Omega = \mathbb{P} = \{2, 3, 5, 7, \ldots\}$ ... all face valencies are prime numbers

...

Bijective Method

Remark.

$M_{\Omega,n}$ ... number of maps in $\mathcal{M}_\Omega$ with n edges

Then there exist positive constants c_Ω and γ_Ω with

$$\boxed{M_{\Omega,n} \sim c_\Omega n^{-5/2} \gamma_\Omega^n, \qquad n \equiv 0 \bmod d},$$

where $d = \gcd\{i : 2i \in \Omega\}$ if Ω contains only even numbers, otherwise $d = 1$.

The exponent $-5/2$ is **universal**.

Mobiles

Definition.

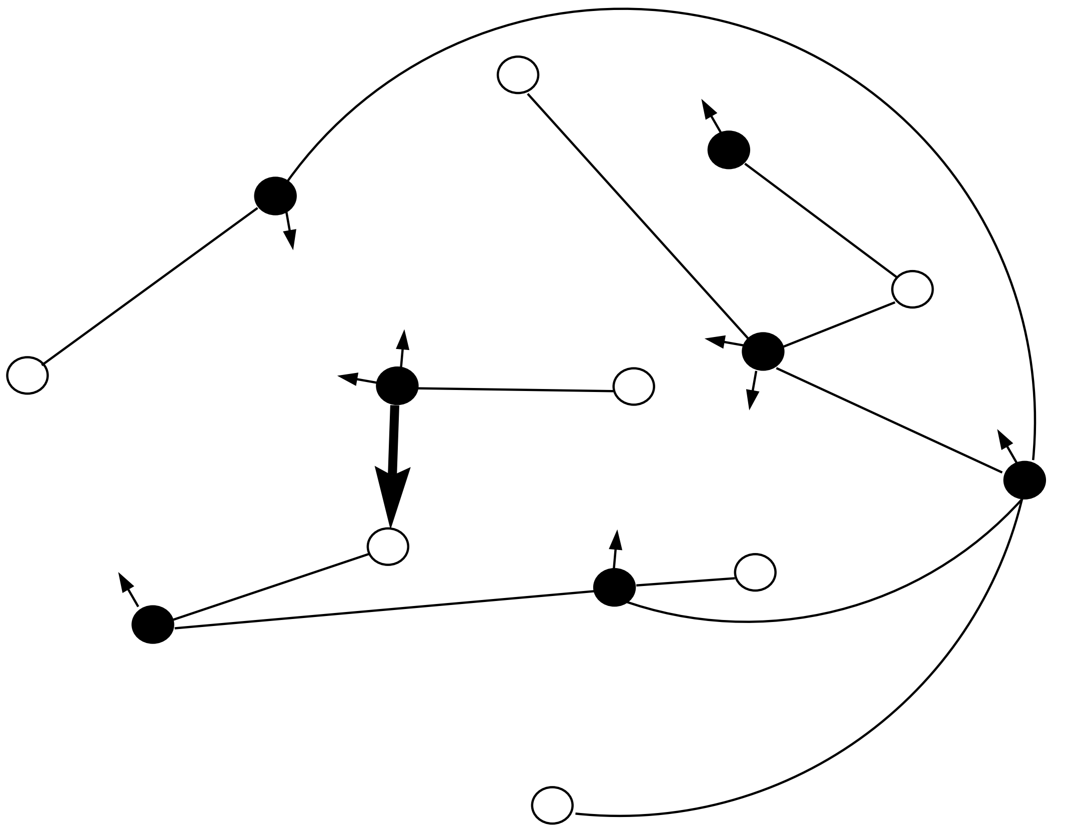
A **mobile** is a **planar tree** – that is, a map with a single face – such that there are **two kinds of vertices** (black and white) with **no white-white edges**, and black vertices additionally have so-called **“legs”** attached to them (which are not considered edges), whose number equals the number of white neighbor vertices.

A **bipartite mobile** is a mobile without black–black edges.

The **degree** of a black vertex is the number of half-edges plus the number of legs that are attached to it.

A mobile is called **rooted** if an edge is distinguished and oriented.

