Enumeration and Succinct Encoding of AVL Trees

J. Chizewer, S. Melczer, J.I. Munro, A. Pun

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Motivation - Storing Binary Trees
Storing Binary Trees

Naive Approach

<table>
<thead>
<tr>
<th>Nodes</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pointers</td>
<td>2, 3</td>
<td>4, ∅</td>
<td>∅, 5</td>
<td>6, 7</td>
<td>8, ∅</td>
<td>∅, ∅</td>
<td>9, ∅</td>
<td>∅, ∅</td>
<td>∅, ∅</td>
</tr>
</tbody>
</table>
Storing Binary Search Trees

Naive approach storage requirements for $n$ nodes

Each pointer requires $O(\log n)$ bits.

There are $2^n$ pointers.

Total Requirement: $O(n \log n)$.
Storing Binary Search Trees

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\[
B(z) = 1 - \sqrt{1 - 4z^2} \\
[n] B(z) = 4n \sqrt{\pi n^3}(1 + o(1)) \\
\log_2 4n \sqrt{\pi n^3}(1 + o(1)) = 2n + o(n)
\] 

Bits are needed to give each tree a unique string.
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$$B(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$

Use analytic tools to get asymptotics of the coefficients:

$$[z^n]B(z) = \frac{4^n}{\sqrt{\pi} n^3} (1 + o(1))$$

So $\log_2 \left( \frac{4^n}{\sqrt{\pi} n^3} (1 + o(1)) \right) = 2n + o(n)$ bits are needed to give each tree a unique string.
Succinct Encodings

Definition
We call an encoding succinct if it uses the information-theoretic minimum number of bits
Step 1: Make sure every internal node has exactly two children by adding external nodes.
Succinct Encoding – Binary Trees

= internal node   = external node
Step 2: Traverse the tree in level order, writing 1 for internal nodes and 0 for external nodes.
Succinct Encoding – Binary Trees

= internal node  = external node

Bitmap:

\[
\begin{array}{ccccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}
\]
Succinct Encoding Analysis

How many bits did we use?
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- We used 1 bit per node in the new tree.

That is $2n + 1$ bits which is succinct.
What if we care about more than just storage costs?

In general, binary trees can be very inefficient.

▶ A random binary tree on $n$ nodes has average depth $\sqrt{n}$.
▶ That means $O(\sqrt{n})$ comparisons on average to find a node.
▶ Average depth in binary search tree is $O(\log n)$.
▶ But worst case $O(n)$. 


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Can we construct a tree with shorter depth?
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AVL Trees
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Enumerating AVL Trees

Is our encoding for binary trees still succinct for AVL trees?
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Let’s enumerate!
Enumerating AVL Trees

Start with a symbolic class:

\[ A_0 = \cdot, \quad A_1 = \cdot \times \cdot, \quad A_{h+2} = A_{h+1} \times (A_{h+1} + 2A_h) \]

\[ A = \sum_{h \geq 0} A_h(z) \]

Where \( A_h(z) \) is the GF for AVL trees with height \( h \).

Note: enumeration is harder because we are recursing on height but enumerating by size.
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Values $\alpha_h$ for even $h$ (red) and odd $h$ (blue).
Enumerating AVL Trees

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Lemma (C., Melczer, Munro, Pun, 2023)

The limit $\alpha = \lim_{h \to \infty} \alpha_h = 0.5219 \ldots$ exists.
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Lemma (C., Melczer, Munro, Pun, 2023)

$A(z)$ converges in the disk $|z| < \alpha$
Enumerating AVL Trees

Theorem (C., Melczer, Munro, Pun, 2023)

There are $\alpha^{-n} \theta(n)$ AVL trees on $n$ nodes where $\theta(n)$ is sub-exponential.
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1. Apply the previous lemmas to conclude $\alpha$ is the singularity of smallest modulus
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2. Conclude $1/\alpha$ is the exponential growth of $a_n$. 

