Lecture hall graphs and the Askey scheme

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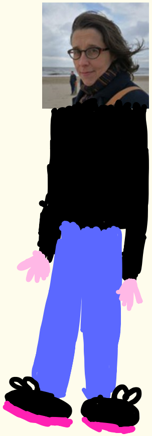
CNRS & UC Berkeley



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COMBINATORIAL ASPECTS OF CONTINUED FRACTIONS

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We show that the universal continued fraction of the Stieltjes-Jacobi type is equivalent to the characteristic series of labelled paths in the plane. The equivalence holds in the set of series in non-commutative indeterminates. Using it, we derive direct combinatorial proofs of continued fraction expansions for series involving known combinatorial quantities: the Catalan numbers, the Bell and Stirling numbers, the tangent and secant numbers, the Euler and Eulerian numbers.... We also show combinatorial interpretations for the coefficients of the elliptic functions, the coefficients of inverses of the Tchebycheff, Charlier, Hermite, Laguerre and Meixner polynomials. Other applications include cycles of binomial coefficients and inversion formulae. Most of the proofs follow from direct geometrical correspondences between objects.

Outline

1. Basics on orthogonal polynomials.
2. Lecture hall partitions and little q-Jacobi poly.
3. Lecture hall graph.
4. Askey-Wilson polynomials
5. Main results
6. Idea of Proof.
7. Asymptotics

Basics on orthogonal polynomials

Def $\{p_n(x)\}_{n\geq 0}$ is an orthogonal polynomial sequence (OPS) with respect to a linear functional $\mathcal{L}$ if

- $\deg p_n(x) = n$
- $\mathcal{L}(p_n(x)p_m(x)) = K_n \delta_{n,m} \quad (K_n \neq 0)$.

Ex

- $\int_0^\pi \cos n\theta \cos m\theta \, d\theta = h_n \delta_{n,m} \quad (h_n \neq 0)$.
- $\cos n\theta$ is a polynomial in $\cos\theta$
- Define $T_n(x)$ by $T_n(\cos\theta) = \cos n\theta$.
- $\mathcal{L}(p(x)) = \int_{-1}^1 p(x)(1-x^2)^{-1/2} dx = \int_0^\pi p(\cos\theta) d\theta$.
- $\mathcal{L}(T_n(x)T_m(x)) = \int_0^\pi \cos n\theta \cos m\theta \, d\theta = h_n \delta_{n,m}$.

Since the linear functional $\mathcal{L}$ is defined on polynomials, it is determined by $\mathcal{L}(x^n)$, $n \geq 0$.

Def The *nth moment* of $\mathcal{L}$ is

$$\mu_n = \mathcal{L}(x^n).$$

The moments of classical orthogonal polynomials have interesting combinatorial meanings.

Ex
- Hermite : $\mu_n = \#$ perfect matchings on $\{1, 2, \dots, n\}$.
- Charlier : $\mu_n = \#$ set partitions on $\{1, 2, \dots, n\}$.
- Laguerre : $\mu_n = \#$ permutations on $\{1, 2, \dots, n\}$.

Thm (Favard)

Monic orthogonal polynomials $p_n(x)$ satisfy a three-term recurrence

$$p_{n+1}(x) = (x - b_n)\, p_n(x) - \lambda_n\, p_{n-1}(x). \qquad (\lambda_n \neq 0)$$

Ex

- Hermite : $H_{n+1}(x) = x H_n(x) - n H_{n-1}(x)$
- Charlier : $C_{n+1}(x) = (x - n - 1)\, C_n(x) - n\, C_{n-1}(x).$
- Laguerre : $L_{n+1}(x) = (x - 2n - 1)\, L_n(x) - n^2 L_{n-1}(x).$

Mixed Moments and Coefficients

Def). $P_n(x)$: OPS

Let $$x^n = \sum_{k=0}^{n} \sigma_{n,k} P_k(x),$$

$$P_n(x) = \sum_{k=0}^{n} \nu_{n,k} x^k.$$

$\sigma_{n,k}$ is the mixed moment.

$\nu_{n,k}$ is the coefficient (dual mixed moment).

(These are the entries of change of bases.)

Note: $$\sigma_{n,k} = \frac{\mathcal{L}(x^n p_k)}{\mathcal{L}(p_k^2)}$$

$$\mu_n = \mathcal{L}(x^n) = \sigma_{n,0}.$$

Multivariate orthogonal polynomials

Def) $F(x_1, \dots, x_n)$ is symmetric if

$$F(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = F(x_1, \dots, x_n) \quad \text{for all } \sigma \in S_n.$$

When we have a basis $\{p_m(x)\}_{m \geq 0}$ of univariate poly,
we can construct a basis $\{P_\lambda(\mathbb{x})\}_\lambda$ of sym. poly.
by

$$P_\lambda(\mathbb{x}) = \frac{\det(p_{\lambda_i+n-i}(x_j))}{\det(p_{n-i}(x_j))}.$$

$\mathbb{x} = (x_1, \dots, x_n)$

Def) The Schur function $s_\lambda(x_1, \dots, x_n)$ is defined
in this way with basis $\{x^m\}_{m \geq 0}$:

$$s_\lambda(\mathbb{x}) = \frac{\det(x_j^{\lambda_i+n-i})}{\det(x_j^{n-i})}.$$

<u>Thm</u> If $\{p_n(x)\}_{n \geq 0}$ is OPS, then $\{P_\lambda(\mathbb{x})\}_{\lambda \in \mathrm{Par}}$ is multivariate OPS.

Mixed moments and coefficients of multivariate OPS

Let
$$P_\lambda(\mathbf{x}) = \frac{\det\left(p_{\lambda_i+n-i}(x_j)\right)}{\det\left(p_{n-i}(x_j)\right)}.$$
where
$$x^n = \sum_k \sigma_{n,k}\, p_k$$
$$p_k = \sum_k \nu_{n,k}\, x^k.$$

Since both P_λ and S_λ are bases for the space of symmetric polynomials, we can expand

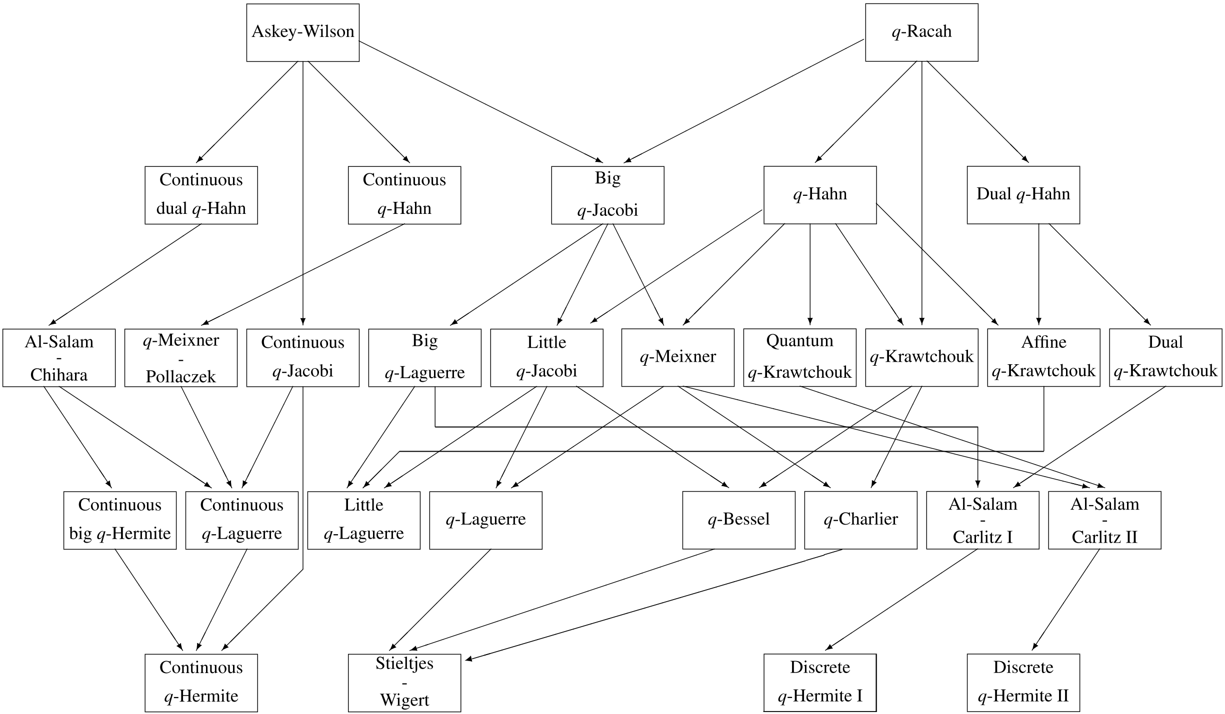
$$S_\lambda = \sum_\mu \underbrace{M_{\lambda\mu}}_{\text{mixed moment}} P_\mu, \qquad P_\lambda = \sum_\mu \underbrace{N_{\lambda\mu}}_{\text{coefficient}} S_\mu.$$

mixed moment

coefficient

Thm $M_{\lambda\mu} = \det\left(\sigma_{\lambda_i+n-i,\,\mu_j+n-j}\right)$, $N_{\lambda\mu} = \det\left(\nu_{\lambda_i+n-i,\,\mu_j+n-j}\right)$.