Mobiles



Mobiles

Theorem [Cori+Vauquelin, Schaeffer, Bouttier+Di Francesco+Guitter, Bernardi+Fusy, Collet+Fusy, ...]

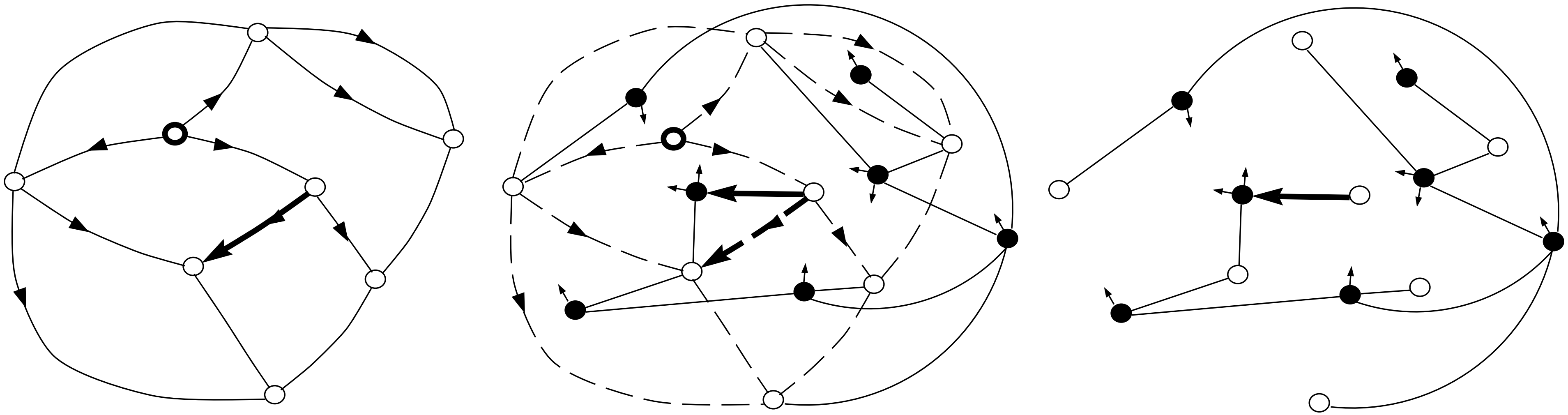
There is a **bijection** *between* **mobiles** *that contain at least one black vertex and* **pointed planar maps***, where* white vertices *in the mobile correspond to* non-pointed vertices *in the equivalent planar map,* black vertices *correspond to* faces *of the map, and the* degrees of the black vertices *correspond to the* face valencies*.*

This bijection induces a bijection on the edge sets *so that the number of edges is the same. (Only the pointed vertex of the map has no counterpart.)*

Similarly, rooted mobiles *that contain at least one black vertex are in bijection to* rooted and vertex-pointed planar maps*.*

Finally, bipartite mobiles *with at least two vertices correspond to* bipartite maps *with at least two vertices, in the unrooted as well as in the rooted case.*

Mobiles



Mobiles and Maps

- $L(t, z, x_1, x_2, \ldots)$ ... mobiles rooted at a black vertex and where an additional edge is attached to the black vertex (the x_i "count" the number of black vertices of degree i)

- $Q(t, z, x_1, x_2, \ldots)$ ... mobiles rooted at a univalent white vertex, which is not counted,

- $R(t, z, x_1, x_2, \ldots)$ ... mobiles rooted at a white vertex and where an additional edge is attached to the root vertex.

$$B_{\ell,m} = \binom{l+2m}{l,m,m}$$

$$B_{\ell,m}^{(+1)} = \binom{l+2m+1}{l,m,m+1}$$

$$\overline{B}_{\ell,m} = \frac{l+m}{l+2m}\binom{l+2m}{l,m,m}$$

Mobiles and Maps

Lemma

The generating functions $L = L(t, z, x_1, x_2, \ldots)$, $Q = Q(t, z, x_1, x_2, \ldots)$, *and* $R = R(t, z, x_1, x_2, \ldots)$ *satisfy the system of equations*

$$\boxed{\begin{array}{rcl} L & = & z\sum_{\ell,m} x_{2m+\ell+1} B_{\ell,m} L^\ell R^m, \\ Q & = & z\sum_{\ell,m} x_{\ell+2m+2} B^{(+1)}_{\ell,m} L^\ell R^m, \\ R & = & \frac{tz}{1-Q}. \end{array}}$$

