The Moment Method Revisited

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Average Case Analysis in Anal. Comb.

$\mathcal{C}$ ... class of combinatorial objects

$c_n = \# \mathcal{C}_n$ ... number of objects in $\mathcal{C}$ of size $n$, $c_n = [z^n] C(z)$

$C(z) = \sum_{n \geq 0} c_n z^n = \sum_{\omega \in \mathcal{C}} z^{\text{size}(\omega)}$ ... GF of $\mathcal{C}$

$c_{n,k} = \# \mathcal{C}_{n,k}$ ... number of objects in $\mathcal{C}_n$, where some parameter of interest has value $k$

$X_n$ ... random variable with $\mathbb{P}[X_n = k] = \frac{c_{n,k}}{c_n}$

$C(z, u) = \sum_{n,k \geq 0} c_{n,k} z^n u^k = \sum_{n \geq 0} \left( \mathbb{E}[u^{X_n}] \right) c_n z^n$ ... bivariate GF

$\mathbb{E}[u^{X_n}] = \sum_{k \geq 0} \frac{c_{n,k}}{c_n} u^k = \frac{[z^n] C(z, u)}{[z^n] C(z, 1)}$

$\mathbb{E} X_n = \left[ z^n \right] C_u(z, u) |_{u=1} = \frac{[z^n] C(z, 1)}{[z^n] C(z, 1)}$
Limiting Distribution

Weak Limit

A sequence of random variables $Y_n$ converges \textbf{weakly} to a random variable $Y$, if

$$\mathbb{E} G(Y_n) \to \mathbb{E} G(Y)$$

for all bounded functionals $G$. \textbf{Notation:} $Y_n \to Y$.

Equivalently we have

$$\mathbb{E} e^{itY_n} \to \mathbb{E} e^{itY}$$

(for all real $t$)

or

$$\mathbb{P}[Y_n \leq t] \to \mathbb{P}[Y \leq t]$$

(for all continuity points of the distribution function $F(t) = \mathbb{P}[Y \leq t]$).
Limiting Distribution

Weak Limit with Moments

**Theorem** (the Moment Method)

Suppose that all moments $E[Y^r]$, $r \geq 1$, of a random variable exist and determine uniquely the distribution of $Y$. Furthermore let $Y_n$ be a sequence of random variables. If for all integers $r \geq 1$

$$E[Y_n^r] \rightarrow E[Y^r]$$

then $Y_n$ **converges to $Y$ weakly**: $Y_n \rightarrow Y$
Examples

Height in binary trees (Flajolet and Odlyzko, 1982)

\[ H_n \ldots \text{height of a binary tree of size } n \]

\[ Y_n = \frac{H_n}{2\sqrt{n}} \ldots \text{normalized height} \]

\[
\mathbb{E}[Y_n^r] \to \mu_r = r(r-1)\Gamma(r/2)\zeta(r) \implies \frac{H_n}{2\sqrt{n}} \to Y. 
\]

\( \mu_r \) are the moments of the theta distribution \( Y \) with distribution function

\[
F(t) = \sum_{k \in \mathbb{Z}} (1 - k^2t^2)e^{-k^2t^2}
\]

and density

\[
f(t) = 4t \sum_{k \geq 1} k^2(2k^2t^2 - 3)e^{-k^2t^2}
\]
Examples

Selected Problems

- Path length in binary trees (Takács, 1992, 1994)
- Cost of linear probing hashing (Flajolet, Plobete, Viola, 1998)
- Maximum degree in triangulations (Gao and Wormald, 2000)
- etc. (many many examples!!!)
Moment Method for Central Limit Theorems

Moments of the Standard Normal Distribution $N(0, 1)$.

$\mu_{2r}^{(N)} = (2r - 1)!!$, $\mu_{2r+1}^{(N)} = 0$

Moment Method for a sequence of random variables $X_n$:

$\mathbb{E} (X_n - \mathbb{E} X_n)^r = \sum_{\ell=0}^{r} (-1)^\ell \binom{r}{\ell} \mathbb{E}[X_n^{r-\ell}](\mathbb{E} X_n)^\ell \sim \mu_{r}^{(N)} (\text{Var } X_n)^{r/2}$.