Goal: Find combinatorial models of $\sigma_{n,k}$ and $\nu_{n,k}$ for the OPS in the q-Askey scheme. such that they induce combinatorial models of $M_{\lambda\mu}$, $N_{\lambda\mu}$.


Askey-Wilson
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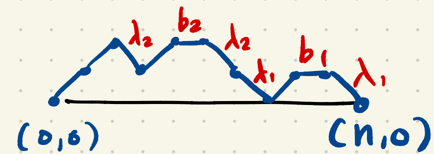
Al-Salam - Carlitz II
Continuous q-Hermite
Stieltjes - Wigert
Discrete q-Hermite I
Discrete q-Hermite II

Viennot's theory of orthogonal polynomials

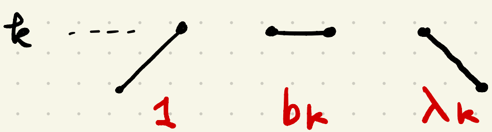
Thm (Flajolet 80, Viennot 83)

$$P_{n+1}(x) = (x-b_n)P_n(x) - \lambda_n P_{n-1}(x).$$

$$\mu_n = \sum_{\pi \in \mathrm{Motz}_n} \mathrm{wt}(\pi).$$

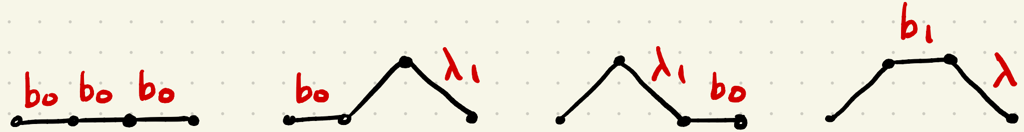


Motz_n = set of Motzkin paths from (0,0) to (n,0).



Ex

$\mu_3 =$



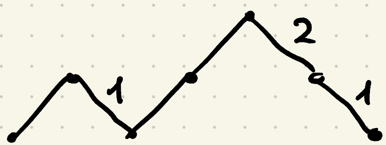
$$= b_0^3 + 2b_0\lambda_1 + b_1\lambda_1.$$

Ex

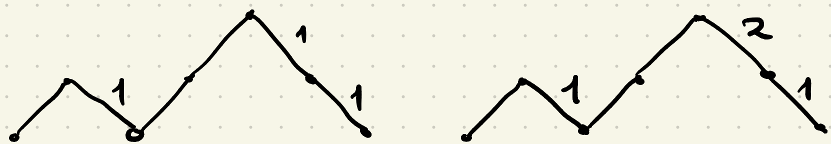
- Hermite : $H_{n+1}(x) = x H_n(x) - n H_{n-1}(x)$

  $\mu_n = \#$ perfect matchings on $\{1, 2, \dots, n\}$.

$n = 6$. $\mu_n = \sum_{\pi \in Motz_6} wt(\pi) = \sum_{\pi \in Dyck_6} wt(\pi)$

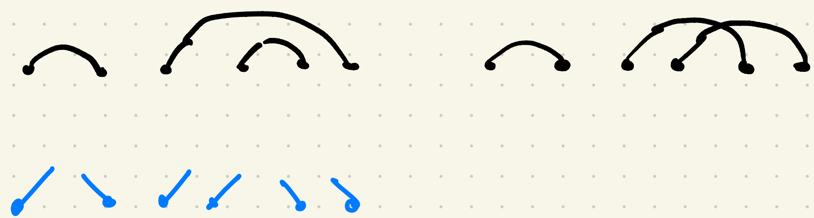


$\longleftrightarrow$



$wt = 1 \cdot 2 \cdot 1$

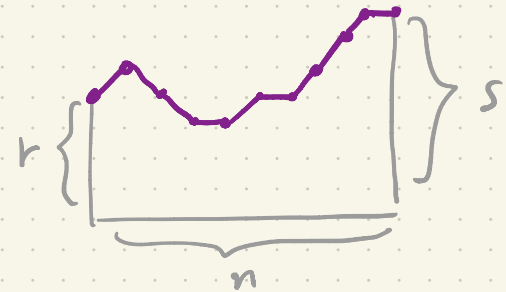
Hermite histories



Thm (Viennot 1983)

$$P_{n+1}(x) = (x - b_n) P_n(x) - \lambda_n P_{n-1}(x).$$

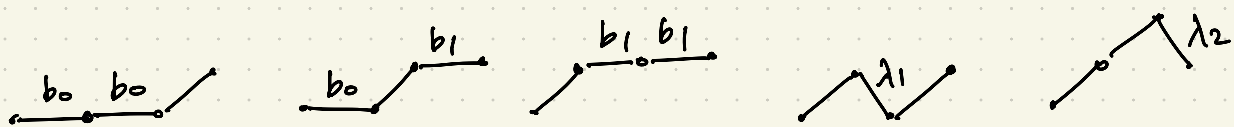
$$\frac{\mathcal{L}(x^n P_r P_s)}{\mathcal{L}(P_s^2)} = \sum_{\pi \in \mathrm{Motz}((0,r)\to(n,s))} \mathrm{wt}(\pi)$$



In particular

$$\sigma_{n,k} = \frac{\mathcal{L}(x^n P_k)}{\mathcal{L}(P_k^2)} = \sum_{\pi \in \mathrm{Motz}((0,0)\to(n,k))} \mathrm{wt}(\pi).$$

Ex $\mu_{3,1} = b_0^2 + b_0 b_1 + b_1^2 + \lambda_1 + \lambda_2.$



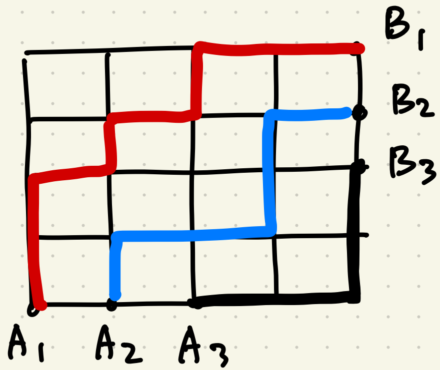
Thm (Viennot). $\nu_{n,k}$ = generating function for Favard tilings.

Lindström–Gessel–Viennot lemma

Ch 32 of the book.

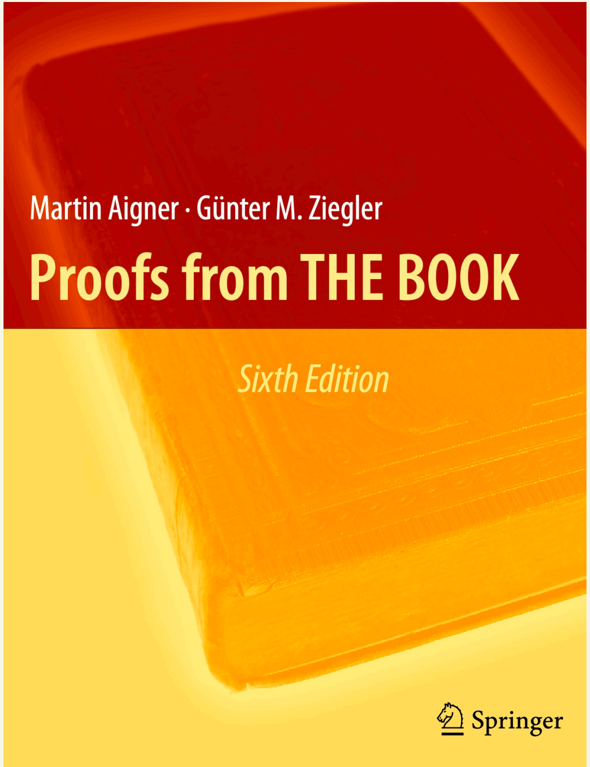
Lemma. *Let $G = (V, E)$ be a finite weighted acyclic directed graph, $\mathcal{A} = \{A_1, \ldots, A_n\}$ and $\mathcal{B} = \{B_1, \ldots, B_n\}$ two n-sets of vertices, and M the path matrix from $\mathcal{A}$ to $\mathcal{B}$. Then*

$$\det M = \sum_{\substack{\mathcal{P} \text{ vertex-disjoint} \\ \text{path system}}} \operatorname{sign} \mathcal{P}\, w(\mathcal{P}). \tag{3}$$



$M = \det(m_{ij})$

$m_{ij} = \#$ paths from A_i to B_j

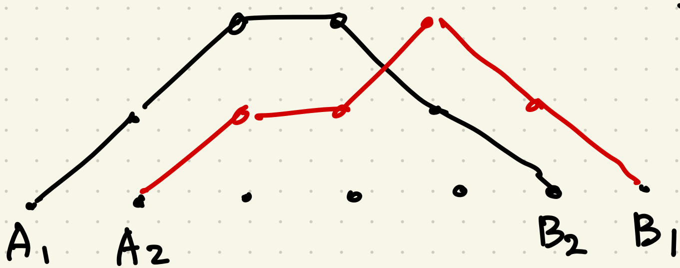


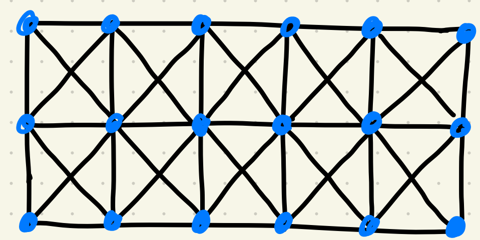
$\Rightarrow$ $M = \#$ vertex-disjoint paths from (A_1, A_2, A_3) to (B_1, B_2, B_3).