Let $T = T(t, z, x_1, x_2, \ldots)$ *be given by*

$$T = 1 + \sum_{\ell,m} x_{2m+\ell} \overline{B}_{\ell,m} L^\ell R^m,$$

Then the generating function $M = M(t, z, x_1, x_2, \ldots)$ *of* **rooted maps** *satisfies*

$$\boxed{\frac{\partial M}{\partial t} = R/z - t + T},$$

where the variable t *corresponds to the number of vertices,* z *to the number of edges, and* x_i, $i \ge 1$, *to the number of faces of valency* i.

Extensions and Limitation

Extensions

• Local limit theorems

Limitations

• There is **no control on adjacent vertices**. Thus, we cannot handle more complicated patterns.

• There is **no control on the level of connectivity**. We cannot handle 2-connected or 3-connected maps.

Quadratic Method

Theorem 2 [D.+Panagiotou 2013]

$X_n^{(r)}$ ... **number of faces of valency** r in a **random planar map** with n edges or in a **random** 2-**connected map** with n edges.

Then we have

$$\mathbb{E}[X_n^{(r)}] = \mu_r n + O(1), \quad \mathbb{V}\mathrm{ar}[X_n^{(r)}] = \sigma_r^2 n + O(1)$$

with constants $\mu_r > 0$, σ_r^2 and

$$\boxed{\frac{X_n^{(r)} - \mathbb{E}[X_n^{(r)}]}{(\mathbb{V}\mathrm{ar}[X_n^{(r)}])^{1/2}} \to N(0,1)}.$$

Remark. The same result holds for **pure** r-**gons** (all vertices are different).

Quadratic Method

Generating functions

$M_{n,k}$ ... number of maps with n edges and outer-face-valency k

$$M(z,v) = \sum_{n,k} M_{n,k} v^k z^n$$

$$\boxed{M(z,v) = 1 + zv^2 M(z,u)^2 + vz\frac{vM(z,v) - M(z,1)}{v-1}}$$

v ... **"catalytic variable"**

Analytic Quadratic Method

(Crucial) Lemma

Suppose that we have a catalytic equation of the form

$$P(z, v, \mathbf{u}, M(z, v, \mathbf{u}), M_1(z, \mathbf{v})) = 0$$

such that for $\mathbf{u} = \mathbf{1}$ *the solution* $M_1(z, \mathbf{1})$ *has a singular behaviour of the form*

$$M_1(z, \mathbf{1}) = g(z) + h(z) \left(1 - \frac{z}{\rho}\right)^{3/2}$$

Then (under weak technical conditions) we have

$$\boxed{M_1(z, \mathbf{u}) = \tilde{g}(z, \mathbf{u}) + \tilde{h}(z, \mathbf{u}) \left(1 - \frac{z}{\tilde{\rho}(\mathbf{u})}\right)^{3/2}}$$

with $\tilde{g}(z, \mathbf{1}) = g(z)$*,* $\tilde{h}(z, \mathbf{1}) = h(z)$*, and* $\tilde{\rho}(\mathbf{1}) = \rho$ *(for* $\mathbf{u}$ *in a small neighborhood of* $\mathbf{1}$*).*

Faces of given valency

$M_{n,k,\ell}$ ... number of maps with n edges, root face valency k and ℓ non-root faces of valency r

$$M(z,v,u) = \sum_{n,k,\ell \ge 0} M_{n,k,l} z^n v^k u^\ell,$$

Lemma [D.+Panagiotou, 2013]

$$M(z,v,u) = 1 + zv^2 M(z,v,u)^2 + zv \frac{M(z,1,u) - vM(z,v,u)}{1-v}$$

$$+ z(u-1)v^{-r+2} \left(M(z,v,u) - \sum_{\ell=0}^{r-2} M_\ell(z,u) v^\ell \right),$$

where $M_\ell(z,u) = [v^\ell] M(z,v,u)$.