$\Rightarrow \frac{X_n - \mathbb{E} X_n}{\sqrt{\text{Var } X_n}} \to N(0, 1)$

Several cancellations of asymptotic leading terms !!
Moment Method for Central Limit Theorems

Asymptotics for Centered Moments [Hwang et al.]
\[ \mathbb{E}(X_n - \mathbb{E}X_n)^r \]

- Recursive random variable: e.g. \( X_n \equiv X_{I_n} + X_{n-1-I_n} + t_n, \) 
  \( (I_n \text{ u.d. on } \{0, 1, \ldots, n-1\}) \)

- Recurrence for scaled moment generating function \( \mathbb{E}[e^{t(X_n-\mathbb{E}X_n)}] \)

- Asymptotic transfer results

- Asymptotics for centered moments

This method is technically highly involved !!
Hwang’s Quasi-Power-Theorem

**Theorem** [Hwang]

Suppose that $X_n$ is a sequence of random variables that satisfies

\[
\mathbb{E}[u^{X_n}] = e^{nf(u) + g(u) + O(1/n)}
\]

uniformly for complex $u$ with $|u - 1| < \eta$ and analytic functions $f(u)$ and $g(u)$ with $f(1) = g(1) = 0$, $f'(1) > 0$, and $f'(1) + f''(1) > 0$. Then we have

\[
\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\mathbb{V}(X_n)}} \rightarrow \mathcal{N}(0, 1),
\]

where

\[
\mathbb{E}(X_n) = f'(1)n + O(1) \quad \text{and} \quad \mathbb{V}(X_n) = (f''(1) + f'(1))n + O(1).
\]
Applications of Hwang’s Quasi-Power-Theorem

Moving Singularities

\[ C(z, u) \approx h(u) F\left(\frac{z}{\rho(u)}\right) \]
with \( F(x) \) singular at \( x = 1 \)

(e.g. \( F(x) = \sqrt{1-x} \))

\[ \Rightarrow \quad [z^n] C(z, u) \sim f_nh(u) \rho(u)^{-n} \]

\[ \Rightarrow \quad \mathbb{E}[u^{X_n}] = \frac{[z^n] C(z, u)}{[z^n] C(z, 1)} \sim \frac{h(u)}{h(1)} \left(\frac{\rho(1)}{\rho(u)}\right)^n \]

(System) of Functional Equations

Unique combinatorial decompositions lead to recurrence relations that rewrite into a (system of) functional equation(s) for \( C(z, u) \)

\[ C(z, u) = G(z, u, C(z, u)) \]

and leads “automatically” to moving singularities (and to a CLT).
Pattern Occurrences in Trees

Occurrence of a pattern $\mathcal{M}$ in a labelled tree.
Pattern Occurences in Trees

Partition of trees in classes (\(\square\) ... out-degree different from 2)
Pattern Occurrences in Trees

Recurrences $A_3 = xA_0A_2 + xA_0A_3 + xA_0A_4$

$a_3 = a_0 + a_2 + a_0 + a_0 + a_0 + a_0$

$A_j(x) = \sum_{n,k} a_{j;n} \frac{x^n}{n!}$

$a_{j;n}$ ... number of trees of size $n$ in class $j$
Pattern Occurrences in Trees

**Recurrences**

\[ A_3 = xuA_0A_2 + xuA_0A_3 + xuA_0A_4 \]

\[ A_j(x, u) = \sum_{n,k} a_{j;n,m} \frac{x^n}{n!} u^m \]

... number of trees of size \( n \) in class \( j \) with \( m \) occurrences of \( \mathcal{M} \)
Pattern Occurrences in Trees

\[ A_0 = A_0(x, u) = x + x \sum_{i=0}^{10} A_i + x \sum_{n=3}^{\infty} \frac{1}{n!} \left( \sum_{i=0}^{10} A_i \right)^n, \]

\[ A_1 = A_1(x, u) = \frac{1}{2} x A_0^2, \]

\[ A_2 = A_2(x, u) = x A_0 A_1, \]

\[ A_3 = A_3(x, u) = x A_0 (A_2 + A_3 + A_4) u, \]

\[ A_4 = A_4(x, u) = x A_0 (A_5 + A_6 + A_7 + A_8 + A_9 + A_{10}) u^2, \]

\[ A_5 = A_5(x, u) = \frac{1}{2} x A_1^2 u, \]

\[ A_6 = A_6(x, u) = x A_1 (A_2 + A_3 + A_4) u^2, \]

\[ A_7 = A_7(x, u) = x A_1 (A_5 + A_6 + A_7 + A_8 + A_9 + A_{10}) u^3, \]

\[ A_8 = A_8(x, u) = \frac{1}{2} x (A_2 + A_3 + A_4)^2 u^3, \]

\[ A_9 = A_9(x, u) = x (A_2 + A_3 + A_4) (A_5 + A_6 + A_7 + A_8 + A_9 + A_{10}) u^4, \]

\[ A_{10} = A_{10}(x, u) = \frac{1}{2} x (A_5 + A_6 + A_7 + A_8 + A_9 + A_{10})^2 u^5. \]
Pattern Occurences in Trees

Result for $\mathcal{M}$ =

Central limit theorem for $(X_n - \mu n)/\sqrt{\sigma^2 n}$ with

$$\mu = \frac{5}{8e^3} = 0.0311169177\ldots$$

and

$$\sigma^2 = \frac{20e^3 + 72e^2 + 84e - 175}{32e^6} = 0.0764585401\ldots.$$ 

**Theorem** [Chyzak+D.+Klausner 2008]

For every tree pattern $\mathcal{M}$ the number of it occurences $X_n$ in a random (labelled) tree of size $n$ satisfies a central limit theorem with (asymptotically) linear mean and variance.
Subgraph Counts in Subcritical Graphs

Examples of subcritical graphs are series-parallel graphs or outer-planar graphs. They behave makroscopically like trees (they have the CRT as a scaling limit).