Recall $M_{\lambda,\mu} = \det\left(\sigma_{\lambda_i+n-i,\, \mu_j+n-j}\right)$.

Motzkin paths are not well behaved with LGV.

These are vertex-disjoint.





lattice for Motzkin paths

Q. Is there a lattice path model for $\sigma_{n,k}$ that gives nice nonintersecting lattice paths using LGV lemma?

Little q-Jacobi polynomials $p_n(x;a,b;q)$

$$p_n(x;a,b;q) = \frac{(ab;q)_n}{(-1)^n q^{-\binom{n}{2}} (abq^{n+1};q)_n} {}_2\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q, qx\right),$$

$${}_2\phi_1\left(\begin{matrix} A, B \\ C \end{matrix}; q, z\right) = \sum_{i\geq 0} \frac{(A;q)_i (B;q)_i}{(q;q)_i (C;q)_i} z^i,$$

$$(A;q)_i = (1-A)(1-Aq)\cdots(1-Aq^{i-1})$$

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Corteel, Kim 2020

There are combinatorial models for $\sigma_{n,k}$, $\nu_{n,k}$.

using lecture hall partitions.

Def) $\lambda \vdash n \iff \lambda$ is a partition of n

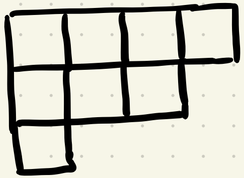
$\iff \lambda = (\lambda_1, \dots, \lambda_k), \quad \lambda_1 + \dots + \lambda_k = n$

$\lambda_1 \geq \dots \geq \lambda_k > 0$

Each $\lambda_i > 0$ is called a part.

ex) $\lambda = (4,3,1,1)$ is a partition of $9 = 4+3+1+1$.

- Young diagram of $\lambda = (4,3,1)$ is

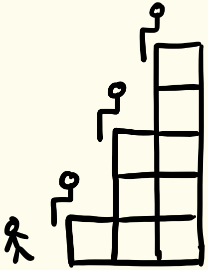


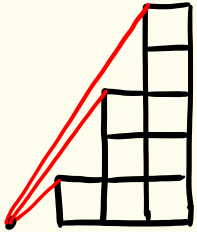
Lecture hall partitions

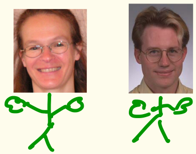
$\lambda = (\lambda_1, \dots, \lambda_n)$ is a lecture hall partition if

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \cdots \geq \frac{\lambda_n}{1} \geq 0$$

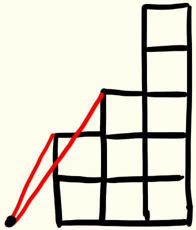
ex) $(5,3,1)$ is LHP







$(5,3,2)$ is NOT LHP



Thm (Bousquet-Mélou, Eriksson, 1997)

$$\sum_{\lambda \in L_n} q^{|\lambda|} = \prod_{i=1}^{n} \frac{1}{1-q^{2i-1}}$$

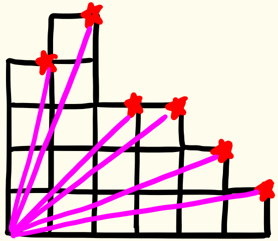
Anti-lecture hall compositions

$\alpha = (\alpha_1, \dots, \alpha_n)$ is an anti-lecture hall composition

(or a planetarium composition)

if $\quad \frac{\alpha_1}{1} \geq \frac{\alpha_2}{2} \geq \cdots \geq \frac{\alpha_n}{n} \geq 0$.

ex)



$(4, 5, 3, 3, 2, 1)$

Thm (Corteel, Savage, 2003)

$$\sum_{\alpha \in A_n} q^{|\alpha|} = \prod_{i=1}^{n} \frac{1+q^i}{1-q^{i+1}}$$

<u>Thm</u> (Corteel, Kim, 2020)

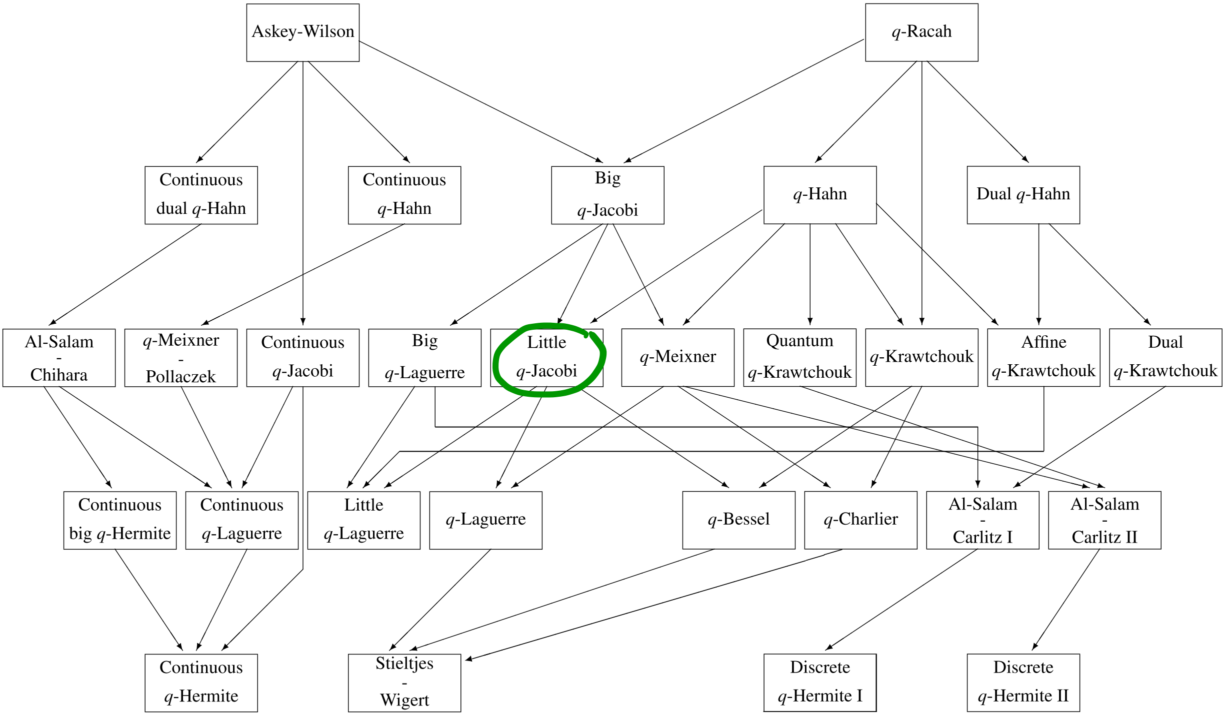
Letting $a = -uv$, $b = -u/v$ for little q-Jacobi,

$$\sigma_{n,k} = \sum_{\frac{\alpha_1}{k+1} \geq \cdots \geq \frac{\alpha_{n-k}}{n} \geq 0} u^{\lfloor \alpha \rfloor} v^{o(\lfloor \alpha \rfloor)} q^{|\alpha|}$$

$$\nu_{n,k} = \sum_{\frac{\lambda_1}{n} > \cdots > \frac{\lambda_{n-k}}{k+1} > 0} u^{\lfloor \lambda \rfloor} v^{o(\lfloor \lambda \rfloor)} q^{|\lambda|}$$

$\Rightarrow$ $\sigma_{n,k}$ = generating function for anti-lecture hall compositions

$\nu_{n,k}$ = " " " lecture hall partitions.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q-Charlier
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Al-Salam - Carlitz II
Continuous q-Hermite
Stieltjes - Wigert
Discrete q-Hermite I
Discrete q-Hermite II

Askey-Wilson polynomials

$$p_n(x;a,b,c,d|q) = \frac{(ab,ac,ad;q)_n}{2^n a^n (abcdq^{n-1};q)_n}\,{}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix};q,q\right),$$

where $x = \cos\theta = (e^{i\theta} + e^{-i\theta})/2$.

$$p_{n+1} = (x - b_n)p_n - \lambda_n p_{n-1}$$

$$b_n = \frac{1}{2}(a + a^{-1} - A_n - C_n), \quad \lambda_n = \frac{1}{4}A_{n-1}C_n$$

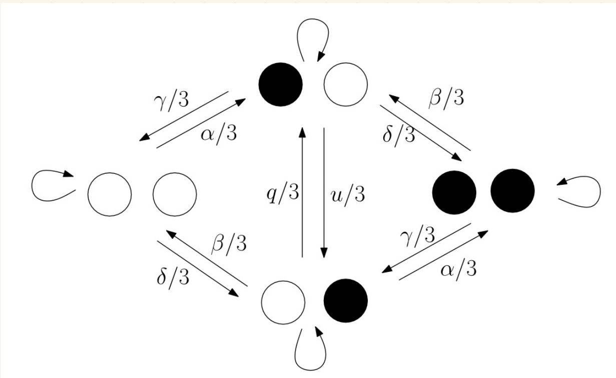
$$A_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})},$$

$$C_n = \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}.$$

Fact: $p_n(x;a,b,c,d|q)$ is symmetric in a,b,c,d.