Remark. The same method works for 2-connected planar maps.

Double-Triangles

Theorem 3 [D.+Yu 2018]

X_n ... **number of double-triangles** in a random planar map with n edges (counted by the number or edges where both adjacent faces have valency 3).

Then we have

$$\mathbb{E}[X_n] = \mu n + O(1), \quad \mathbb{V}\mathrm{ar}[X_n] = \sigma^2 n + O(1)$$

with constants $\mu > 0$, $\sigma^2 > 0$ and

$$\boxed{\frac{X_n - \mathbb{E}[X_n]}{(\mathbb{V}\mathrm{ar}[X_n])^{1/2}} \to N(0,1)}.$$

Double-Triangles

$D_{n,k,\ell}$ ... number of maps with n edges, root face valency k and ℓ edges outside the root face, where both adjacent faces are triangles

$$D(z,v,u) = \sum_{n,k,\ell\ge 0} D_{n,k,\ell} z^n v^k u^\ell,$$

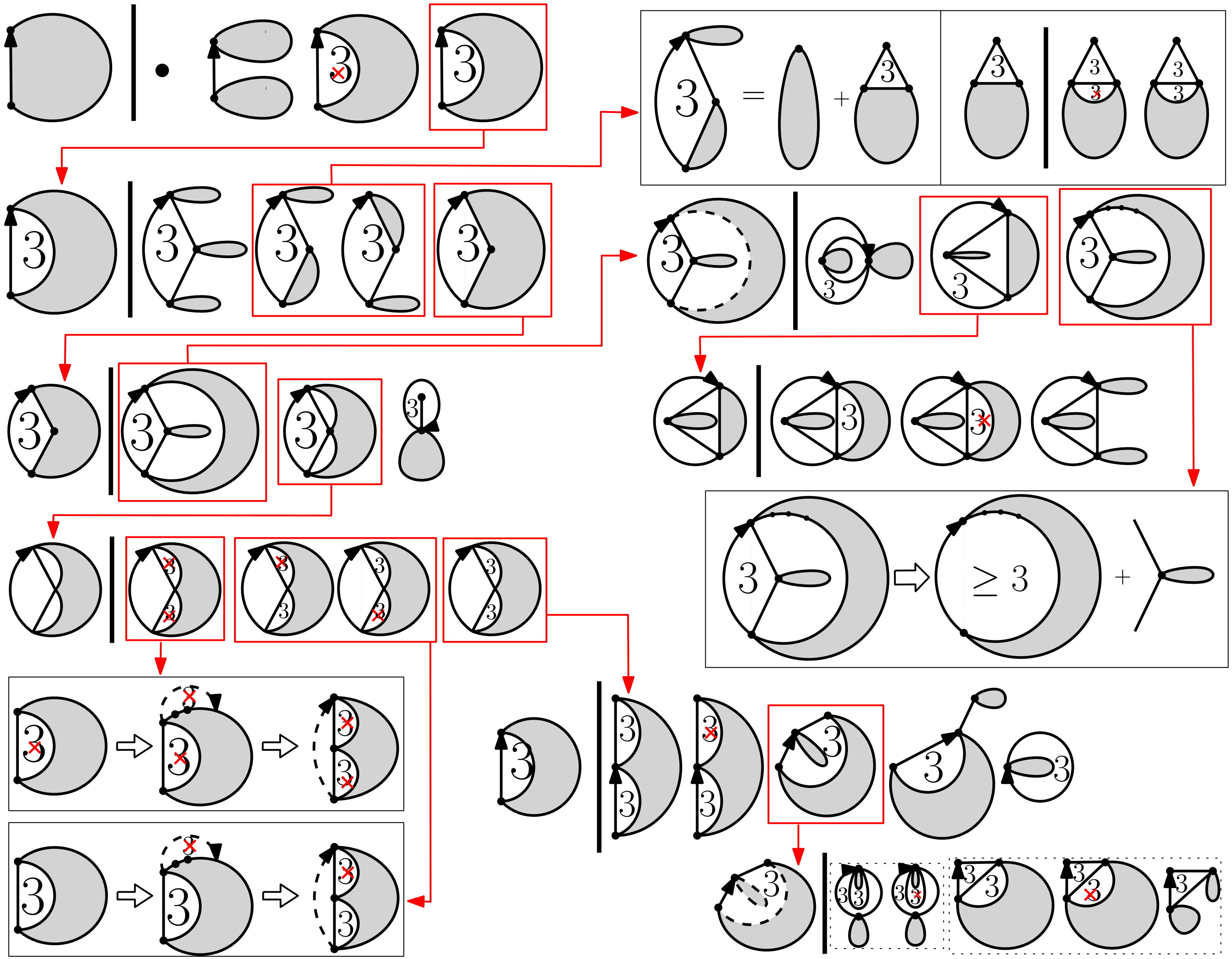
Lemma

$$\begin{aligned}
D &= 1 + zv^2D^2 + D_{\not\triangleright} + D_{\triangleright},\\
D_{\not\triangleright} &= zv\frac{D(1)-vD}{1-v} - zv^{-1}\left(D - 1 - v[v^1]D\right),\\
D_{\triangleright} &= zv^{-1}\left(D - 1 - v[v^1]D\right) + (u-1)\Big[z^2vD - z^2v(u-1)DD_{\triangleright}\\
