Enumeration and Succinct Encoding of AVL Trees

J. Chizewer, S. Melczer, J.I. Munro, A. Pun

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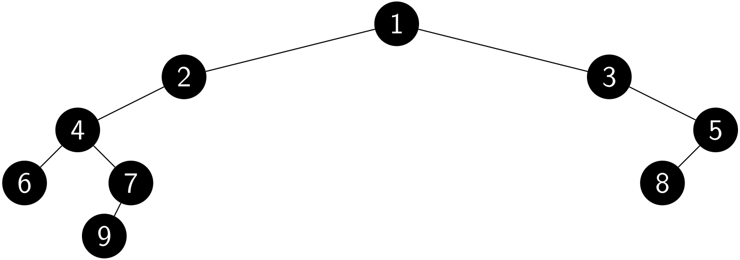
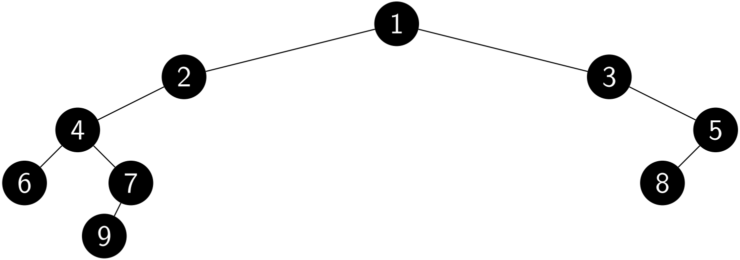
Motivation - Storing Binary Trees



Storing Binary Trees



Naive Approach

Nodes	1	2	3	4	5	6	7	8	9
Pointers	2, 3	4, ∅	∅, 5	6, 7	8, ∅	∅, ∅	9, ∅	∅, ∅	∅, ∅

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Let's enumerate!

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Use analytic tools to get asymptotics of the coefficients:

$$[z^n]B(z) = \frac{4^n}{\sqrt{\pi n^3}}(1 + o(1))$$

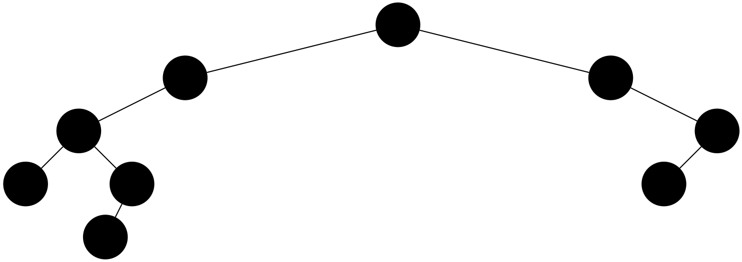
So $\log_2 \left(\frac{4^n}{\sqrt{\pi n^3}}(1 + o(1)) \right) = \boxed{2n + o(n)}$ bits are needed to give each tree a unique string

Succinct Encodings

Definition

We call an encoding *succinct* if it uses the information-theoretic minimum number of bits

Succinct Encoding – Binary Trees



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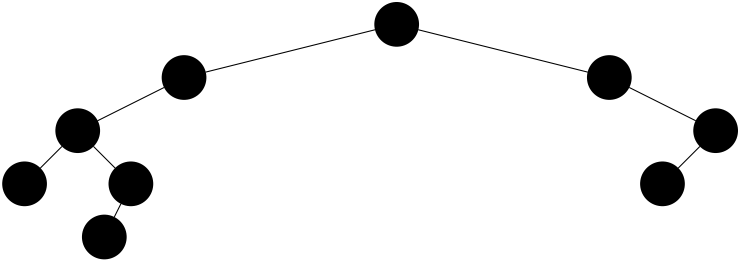

Step 1: Make sure every internal node has exactly two children by adding external nodes.

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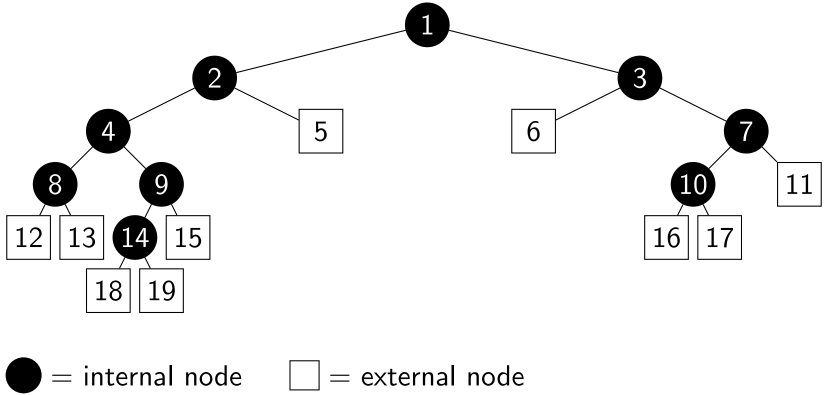