Thm (Corteel, Williams, 2011)

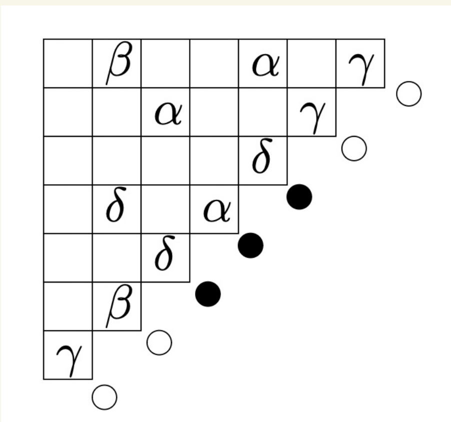
The Askey-Wilson moment μ_n is a partition function for ASEP.



state diagram of ASEP (asymmetric simple exclusion process)

Thm (Corteel, Stanley, Stanton, Williams. 2012)

$$
\begin{aligned}
&2^n (abcd;q)_n \mu_n(a,b,c,d;q) \\
&\quad = i^{-n} \sum_{T\in\mathcal{T}(n)} (-1)^{\mathrm{b}(T)} (1-q)^{A(T)+B(T)+C(T)+D(T)-n} q^{E(T)} \\
&\qquad \times (ac)^{C(T)} (bd)^{D(T)} \big((1+ai)(1+ci)\big)^{n-A(T)-C(T)} \big((1-bi)(1-di)\big)^{n-B(T)-D(T)}.
\end{aligned}
$$



staircase tableaux

After rescaling & reparametrization

μ_n is a polynomial in $\alpha, \beta, \gamma, \delta$.

$$
\alpha = \frac{1-q}{(1+ai)(1+ci)}, \qquad \beta = \frac{1-q}{(1-bi)(1-di)},
$$

$$
\gamma = \frac{ac(1-q)}{(1+ai)(1+ci)}, \qquad \delta = \frac{bd(1-q)}{(1-bi)(1-di)}.
$$

Motivation

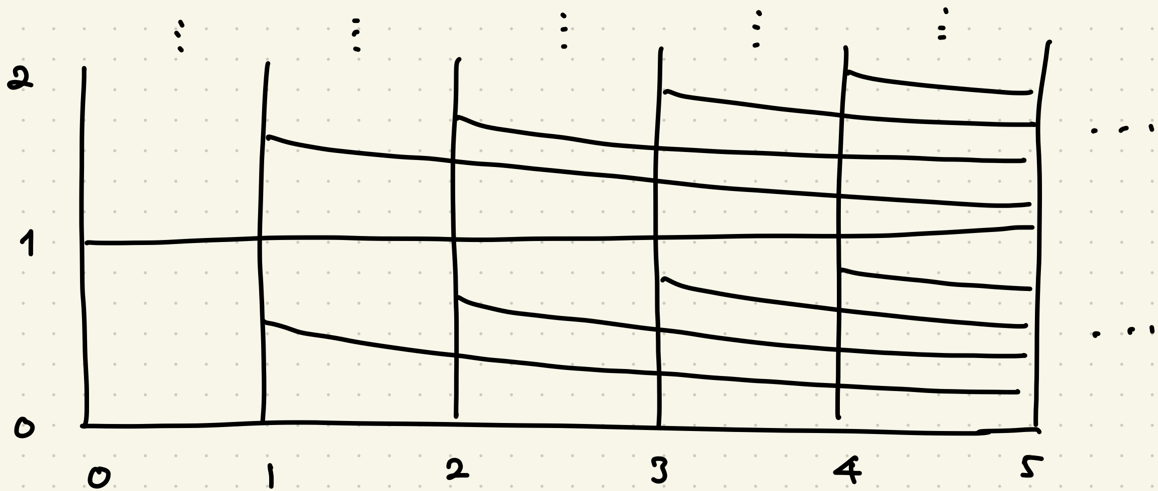
① Find a lecture hall partition model for Askey-Wilson.

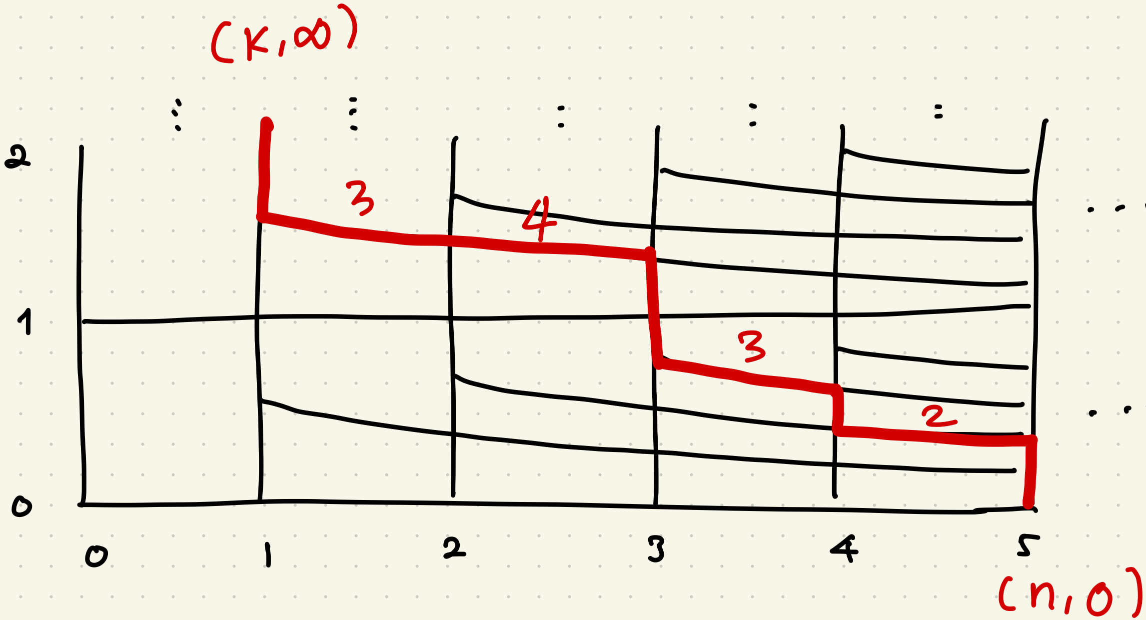
② Find a combinatorial proof of the symmetry in a, b, c, d.

Q. How can we generalize Corteel-Kim result on little q-Jacobi to Askey-Wilson?

$\rightarrow$ We need a new approach to "discover" a combinatorial model.

Lecture hall graph (Introduced by Corteel-Kim 2020
Further studied by Corteel-Keating-Nicoletti 2021)





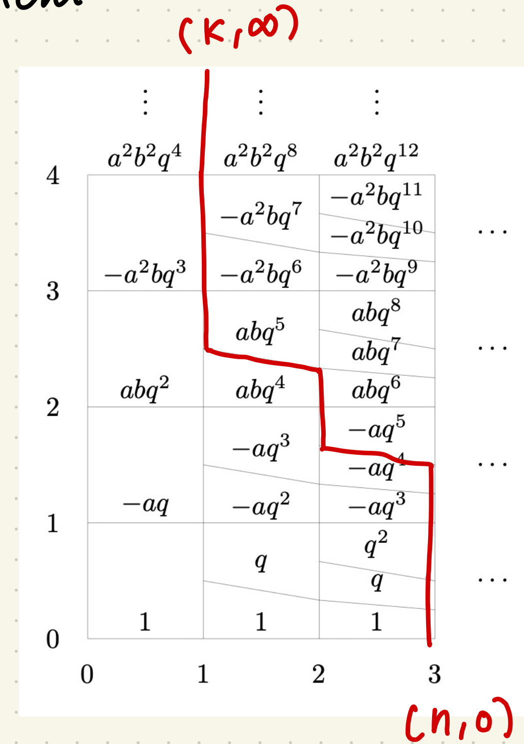
$$\frac{3}{2} \geqslant \frac{4}{3} \geqslant \frac{3}{4} \geqslant \frac{2}{5} \qquad (k=1,\ n=5).$$

$$\frac{\lambda_{k+1}}{k+1} \geqslant \frac{\lambda_{k+2}}{k+2} \geqslant \cdots \geqslant \frac{\lambda_n}{n} \quad \longleftrightarrow \quad \text{path from } (k,\infty) \text{ to } (n,0)$$