&\quad + (u+1)\left(zv^{-1}D_{\triangleright} - z[v^1]D_{\triangleright}\right) - (u-1)P(D_{\triangleright})\Big]
\end{aligned}$$

with

$$P(D_{\triangleright}) = z^2\frac{D_{\triangleright}(1)-vD_{\triangleright}}{1-v} - z^2D_{\triangleright}(1) - z^2v^{-2}\left(D_{\triangleright} - v[v^1]D_{\triangleright} - v^2[v^2]D_{\triangleright}\right).$$

Double-Triangles



Double-Triangles

Proof strategy

- Combinatorics leads to a **system of catalytic equations**
- Extension of **quadratic method**
- This leads to a **central limit theorem** with the help of the **Crucial Lemma** and the **Quasi-Power-Theorem**.

General Pattern

P ... pattern

$X_n^{(P)}$ ... number of occurences of P in a random planar map of size n

M_n ... number ob maps of size n

$M_n^{(r)}$ ... number of maps of size n with r distinguished (and ordered) occurences of P

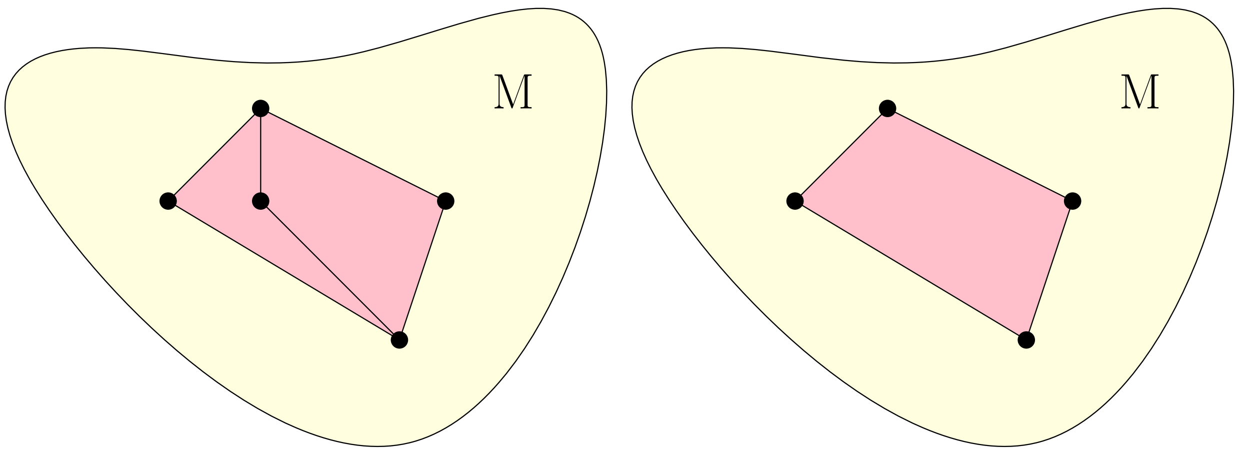
Then

$$\boxed{\mathbb{E}[(X_n^{(P)})_r] = \frac{M_n^{(r)}}{M_n}},$$

where $\mathbb{E}[(X)_r] = \mathbb{E}[X(X-1)\cdots(X-r+1)]$ denotes the r-th factorial moment.