**Theorem** [D.+Ramos+Rue 2017]

Let \( G \) be a given subcritical class of (labelled) graphs. Then for every given graph \( H \) the number of its occurrences \( X_n \) (as subgraphs) in a graph \( G \) of size \( n \) satisfies a central limit theorem with (asymptotically) linear mean and variance.

Here we need an infinite system of equations (that can be analyzed since the Jacobian of the system is a compact operator).
Factorial Moments

Problem.

What can we do if we expect a central limit theorem but the bivariate generating function $C(z,u)$ cannot be described in a proper way (explicitly or implicitly)?

Observation

Usually we can compute (factorial) moments.
Factorial Moments

\[(x)_r = x(x - 1)(x - 2) \cdots (x - r + 1) \ldots \text{falling factorials}\]

Factorial Moments

\[
\mathbb{E}(X)_r = \mathbb{E}[X(X - 1)(X - 2) \cdots (X - r + 1)] = \left. \frac{\partial^r}{\partial u^r} \mathbb{E}[u^X] \right|_{u=1}
\]

They can be computed by the **Bivariate generating functions**

\[
C(z, u) = \sum_{n,k \geq 0} c_{n,k} z^n u^k = \sum_{n \geq 0} \left( \mathbb{E}[u^{X_n}] \right) c_n z^n
\]

\[\implies \mathbb{E}((X_n)_r) = \left. \left[ z^n \frac{\partial^r}{\partial u^r} C(z, u) \right] \right|_{u=1} \]

\[\frac{[z^n] \frac{\partial^r}{\partial u^r} C(z, u) |_{u=1}}{[z^n] C(z, 1)}\]
Factorial Moments

... or by a

Combinatorial interpretation

Suppose that the parameter of interest is a counting parameter, e.g. the number of leaves in a tree or the number of triangles in a graph.

The factorial moment

\[ \mathbb{E}[(X_n)_r] = \frac{1}{c_n} \sum_{k \geq 0} k(k-1) \cdots (k-r+1)c_{n,k} \]

is also the number of objects of size \( n \), where \( r \) different appearances of the parameter (that is considered) are marked (and the order or marks is important) divided by the number of objects of size \( n \).
Factoral Moments of the Binomial Distr.

\[ X_n \ldots Bi(n, p), \quad \frac{X_n - np}{\sqrt{p(1-p)n}} \to N(0, 1). \]

\[ \mathbb{E} u^{X_n} = (1 - p + up)^n \]

\[ \mathbb{E}[(X_n)_r] = \frac{\partial^r}{\partial u^r} (1 - p + up)^n \bigg|_{u=1} = n(n-1) \cdots (n-r+1)p^r \sim (np)^r e^{-r^2/(2n)} \]

for \[ r = O(\sqrt{n}) \]

**QUESTION.** Suppose that \( \mathbb{E} (X_n)_r \sim (np)^r e^{-r^2/(2n)} \) for \( r = O(\sqrt{n}) \).

Does it follow that

\[ \frac{X_n - np}{\sqrt{p(1-p)n}} \to N(0, 1) \]
Factorial Moment Method by Gao and Wormald

\[ \mathbb{E}[(X)_r] = \mathbb{E}[X(X - 1) \cdots (X - r + 1)] \ldots \text{r-th factorial moment} \]

**Lemma** [Gao+Wormald 2004]

Suppose that \( \mathbb{E}[X_n] = \mu_n \to \infty, \mathbb{V}ar[X_n] = \sigma_n^2 = o(\mu_n^2/\log^4 n), \mu_n = o(\sigma_n^3) \) and \((X_n)_{n \geq 1} \geq 0\) satisfies

\[
\mathbb{E}[(X_n)_r] \sim \mu_n^r \exp \left( \frac{r^2 \sigma_n^2 - \mu_n}{2 \mu_n^2} \right)
\]

uniformly for all \( r \) in the range \( c \mu_n/\sigma_n \leq r \leq c' \mu_n/\sigma_n \) for some constants \( c' > c > 0 \). Then

\[
\frac{X_n - \mu_n}{\sigma_n} \to \mathcal{N}(0, 1).
\]