Thm (Corteel, Kim, 2020)

little q-Jacobi mixed moment

$$\sigma_{n,k} = \sum_{p:(k,\infty)\to(n,0)} wt(p)$$

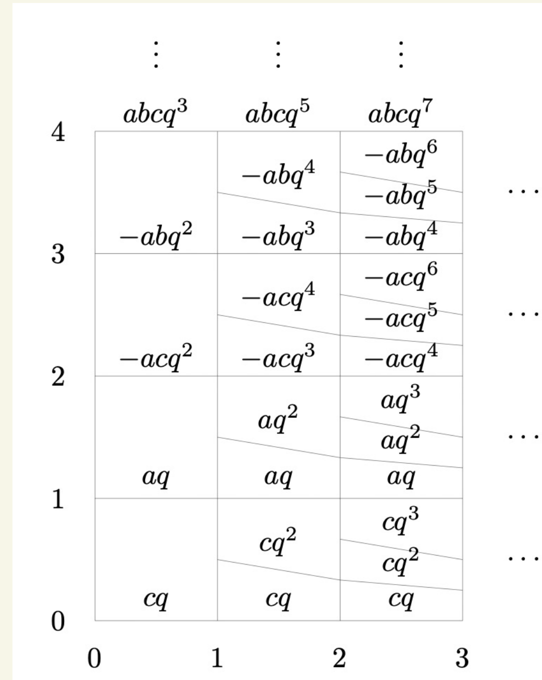


Thm (CJKK, 2023)

$\boxed{\text{big } q\text{-Jacobi}}$

$$p_n(x;a,b,c|q) = \frac{(aq,cq;q)_n}{(abq^{n+1};q)_n}\, {}_3\phi_2\left(\begin{matrix} q^{-n},\ abq^{n+1}x \\ aq,\ cq \end{matrix}; q, q\right)$$

$$\sigma_{n,k} = \sum_{p:(k,\infty)\to(n,0)} \mathrm{wt}(p)$$



Cor Let $\tilde{\sigma}_{n,k} = (-1)^{n-k}\sigma_{n,k}(-a,-b,-c)$.

$\Rightarrow$ $(\tilde{\sigma}_{n,k})_{n,k=0}^{\infty}$ is totally positive.

Equivalently multivariate big q-Jacobi is Schur positive.

Thm (CJKK, 2023)

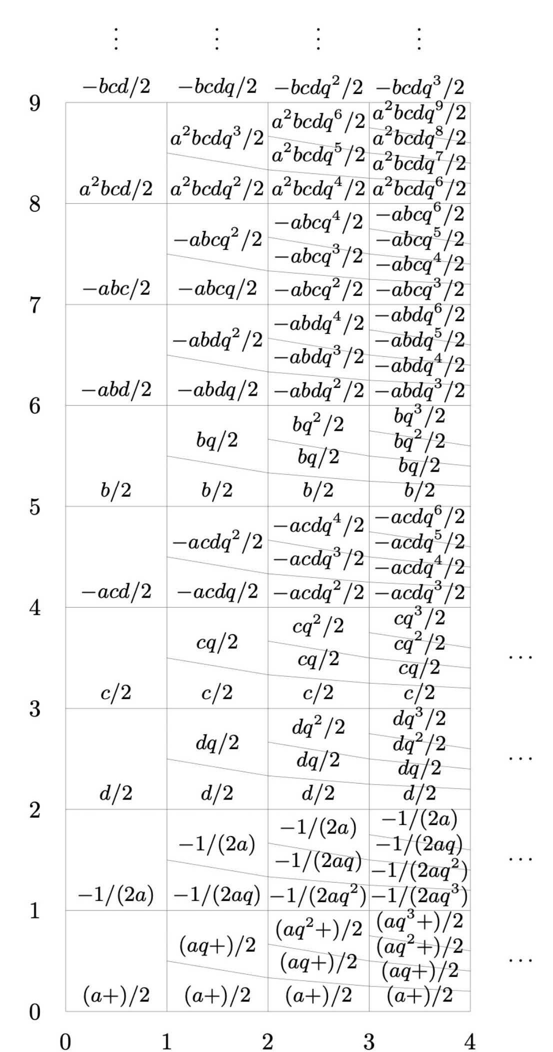
Askey-Wilson mixed moment

$$\sigma_{n,k} = \sum_{p:(k,\infty)\to(n,0)} wt(p)$$

$$\nu_{n,k} = \sum_{\substack{p:(k,0)\to(n,\infty)\\ \text{no consecutive east step.}}} wt(p)$$

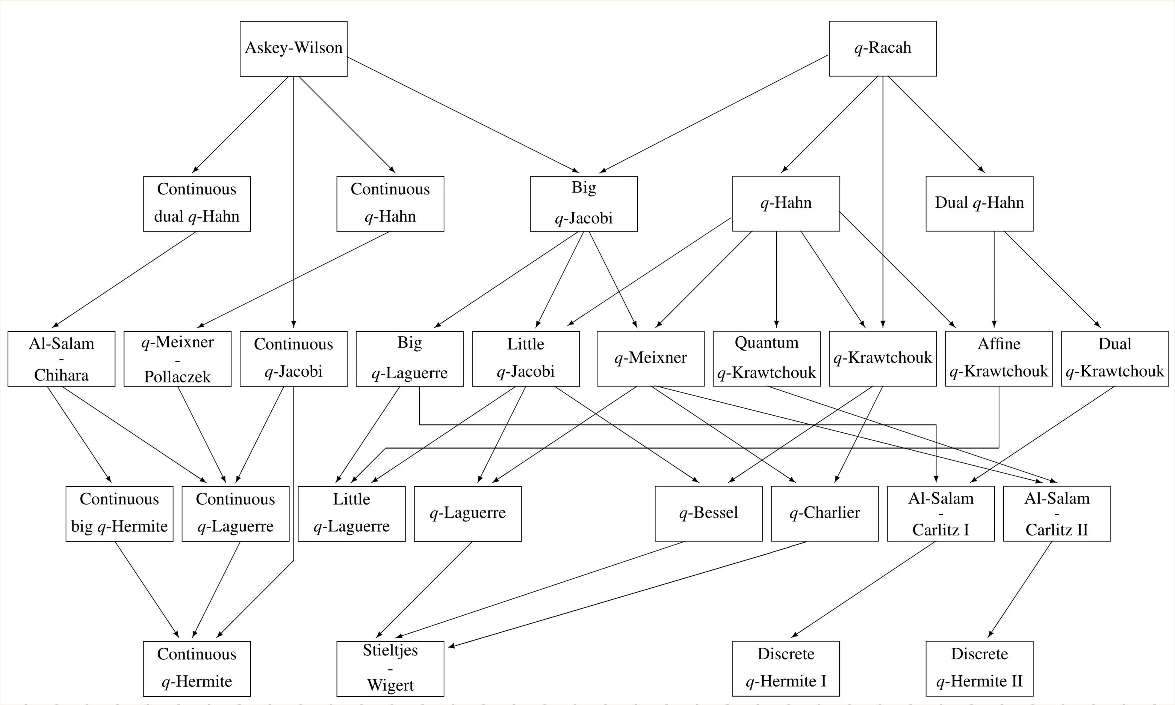
$$(\sigma_{n,k})(\nu_{n,k}) = I$$

$(x+) = x + x^{-1}$



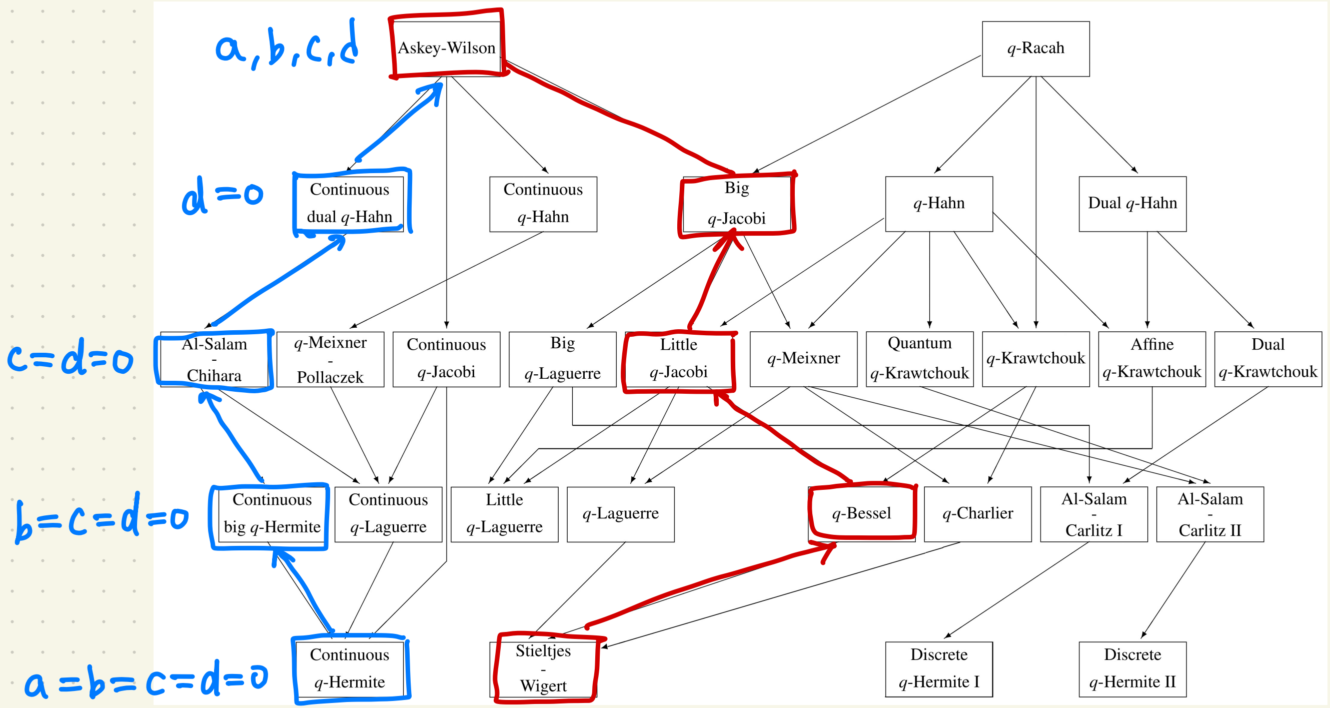
Thm (CJKK, 2023).