General Pattern

- Delete interior edges of pattern
- Count maps with distinguished pure polygon face
- Insert interior edges back



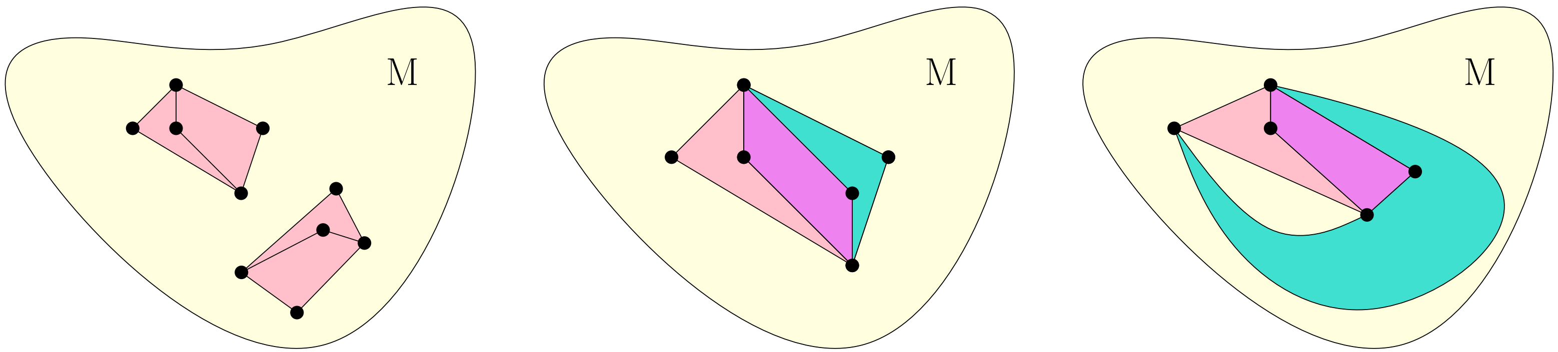
Moment Method

$M(z, v, u)$ ... generating function, where u counts pure 4-gons

$$
\begin{aligned}
\mathbb{E}[X_n^{(P)}] &= \left.\frac{[z^n]\, 2\, z^2\, \partial_u M(z,v,u)}{[z^n] M(z,v,u)}\right|_{v=u=1} \\
&= 2\, \left.\frac{[z^{n-2}]\partial_u M(z,v,u)}{[z^{n-2}] M(z,v,u)} \cdot \frac{[z^{n-2}] M(z,v,u)}{[z^n] M(z,v,u)}\right|_{v=u=1} \\
&\sim 2\mu_4 \frac{n-2}{12^2}
\end{aligned}
$$

Moment Method

$M(z, v, u, u_2)$ ... generating function, where u counts pure 4-gons and u_2 pure 2-gons



$$\mathbb{E}[X_n^{(P)}(X_n^{(P)} - 1)] = \left(2! \frac{[z^n] 4z^4 (2!)^{-1} \partial_u^2 M(z, v, u, u_2)}{[z^n] M(z, v, u, u_2)} \right) \Big|_{v=u=u_2=1}$$
$$+ \left(2! \frac{[z^n] 2z^4 \partial_u M(z, v, u, u_2)}{[z^n] M(z, v, u, u_2)} \right) \Big|_{v=u=u_2=1}$$
$$+ \left(2! \frac{[z^n] z^4 \partial_{u_2} M(z, v, u, u_2)}{[z^n] M(z, v, u, u_2)} \right) \Big|_{v=u=u_2=1}$$

Moment Method

Remark.