□
Generalizing the Main Theorem

Let $\mathcal{F} = \bigcup_{h=0}^{\infty} \mathcal{F}_h$ such that $F_h(z)$ are non-constant and

$$F_h(z) = f(F_{h-1}(z), F_{h-2}(z), \ldots, F_{h-c}(z)) \quad \text{for all} \quad h \geq c$$

where $c$ is a positive integer and $f$ has non-negative coefficients.

\[\text{Theorem (C., Melczer, Munro, Pun, 2023)}\]

If $f$ has positive fixed point $B$ and $\beta_h$ is the positive root of $B = F_h(z)$ then the limit $\lim_{h \to \infty} \beta_h$ exists\(^1\) and there are $\beta^{-n}\theta(n)$ objects in $\mathcal{F}$ of size $n$ where $\theta(n)$ is sub-exponential.

\[^1f\text{ must satisfy an additional technical condition}\]
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\(^1\) $f$ must satisfy an additional technical condition
Back to Encoding AVL Trees

Using the enumeration result, we know that

$$\log_2 (\alpha^{-n} \theta(n)) = (0.938 \ldots) n + o(n)$$

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Can we improve on the $2n + 1$ bits used in the general binary tree encoding?
Succinct Encoding for AVL Trees

\[ n = 32 \]
Succinct Encoding for AVL Trees

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Identify subtree \( \tau \) (in red) of all nodes whose parents are the roots of large subtrees.
Succinct Encoding for AVL Trees

\[ n = 32 \]

Name the shapes of subtrees rooted at leaves (in blue) of \( \tau \) (in red/blue)
Succinct Encoding for AVL Trees

\[ n = 32 \]

00 =

01 =

10 =
Succinct Encoding for AVL Trees

\( n = 32 \)

Write \( \tau \), codeword mapping, and names of leaf trees in leaf order.

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Succinct Encoding for AVL Trees

Algorithm Summary

1. Identify subtree $\tau$ of all nodes whose parents are the roots of large subtrees.
2. The leaves of $\tau$ are the roots of small subtrees.
3. Write a unique code word for each subtree in a lookup table.
4. Store $\tau$, the lookup table, and the codewords in order.
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1. The lookup table has size $o(n)$ because there are few distinct shapes
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2. The tree $\tau$ has size $o(n)$
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2. The tree $\tau$ has size $o(n)$
3. The names of the codewords are asymptotically optimal in bits per node used.
Generalizing the Encoding

Definition
A class $\mathcal{T}$ of trees is \textit{weakly tame}\textsuperscript{2} if

1. All subtrees of a tree in the class are also in the class
2. The subtree $\tau$ defined previously satisfies $|\tau| = o(n)$
3. $\log |\mathcal{T}_n| = c \cdot n + o(n)$ for some constant $c$.

\textsuperscript{2}Following the work of J. Ian Munro, Patrick K. Nicholson, Louisa Seelbach Benkner, and Sebastian Wild.
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Theorem (C., Melczer, Munro, Pun, 2023)

There exists a static succinct encoding for any weakly tame class of trees.

\(^2\)Following the work of J. Ian Munro, Patrick K. Nicholson, Louisa Seelbach Benkner, and Sebastian Wild.
Conclusion

Theorem (C., Melczer, Munro, Pun, 2023)

If $f$ has positive fixed point $B$ and $\beta_h$ is the positive root of $B = F_h(z)$ then the limit $\beta = \lim_{h \to \infty} \beta_h$ exists\(^3\) and there are $\beta^{-n} \theta(n)$ objects in $\mathcal{F}$ of size $n$ where $\theta(n)$ is sub-exponential.

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There exists a static succinct encoding for any weakly tame class of trees.

\(^3f\) must satisfy an additional technical condition
Future Work

1. Construct dynamic succinct encodings
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2. Apply enumeration tools to other balanced data structures.
Thank you!