There is a lecture hall graph model for $\mu_{n,k}$ and $\nu_{n,k}$ for every orthogonal polynomial in Askey scheme



Idea of proof

Two bootstrapping methods



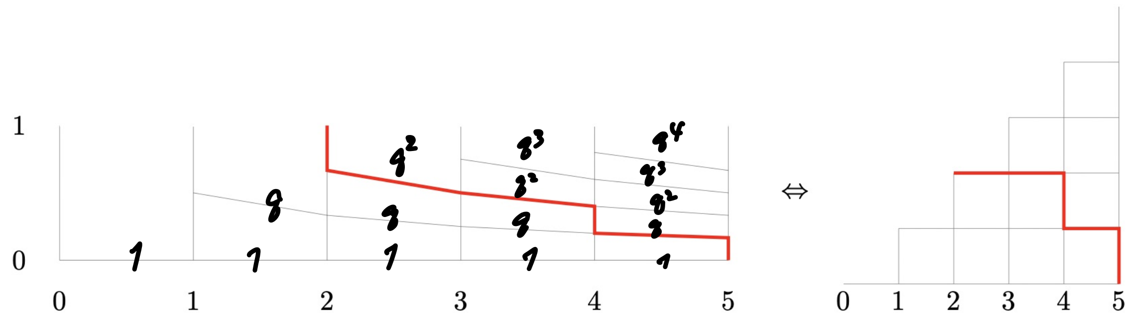
4.1. **Stieltjes–Wigert.** The monic *Stieltjes–Wigert polynomials* are defined by

$$S_n(x;q) = (-1)^n q^{-n^2} {}_1\phi_1 \left(\begin{matrix} q^{-n} \\ 0 \end{matrix} ; q, -q^{n+1}x \right).$$

The mixed moments $\sigma_{n,k}$ and the coefficients $\nu_{n,k}$ of Stieltjes–Wigert polynomials are given by

$$\sigma_{n,k} = q^{k^2-n^2+\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q,$$

$$\nu_{n,k} = (-1)^{n-k} q^{k^2-n^2} \begin{bmatrix} n \\ k \end{bmatrix}_q.$$



$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{p:(k,1)\to(n,0)} wt(p)$$

Weight system of Height 1 is unique.

Proposition 3.1. *Let $\{a_{n,k}\}_{n\geq k\geq 0}$ be a triangular array of indeterminates and let $A = (a_{n,k})_{n,k\geq 0}$, where $a_{n,k} = 0$ if $n < k$. Then there is a unique weight system w of height 1 satisfying $h^w_{n,k} = a_{n,k}$ for all $n \geq k \geq 0$. Moreover, each $w(0; i, j)$ is the rational function in the indeterminates $a_{n,k}$ given by*

$$w(0; i, j) = \frac{A_{j+1}(i-j+1, 0)A_j(i-j, 0)}{A_j(i-j+1, 0)A_{j+1}(i-j, 0)}. \tag{3.1}$$

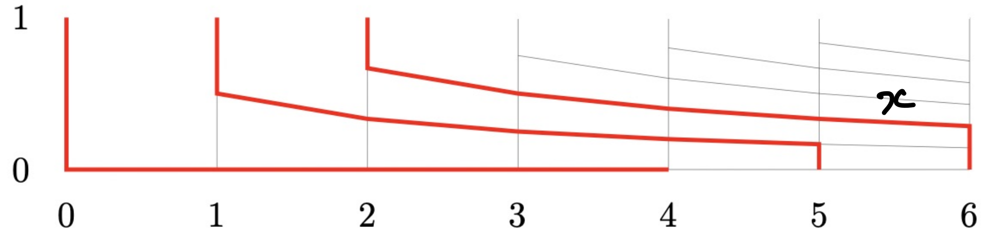
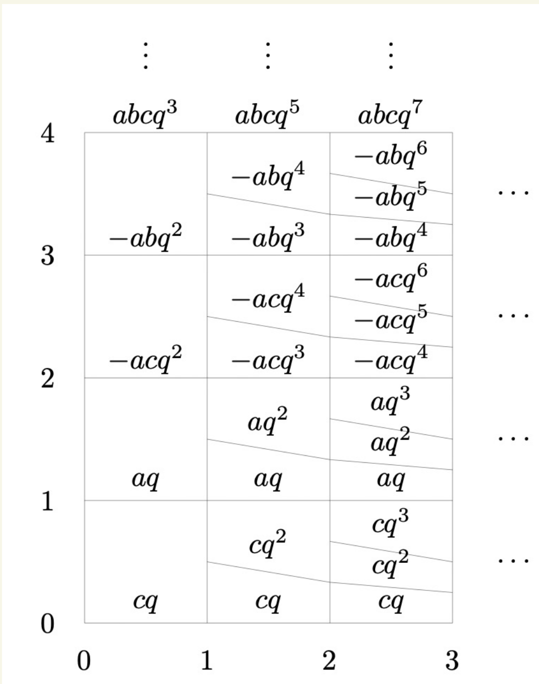


FIGURE 6. The unique family of nonintersecting paths from $(0,1), (1,1), (2,1)$ to $(4,0), (5,0), (6,0)$.

Weight system of infinite height is not unique!

How can we find a model like this?



big q-Jacobi.

$$\sigma_{2,1} = cq + cq^2 + aq + aq^2 - acq^3 - acq^4 + \cdots$$

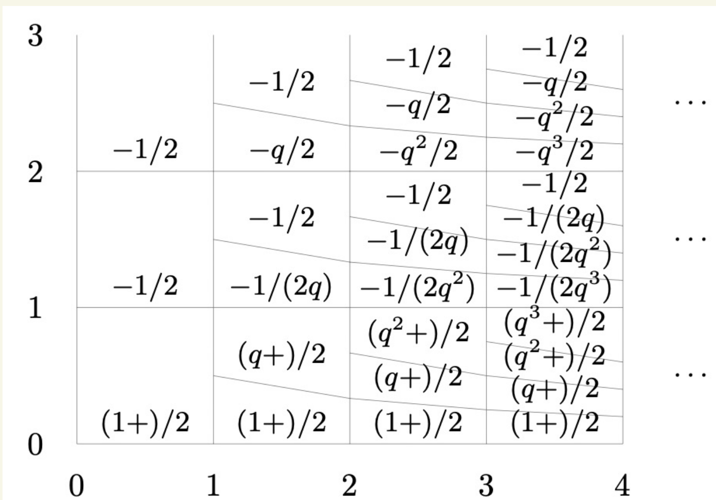
Simply order the monomials lexicographically $q < c < a$.

Then we can prove this "conjecture" by induction.

Thm (CJKK)

Continuous q-Hermite

$$\sigma_{n,k} = \sum_{p:(k,3)\to(n,0)} wt(p)$$



$(x+) = x + x^{-1}.$

<u>Open problem</u>

Find a combinatorial proof that $\mu_{n,k}$ is a polynomial in q.

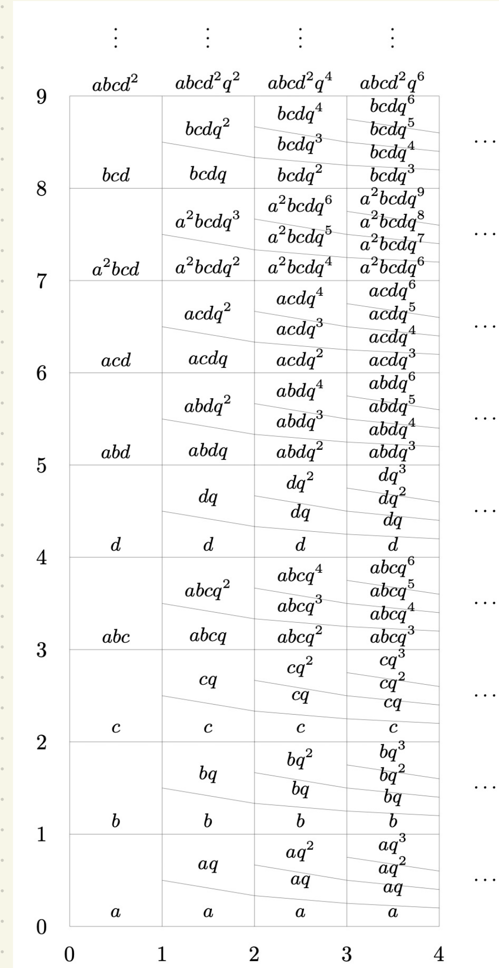
Def) mixed moment of AW with respect to q-Hermite

$$H_n(x) = \sum_k \tilde{\sigma}_{n,k}\, p_k^{AW}(x)$$

<u>Thm</u> (CJKK)

$$\frac{2^{n-k}}{i}\, \tilde{\sigma}_{n,k}(ai, bi, ci, di) = \sum_{p:(k,\infty)\to(n,0)} wt(p)$$

<u>Cor</u> Askey-Wilson is symmetric in a, b, c, d.