$M(z,v,u)$ ... generating function, where u counts pure 4-gons

By the D.+Panagiotou-method we have the property that

$$\boxed{\mathbb{E}[u^{X_n^{(4)}}] = \frac{\tilde{h}(1,u)}{\tilde{h}(1,1)} \left(\frac{\tilde{\rho}(1,1)}{\tilde{\rho}(1,u)}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right)}$$

which implies that

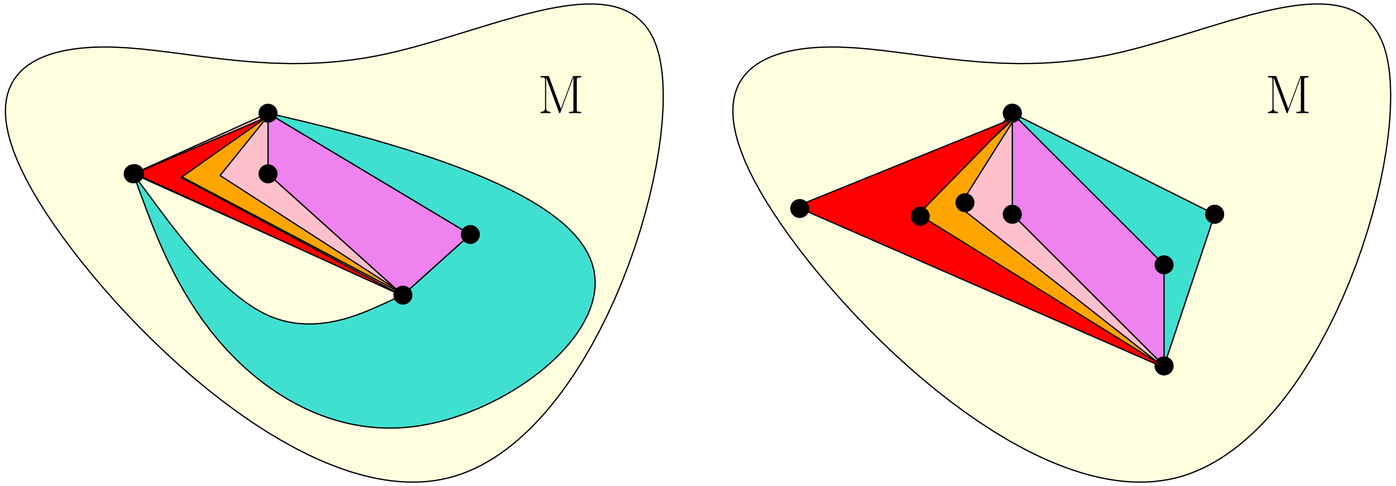
$$\boxed{\mathbb{E}[(X_n^{(4)})_r] \sim (nf'(1))^k e^{\frac{r^2}{2n}\frac{f''(1)}{(f'(1))^2}}}$$

uniformly for $0 \le r \le C\sqrt{n}$.

With **factorial moments** of $X_n^{(4)}$ (and more ...) we can then compute the **factorial moments** of $X_n^{(P)}$.