**Remark.** \( X_n \sim Bi(n, p), \mu_n = np, \sigma_n^2 = p(1 - p)n, r = \Theta(\sqrt{n}) \).
Quasi-Powers and Factorial Moments

Lemma

Suppose that $X_n$ is a sequence of random variables that satisfies

$$\mathbb{E}[u^{X_n}] = e^{nf(u) + g(u) + O(1/n)}$$

uniformly for complex $u$ with $|u - 1| < \eta$ and analytic functions $f(u)$ and $g(u)$ with $f(1) = g(1) = 0$ and $f'(1) > 0$. Then we have

$$\mathbb{E}[(X_n)_r] \sim (nf'(1))^r \exp \left( \frac{r^2 f''(1)}{2n (f'(1))^2} \right)$$

uniformly for $0 \leq r \leq C\sqrt{n}$, where $C > 0$ is an arbitrary constant.

Remark. This is consistent with Hwang’s Quasi-Power-Theorem.
Quasi-Powers and Factorial Moments

Proof

\[ E[(X_n)_r] = r! \frac{1}{2\pi i} \int_{\gamma} \frac{E[u^{X_n}]}{(u - 1)^{r+1}} \, du, \]

\( \gamma \) is a cycle with center 1 and radius \( \rho = r/(nf'(1)) \): \( u = 1 + \rho e^{i\varphi} \)

\[ E[(X_n)_r] = \frac{r!}{2\pi} \int_{-\pi}^{\pi} e^{nf'(1)\rho e^{i\varphi} + \frac{n}{2} f''(1)\rho e^{2i\varphi}} + O(n\rho^3 + \rho + 1/n) \rho - r e^{-ir\varphi} \, d\varphi \]

\[ = (nf'(1)) r e^{\frac{r^2}{2n(f'(1))^2}} \frac{r!}{2\pi r^r e^{-r}} \times \int_{-\pi}^{\pi} r(e^{i\varphi} - 1 - i\varphi) + \frac{r^2}{2n(f'(1))^2} (e^{2i\varphi} - 1) + O\left(\frac{r^3}{n^2} + \frac{r+1}{n}\right) \, d\varphi. \]

The last integral evaluates (uniformly for \( r \leq C\sqrt{n} \)) by Laplace’s method

\[ \int_{-\pi}^{\pi} e^{r(e^{i\varphi} - 1 - i\varphi)} + \frac{r^2}{2n(f'(1))^2} (e^{2i\varphi} - 1) + O\left(\frac{r^3}{n^2} + \frac{r+1}{n}\right) \, d\varphi \sim \int_{-\infty}^{\infty} e^{-\frac{r}{2}\varphi^2} \, d\varphi = \sqrt{\frac{2\pi}{r}}. \]
Applications of the Factorial Moment Method

- Gao+Wormald 2004: submap counts in random planar triangulations
- Cai+Devroye 2017: subtrees in conditional Galton-Watson trees
- Hitczenko+Wormald 2023+: balls in bins in a classical allocation scheme (multivariate version)
- Ojeda+Holmgren+Janson 2023+: Fringe trees for random trees with given vertex degrees (multivariate version)
- **NEW**: D.+Hainzl+Wormald 2024+: pattern counts in random planar maps
A **planar map** is a connected planar graph, possibly with loops and multiple edges, together with an embedding in the plane. A map is **rooted** if a vertex \( v \) and an edge \( e \) incident with \( v \) are distinguished, and are called the root-vertex and root-edge, respectively. The face to the right of \( e \) is called the root-face and is usually taken as the **outer face**.
Planar Maps

$M_n$ ... number of rooted maps with $n$ edges [Tutte]

\[
M_n = \frac{2(2n)!}{(n + 2)!n!}3^n
\]

The proof is given with the help of generating functions and the so-called quadratic method.

Asymptotics:

\[
M_n \sim \frac{2}{\sqrt{\pi}} \cdot n^{-5/2} 12^n
\]

Generating Function:

\[
M(z) = \sum_{n \geq 0} M_n z^n = -\frac{1}{54z^2} \left(1 - 18z - (1 - 12z)^{3/2}\right)
\]
Planar Maps

Quadratic Method

\( M_{n,k} \) ... number of maps with \( n \) edges and outer-face-valency \( k \)

\[
M(z, v) = \sum_{n,k} M_{n,k} v^k z^n
\]

\[
M(z, v) = 1 + zv^2 M(z, v)^2 + vz \frac{vM(z, v) - M(z, 1)}{v - 1}.
\]

By binding \( z \) and \( v \) by a proper function \( v = v(z) \) this equation can be solved and we get

\[
M(z) = M(z, 1) - \frac{1}{54z^2} \left( 1 - 18z - (1 - 12z)^{3/2} \right).
\]
Random Planar Maps