Advantages of lecture hall graph models.

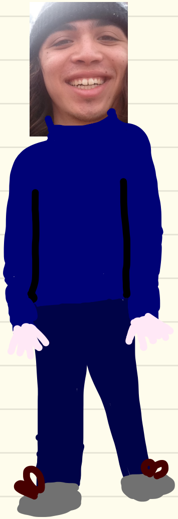
① a model for $\sigma_{n,k}$ gives a model for $\nu_{n,k}$.

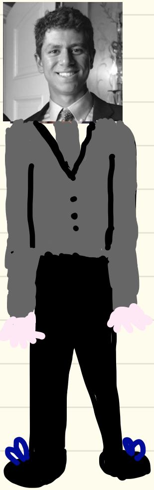
② multivariate moments can be obtained by Lindström-Gessel-Viennot Lemma

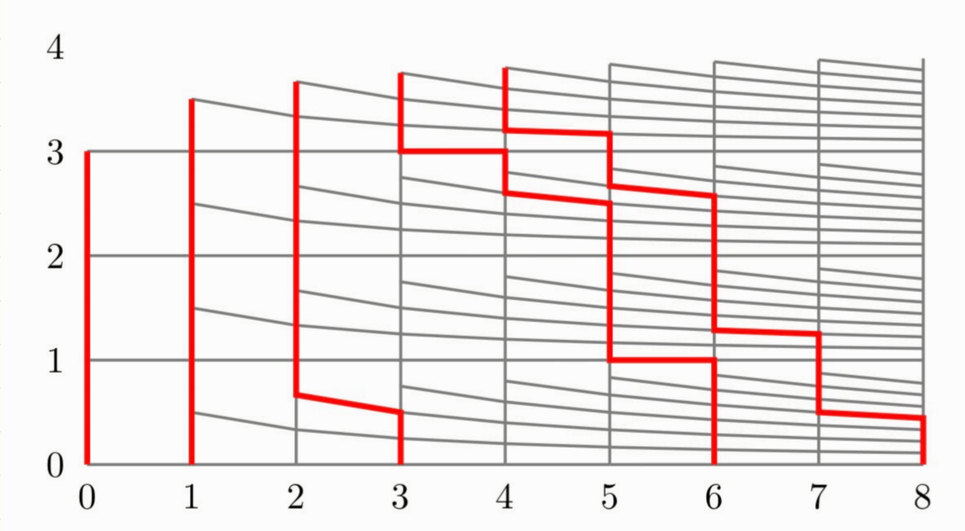
③ Total positivity

④ Applicable to all OPS in Askey scheme.