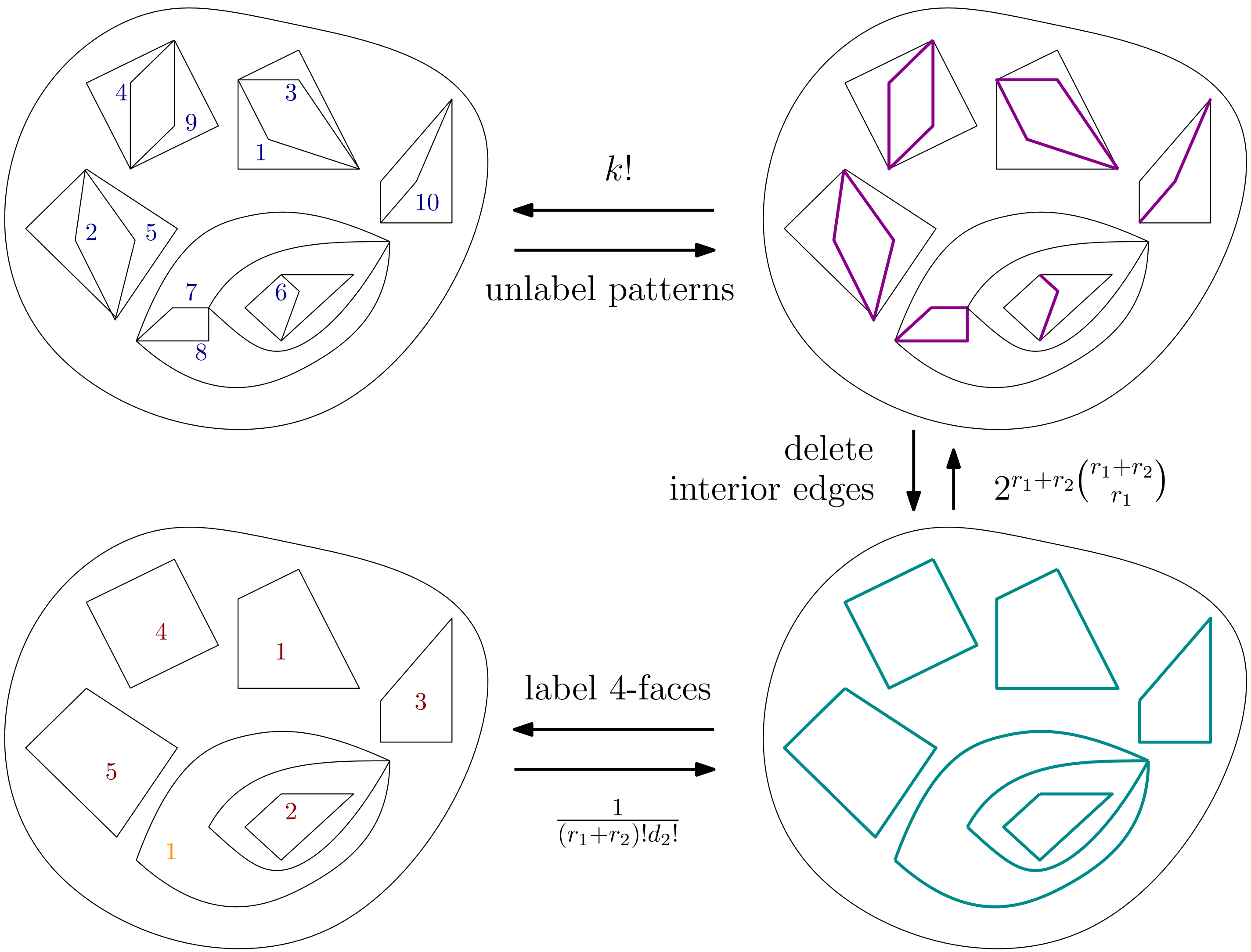
Moment Method

A Problem. Overlaps of several occurences of P



Solution. We only have to take care of **double occurences**. The other cases do not contribute to the asympotic leading term.

Moment Method



Moment Method

Theorem 4 [D.+Hainzl+Wormald 2024+]

Let P be a pattern with a *simple boundary* (in particular without cut-vertices or loops) and let $X_n^{(P)}$ be the number of occurences of P in a random planar map. Then

$$\frac{X_n^{(P)} - \mathbb{E}[X_n^{(P)}]}{\sqrt{\mathbb{V}[X_n^{(P)}]}} \to \mathcal{N}(0,1).$$

Remark. It is very likely that this method can be generalized to general pattern (ongoing work).

General Framework

Let f be **proper local functional** on a structure G_n with a **Benjamini-Schramm limit** and let $\boxed{S_n = \sum_{x \in G_n} f(x)}$ be the sum functional on a random structure G_n of size n.

Then (under natural assumptions) we know that

$$\mathbb{E}\, S_n \sim cn$$

Which additional condition on the Benjamini-Schramm limit implies a CLT for S_n??

For example, it is not known if the number of vertices of given degree k in random planar graphs with n vertices satisfy a CLT (although a Benjamini-Schramm limit exists).

Thank You!