Picture by Nicolas Curien
Classes of Planar Maps

- Bipartite / Eulerian planar maps
- Quadrangulations / 4-regular planar maps
- Triangulations / 3-regular planar maps
- 2-connected planar maps
- 3-connected planar maps
- ...
Scaling Limit of Planar Maps

\( \mathcal{M}_n \) ... random planar map with \( n \) edges
\( Q_n \) ... random quadrangulation with \( n \) edges

**Theorem** [Miermont 13, Le Gall 13, Bettinelli+Jacob+Miermont 14]

*We have in distribution for the Gromov-Hausdorff topology*

\[
\left( \frac{9}{8n} \right)^{1/4} \mathcal{M}_n \to S, \quad \left( \frac{9}{8n} \right)^{1/4} Q_n \to S,
\]

*where \( S \) denotes the Brownian Map.*

In particular, the typical distance of two vertices is of order \( n^{1/4} \)
Local Limit of Planar Maps

$U_r(M)$ ... rooted map induced by those vertices of the rooted map $M$ with distance $< r$ from the root vertex of $M$.

**Theorem** [Ménard+Nolin 2014, Stephenson 2016]
We have for any rooted map $M$ and for all $r > 0$

$$\lim_{n \to \infty} \mathbb{P}[U_r(M_n) = M] = \mathbb{P}[U_r(s) = M],$$

where $s$ denotes the **Uniform Infinite Planar Map**.

**Remark.** There are similar results for triangulations by [Angel+Schramm 2003], for quadrangulations by Krikum 2005+ and by [Curien+Ménard+Miermont 2013], and for bipartite maps by [Björnberg+Stefánsson 2014].
Local Limit of Planar Maps

$\mathcal{M}_n^\bullet$ ... random planar map with $n$ edges and a randomly chosen distinguished vertex $v$

$U_r^v(M)$ ... rooted map induced by those vertices of $M$ the with distance $< r$ from the vertex $v$ (of $M$)

**Corollary** [D. + Stufler 2019]

We have for every vertex rooted map $\tilde{M}$ and for all $r > 0$

$$\lim_{n \to \infty} P[U_r^v(\mathcal{M}_n^\bullet) = \tilde{M}] = P[U_r^v(s^*) = \tilde{M}],$$

where $s^*$ denotes the corresponding Benjamini-Schramm limit.
Pattern in a Planar Map

A **pattern** $P$ in a planar map $M$ is a planar map, if $M$ can be constructed by adding successively faces to the outer face of $P$ and to the outer faces of the appearing maps.

Simplest pattern: face of valency $r$

Coloured pictures by Eva-Maria Hainzl
Local Limit of Planar Maps

$P$ ... planar pattern

$X_n^{(P)}$ ... number of occurrences of $P$ in $M_n$

**Theorem** [D.+Stufler 2019]

*There exists $c(P) > 0$ with*

$$
\mathbb{E} X_n^{(P)} \sim c(P) n.
$$

**Problem.** What can be said about the difference $X_n^{(P)} - \mathbb{E} X_n^{(P)}$?

**Is there always a Central Limit Theorem?**
Results

- Submaps counts of a 3-connected map $N$ in 2-connected triangulations satisfy a CLT. [Gao+Wormald 2004]

- The number of faces of degree $r$ in (2-connected) random planar maps satisfy a CLT. [D+Panagiotou 2013, Collet+D.+Klausner+Kok 2019]

- Double-triangles in random planar maps [D.+Yu 2018]

- **NEW**: pattern with simply boundary [D.+Hainzl+Wormald]
Methods

Proof Methods for CLT

- **Bijective method** with mobiles (restricted to face valencies, no other pattern, no connectivity assumptions): **Quasi-Power-Th.**

- **Quadratic method** (face valencies, pattern without self-intersections, several map classes): **Quasi-Power-Th.**

- NEW; Gao-Wormald-Moment-Method with “Quasi-Power-Preprocessing” (pattern with simple boundary)
Bijective Method

**Theorem 1** [Collet+D.+Klausner 2019]

\( \Omega \) ... an arbitrary set of positive integers, not a subset of \( \{1, 2\} \)

\( \mathcal{M}_\Omega \) ... planar rooted maps such that all face valencies are in \( \Omega \)

\( X_n^{(r)} \) ... **number of faces of valency** \( r \) in a random planar map in \( \mathcal{M}_\Omega \). \( (r \in \Omega) \)

Then we have

\[
\mathbb{E}[X_n^{(r)}] = \mu_r n + O(1), \quad \text{Var}[X_n^{(k)}] = \sigma_r^2 n + O(1)
\]

for certain constants \( \mu_r > 0, \sigma_r^2 \geq 0 \), and

\[
\frac{X_n^{(r)} - \mathbb{E}[X_n^{(r)}]}{\sqrt{n}} \to N(0, \sigma_r^2).
\]
Bijective Method

Examples.