Asymptotics (Corteel, Keating Nicoletti) 2020





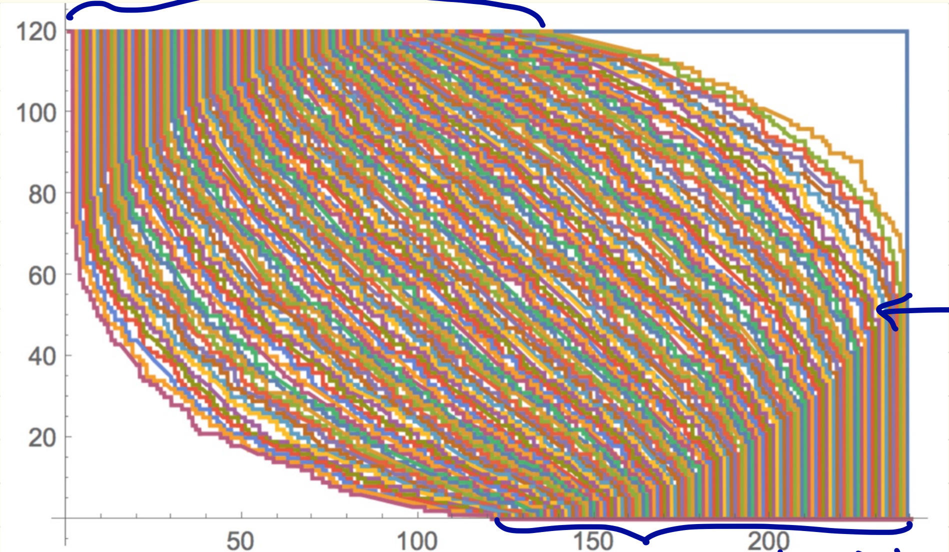


Matthew Nicoletti
(MIT)
↳ Berkeley & Stanford

David Keating
Wisconsin → Berkeley & Stanford

Find starting & ending points, pick a random path system

starting points



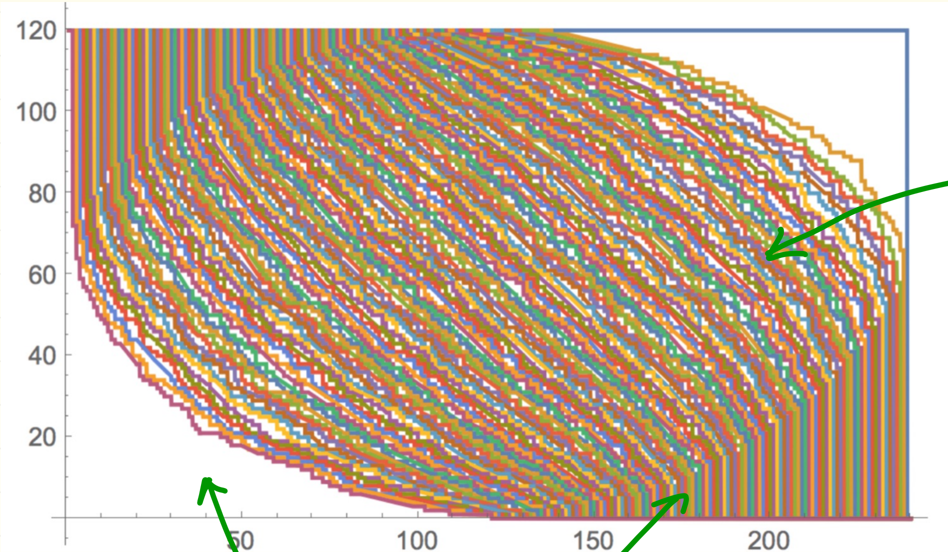
non intersecting paths

arrival points

λ a partition

Arctic curve phenomenon

Sharp phase separation

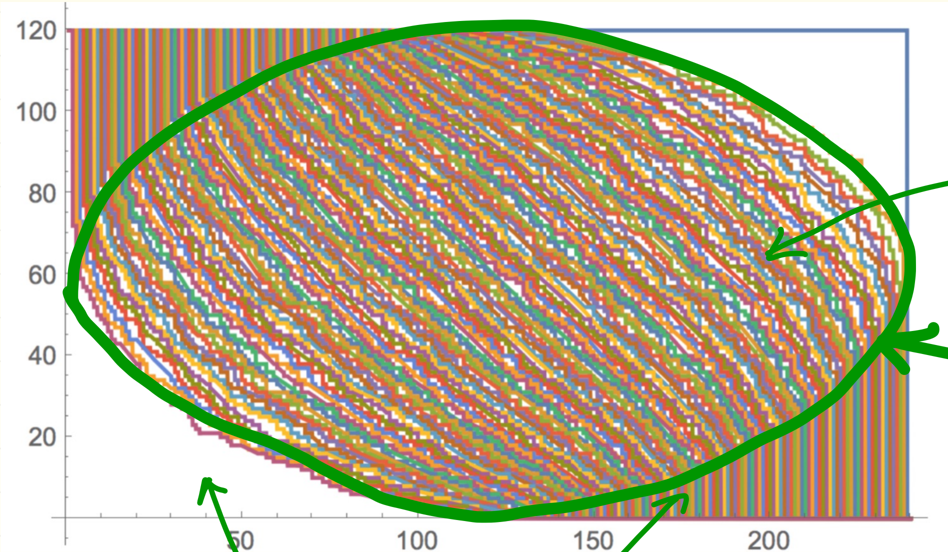


liquid

frozen

Arctic curve phenomenon

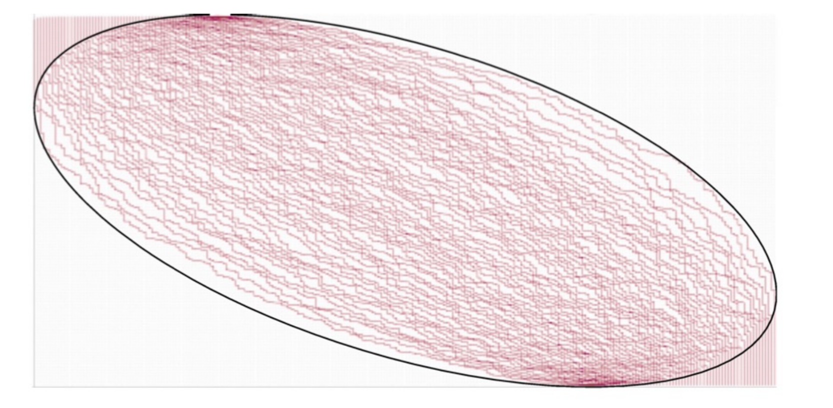
Sharp phase separation



liquid

arctic curve

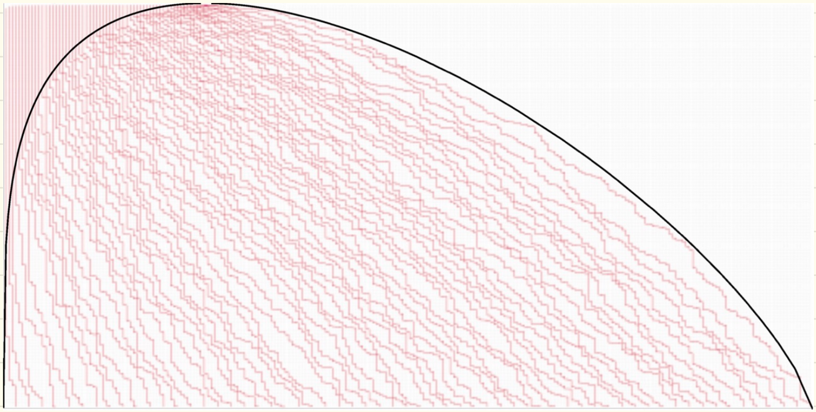
frozen



$p \geq 1$

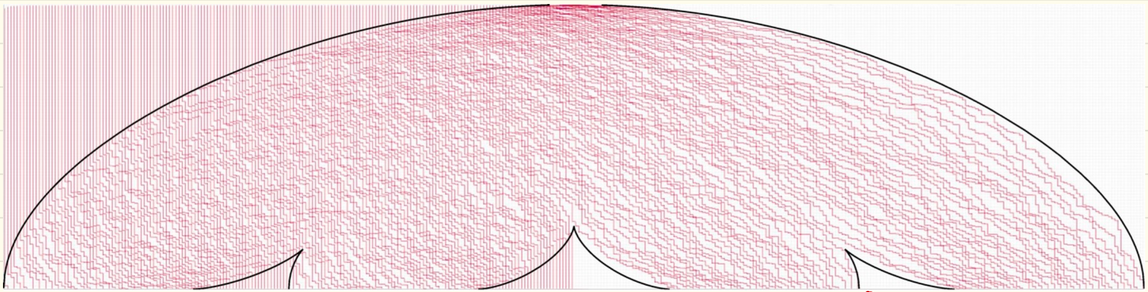
$\lambda = (pn, pn, \ldots, pn)$

Algebraic curve of degree 2



$\lambda = (pn, p(n-1), \ldots, 2p, p)$

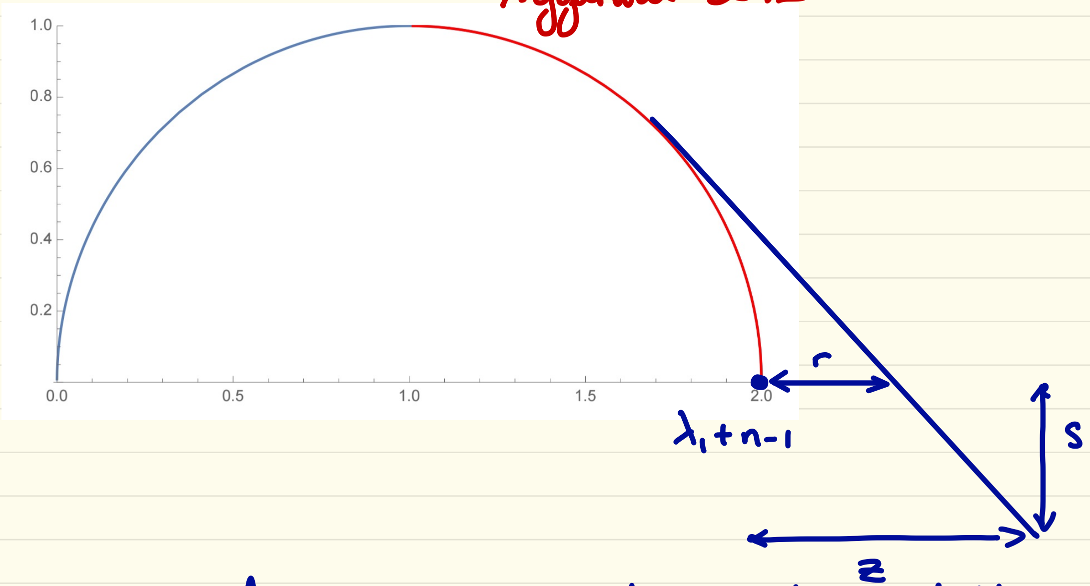
Algebraic curve of degree p.



cusps

Tangent method (Colomo & Sportiello) 2016

Aggarwal 2019

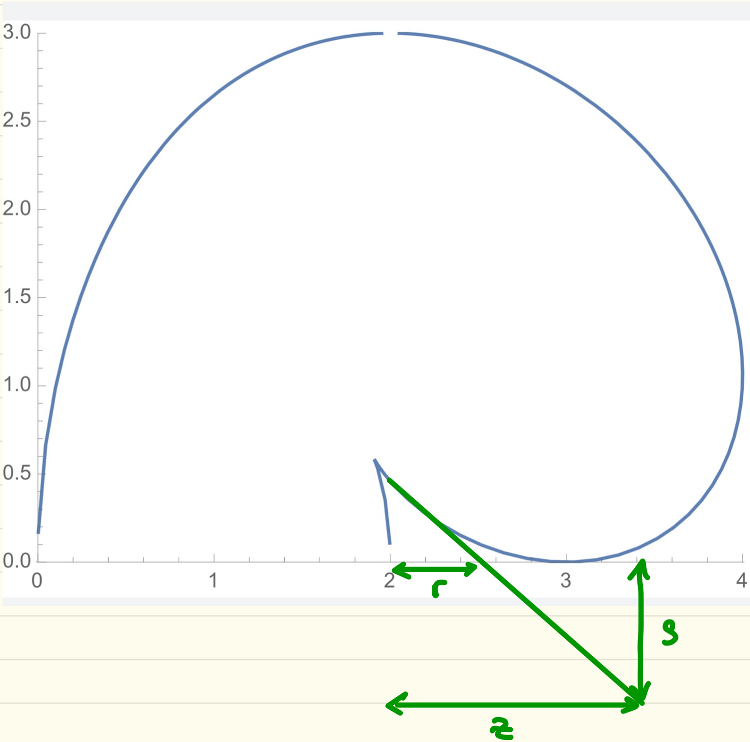


Philosophy: fix z, s, compute most probable r

$\Rightarrow$ gives a straight line tangent to the arctic curve

Vary $z \Longrightarrow$ give a parametrization of the curve

Cusps

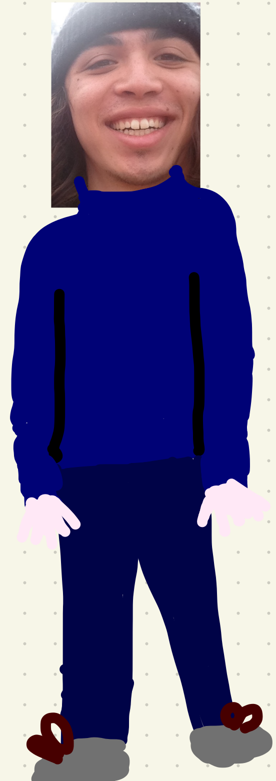


open question:

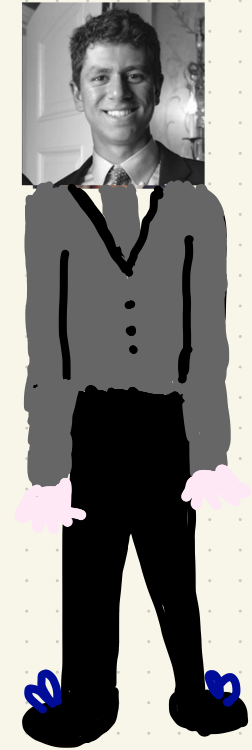
Can we study the asymptotic behavior of a path systems using the weight of the Askey-Wilson polynomials?

Merci!

Bhargavi Jonnadula



David Keating



Matthew Nicoletti

Let $$P_\lambda(x) = \frac{\det\left(p_{\lambda_i+n-i}(x_j)\right)}{\prod_{i<j}(x_i - x_j)}.$$

If $$\sigma_{n,k} = \sum_{\pi : (k,0) \to (n,\infty)} wt(\pi) \quad \text{in lecture hall graph}$$

then

$$M_{\lambda,\mu} = \sum_{\substack{T : LHT(\lambda/\mu) \\ \text{type } (n, \geq, >)}} wt(T)$$

			25	25	21
	16	18	21	10	4
8	9	2	0		
4	4	0