\( \Omega = \{3\} \) ... triangulations

\( \Omega = \{4\} \) ... quadrangulations

\( \Omega = 2\mathbb{N} \) ... bipartite maps

\( \Omega = \mathbb{N} \) ... all maps

\( \Omega = \mathbb{P} = \{2, 3, 5, 7, \ldots\} \) ... all face valencies are prime numbers

...
**Bijective Method**

Remark.

\( M_{\Omega,n} \ldots \) number of maps in \( \mathcal{M}_\Omega \) with \( n \) edges

Then there exist positive constants \( c_\Omega \) and \( \gamma_\Omega \) with

\[
M_{\Omega,n} \sim c_\Omega n^{-5/2} \gamma_\Omega^n, \quad n \equiv 0 \mod d,
\]

where \( d = \gcd\{i : 2i \in \Omega\} \) if \( \Omega \) contains only even numbers, otherwise \( d = 1 \).

The exponent \(-5/2\) is **universal**.
Mobiles

Definition.

A mobile is a planar tree — that is, a map with a single face — such that there are two kinds of vertices (black and white) with no white-white edges, and black vertices additionally have so-called “legs” attached to them (which are not considered edges), whose number equals the number of white neighbor vertices.

A bipartite mobile is a mobile without black–black edges.

The degree of a black vertex is the number of half-edges plus the number of legs that are attached to it.

A mobile is called rooted if an edge is distinguished and oriented.
Mobiles
Mobiles

**Theorem** [Cori+Vauquelin, Schaeffer, Bouttier+Di Francesco+Guitter, Bernardi+Fusy, Collet+Fusy, ...]

There is a bijection between mobiles that contain at least one black vertex and pointed planar maps, where white vertices in the mobile correspond to non-pointed vertices in the equivalent planar map, black vertices correspond to faces of the map, and the degrees of the black vertices correspond to the face valencies.

This bijection induces a bijection on the edge sets so that the number of edges is the same. (Only the pointed vertex of the map has no counterpart.)

Similarly, rooted mobiles that contain at least one black vertex are in bijection to rooted and vertex-pointed planar maps.

Finally, bipartite mobiles with at least two vertices correspond to bipartite maps with at least two vertices, in the unrooted as well as in the rooted case.
Mobiles


Mobiles and Maps

- $L(t, z, x_1, x_2, \ldots)$ ... mobiles rooted at a black vertex and where an additional edge is attached to the black vertex (the $x_i$ “count” the number of black vertices of degree $i$)

- $Q(t, z, x_1, x_2, \ldots)$ ... mobiles rooted at a univalent white vertex, which is not counted,

- $R(t, z, x_1, x_2, \ldots)$ ... mobiles rooted at a white vertex and where an additional edge is attached to the root vertex.

\[
B_{l,m} = \binom{l + 2m}{l, m, m} \\
B_{l,m}^{(+1)} = \binom{l + 2m + 1}{l, m, m + 1} \\
\overline{B}_{l,m} = \frac{l + m}{l + 2m} \binom{l + 2m}{l, m, m}
\]
Mobiles and Maps

Lemma

The generating functions $L = L(t, z, x_1, x_2, \ldots)$, $Q = Q(t, z, x_1, x_2, \ldots)$, and $R = R(t, z, x_1, x_2, \ldots)$ satisfy the system of equations

\[
\begin{align*}
L &= z \sum_{\ell, m} x_{2m + \ell + 1} B_{\ell, m} L^\ell R^m, \\
Q &= z \sum_{\ell, m} x_{\ell + 2m + 2} B_{\ell, m}^{(+1)} L^\ell R^m, \\
R &= \frac{tz}{1-Q}.
\end{align*}
\]

Let $T = T(t, z, x_1, x_2, \ldots)$ be given by

\[T = 1 + \sum_{\ell, m} x_{2m + \ell} B_{\ell, m} \ell L^\ell R^m,\]

Then the generating function $M = M(t, z, x_1, x_2, \ldots)$ of rooted maps satisfies

\[
\frac{\partial M}{\partial t} = \frac{R}{z} - t + T,
\]

where the variable $t$ corresponds to the number of vertices, $z$ to the number of edges, and $x_i$, $i \geq 1$, to the number of faces of valency $i$. 
Extensions and Limitation

Extensions

• Local limit theorems

Limitations

• There is no control on adjacent vertices. Thus, we cannot handle more complicated patterns.

• There is no control on the level of connectivity. We cannot handle 2-connected or 3-connected maps.
Quadratic Method

**Theorem 2** [D. + Panagiotou 2013]

$X_n^{(r)}$ ... number of faces of valency $r$ in a random planar map with $n$ edges or in a random 2-connected map with $n$ edges.

Then we have

$$\mathbb{E}[X_n^{(r)}] = \mu_r n + O(1), \quad \text{Var}[X_n^{(r)}] = \sigma_r^2 n + O(1)$$

with constants $\mu_r > 0$, $\sigma_r^2$ and

$$\frac{X_n^{(r)} - \mathbb{E}[X_n^{(r)}]}{(\text{Var}[X_n^{(r)]})^{1/2}} \rightarrow N(0, 1).$$

**Remark.** The same result holds for pure $r$-gons (all vertices are different).
Quadratic Method

Generating functions

$M_{n,k}$ ... number of maps with $n$ edges and outer-face-valency $k$

\[ M(z, v) = \sum_{n,k} M_{n,k} v^k z^n \]

\[
M(z, v) = 1 + zv^2 M(z, u)^2 + vz \frac{vM(z, v) - M(z, 1)}{v - 1}
\]

$v$ ... “catalytic variable”
(Crucial) Lemma

Suppose that we have a catalytic equation of the form

\[ P(z, v, u, M(z, v, u), M_1(z, v)) = 0 \]

such that for \( u = 1 \) the solution \( M_1(z, 1) \) has a singular behaviour of the form

\[ M_1(z, 1) = g(z) + h(z) \left(1 - \frac{z}{\rho}\right)^{3/2} \]

Then (under weak technical conditions) we have

\[ M_1(z, u) = \tilde{g}(z, u) + \tilde{h}(z, u) \left(1 - \frac{z}{\tilde{\rho}(u)}\right)^{3/2} \]

with \( \tilde{g}(z, 1) = g(z) \), \( \tilde{h}(z, 1) = h(z) \), and \( \tilde{\rho}(1) = \rho \) (for \( u \) in a small neighborhood of \( 1 \)).
Faces of given valency

\( M_{n,k,\ell} \) ... number of maps with \( n \) edges, root face valency \( k \) and \( \ell \) non-root faces of valency \( r \)

\[
M(z, v, u) = \sum_{n,k,\ell \geq 0} M_{n,k,\ell} z^n v^k u^\ell ,
\]

**Lemma** [D. + Panagiotou, 2013]

\[
 M(z, v, u) = 1 + zv^2M(z, v, u)^2 + zv \frac{M(z, 1, u) - vM(z, v, u)}{1 - v} \\
+ z(u - 1)v^{-r+2} \left( M(z, v, u) - \sum_{\ell=0}^{r-2} M_\ell(z, u)v^\ell \right),
\]

where \( M_\ell(z, u) = [v^\ell]M(z, v, u) \).

**Remark.** The same method works for 2-connected planar maps.
Double-Triangles

**Theorem 3** [D.+Yu 2018]

\(X_n\) ... **number of double-triangles** in a random planar map with \(n\) edges (counted by the number or edges where both adjacent faces have valency 3).

Then we have

\[
E[X_n] = \mu n + O(1), \quad \text{Var}[X_n] = \sigma^2 n + O(1)
\]

with constants \(\mu > 0, \sigma^2 > 0\) and

\[
\frac{X_n - E[X_n]}{(\text{Var}[X_n])^{1/2}} \rightarrow N(0,1)
\].
Double-Triangles

\( D_{n,k,\ell} \) ... number of maps with \( n \) edges, root face valency \( k \) and \( \ell \) edges outside the root face, where both adjacent faces are triangles

\[
D(z, v, u) = \sum_{n,k,\ell \geq 0} D_{n,k,\ell} z^n v^k u^\ell,
\]

**Lemma**

\[
D = 1 + zv^2 D^2 + D_{\not\bowtie} + D_{\triangleright},
\]

\[
D_{\not\bowtie} = zv \frac{D(1) - vD}{1 - v} - zv^{-1} \left( D - 1 - v[v^1]D \right),
\]

\[
D_{\triangleright} = zv^{-1} \left( D - 1 - v[v^1]D \right) + (u - 1) \left[ z^2 vD - z^2 v(u - 1) DD_{\triangleright} \right.
\]

\[
+ (u + 1) \left( zv^{-1} D_{\triangleright} - z[v^1] D_{\triangleright} \right) - (u - 1) P(D_{\triangleright}) \right]
\]

with

\[
P(D_{\triangleright}) = z^2 \frac{D_{\triangleright}(1) - vD_{\triangleright}}{1 - v} - z^2 D_{\triangleright}(1) - z^2 v^{-2} \left( D_{\triangleright} - v[v^1]D_{\triangleright} - v^2 [v^2]D_{\triangleright} \right).
Double-Triangles
Double-Triangles

Proof strategy

- Combinatorics leads to a system of catalytic equations

- Extension of quadratic method

- This leads to a central limit theorem with the help of the Crucial Lemma and the Quasi-Power-Theorem.
General Pattern

\( P \) ... pattern
\( X_{n}^{(P)} \) ... number of occurrences of \( P \) in a random planar map of size \( n \)

\( M_{n} \) ... number of maps of size \( n \)
\( M_{n}^{(r)} \) ... number of maps of size \( n \) with \( r \) distinguished (and ordered) occurrences of \( P \)

Then

\[
\mathbb{E}[(X_{n}^{(P)})_{r}] = \frac{M_{n}^{(r)}}{M_{n}},
\]

where \( \mathbb{E}[(X)_{r}] = \mathbb{E}[X(X - 1) \cdots (X - r + 1)] \) denotes the \( r \)-th factorial moment.
General Pattern

- Delete interior edges of pattern

- Count maps with distinguished pure polygon face

- Insert interior edges back
Moment Method

\[ M(z, v, u) \] ... generating function, where \( u \) counts pure 4-gons

\[
\mathbb{E}[X_n^{(P)}] = \frac{[z^n] 2 z^2 \partial_u M(z, v, u)}{[z^n] M(z, v, u)} \bigg|_{v=u=1} = 2 \frac{[z^{n-2}] \partial_u M(z, v, u)}{[z^{n-2}] M(z, v, u)} \cdot \frac{[z^{n-2}] M(z, v, u)}{[z^n] M(z, v, u)} \bigg|_{v=u=1}
\sim 2\mu_4 \frac{n - 2}{12^2}
\]
Moment Method

\[ M(z, v, u, u_2) \ldots \text{ generating function, where } u \text{ counts pure 4-gons and } u_2 \text{ pure 2-gons} \]

\[ \mathbb{E}[X_n^{(P)}(X_n^{(P)} - 1)] = \left( 2! \frac{[z^n]4z^4(2!)^{-1}\partial_u^2 M(z, v, u, u_2)}{[z^n]M(z, v, u, u_2)} \right) \bigg|_{v=u=u_2=1} + \left( 2! \frac{[z^n]2z^4\partial_u M(z, v, u, u_2)}{[z^n]M(z, v, u, u_2)} \right) \bigg|_{v=u=u_2=1} + \left( 2! \frac{[z^n]z^4\partial_{u_2} M(z, v, u, u_2)}{[z^n]M(z, v, u, u_2)} \right) \bigg|_{v=u=u_2=1} \]
Moment Method

Remark.

\( M(z, v, u) \) ... generating function, where \( u \) counts pure 4-gons

By the D. + Panagiotou-method we have the property that

\[
\mathbb{E}[uX_n^{(4)}] = \frac{\tilde{h}(1, u)}{\tilde{h}(1, 1)} \left( \frac{\tilde{\rho}(1, 1)}{\tilde{\rho}(1, u)} \right)^n \left( 1 + O \left( \frac{1}{n} \right) \right)
\]

which implies that

\[
\mathbb{E}[(X_n^{(4)})_r] \sim (nf'(1))^{r^2} f''(1) e^{2n(f'(1))^2}
\]

uniformly for \( 0 \leq r \leq C\sqrt{n} \).

With **factorial moments** of \( X_n^{(4)} \) (and more ...) we can then compute the **factorial moments** of \( X_n^{(P)} \).
Moment Method

A Problem. Overlaps of several occurrences of $P$

Solution. We only have to take care of double occurrences. The other cases do not contribute to the asymptotic leading term.
Moment Method

1. Unlabel patterns
2. Delete interior edges
3. Label 4-faces

$k! (2r_1+r_2) (r_1+r_2) r_1\frac{1}{(r_1+r_2)!d_2!}$
Moment Method

**Theorem 4** [D. + Hainzl + Wormald 2024+]

Let $P$ be a pattern with a *simple boundary* (in particular without cut-vertices or loops) and let $X_n^{(P)}$ be the number of occurrences of $P$ in a random planar map. Then

$$
\frac{X_n^{(P)} - \mathbb{E}[X_n^{(P)}]}{\sqrt{\mathbb{V}[X_n^{(P)}]}} \rightarrow \mathcal{N}(0, 1).
$$

**Remark.** It is very likely that this method can be generalized to general pattern (ongoing work).
General Framework

Let \( f \) be proper local functional on a structure \( G_n \) with a Benjamini-Schramm limit and let \( S_n = \sum_{x \in G_n} f(x) \) be the sum functional on a random structure \( G_n \) of size \( n \).

Then (under natural assumptions) we know that

\[ \mathbb{E} S_n \sim cn \]

Which additional condition on the Benjamini-Schramm limit implies a CLT for \( S_n \)?

For example, it is not known if the number of vertices of given degree \( k \) in random planar graphs with \( n \) vertices satisfy a CLT (although a Benjamini-Schramm limit exists).
Thank You!