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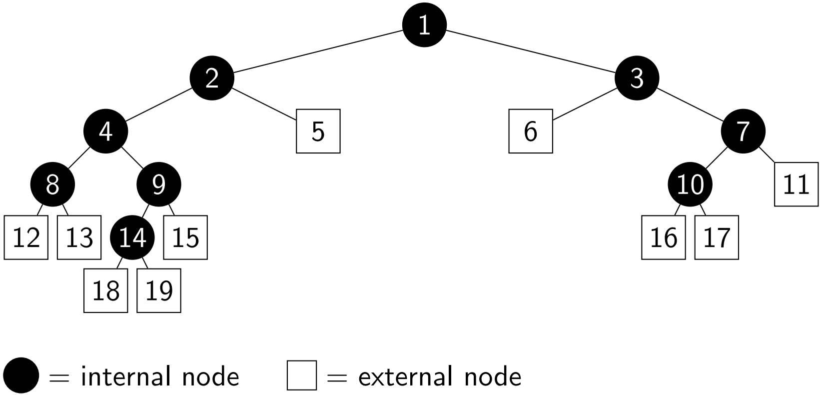
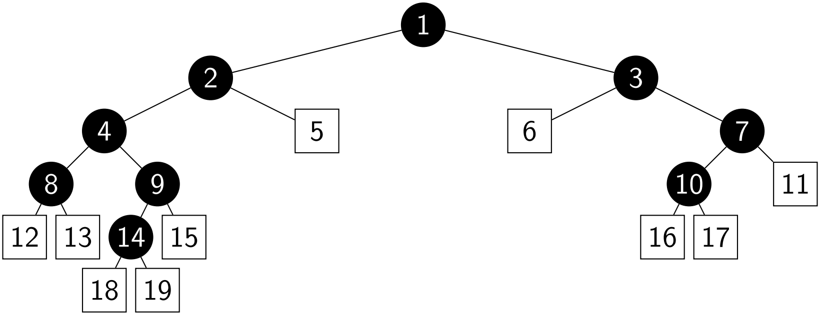


Step 2: Traverse the tree in level order, writing 1 for internal nodes and 0 for external nodes.

Succinct Encoding – Binary Trees



● = internal node   □ = external node

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Bitmap:	1	1	1	1	0	0	1	1	1	1	0	0	0	1	0	0	0	0	0

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That is $2n+1$ bits which is succinct.

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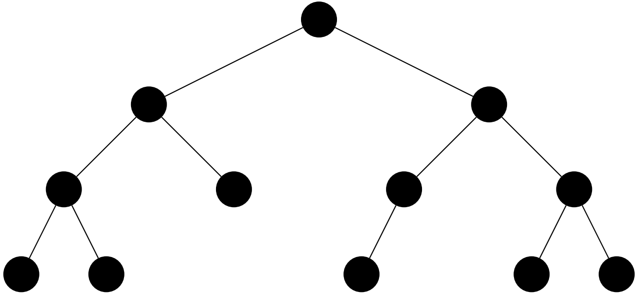

Enumerating AVL Trees

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Note: enumeration is harder because we are recursing on height but enumerating by size.

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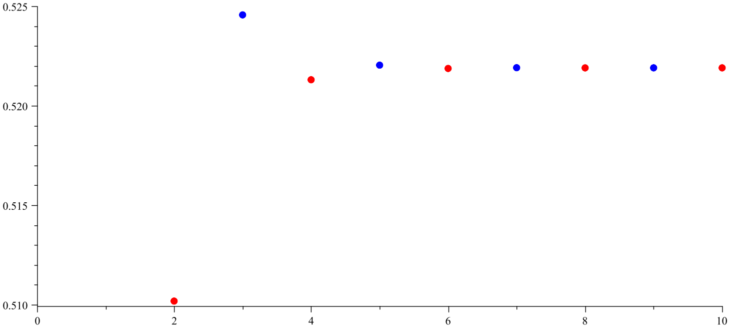

Values α_h for even h (red) and odd h (blue).

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Lemma (C., Melczer, Munro, Pun, 2023)

The limit $\alpha = \lim_{h\to\infty} \alpha_h = 0.5219\ldots$ *exists.*

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□

Lemma (C., Melczer, Munro, Pun, 2023)

$A(z)$ *converges in the disk* $|z| < \alpha$

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There are $\alpha^{-n}\,\theta(n)$ *AVL trees on* n *nodes where* $\theta(n)$ *is sub-exponential.*

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2. Conclude $1/\alpha$ is the exponential growth of a_n. $\square$

Generalizing the Main Theorem

Let $\mathcal{F} = \bigsqcup_{h=0}^{\infty} \mathcal{F}_h$ such that $F_h(z)$ are non-constant and

$$F_h(z) = f(F_{h-1}(z), F_{h-2}(z), \ldots, F_{h-c}(z)) \quad \text{for all} \quad h \geq c$$

where c is a positive integer and f has non-negative coefficients.

[1] f must satisfy an additional technical condition

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If f has positive fixed point B and β_h is the positive root of $B = F_h(z)$ then the limit $\beta = \lim_{h\to\infty} \beta_h$ exists[1] *and there are $\beta^{-n}\theta(n)$ objects in $\mathcal{F}$ of size n where $\theta(n)$ is sub-exponential.*

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Back to Encoding AVL Trees

Using the enumeration result, we know that

$$\log_2 \left(\alpha^{-n}\,\theta(n)\right) = (0.938\ldots)\,n + o(n)$$

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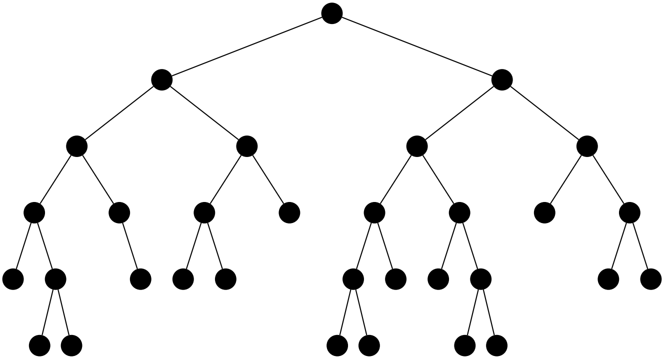
Can we improve on the $2n+1$ bits used in the general binary tree encoding?

Succinct Encoding for AVL Trees

$n = 32$



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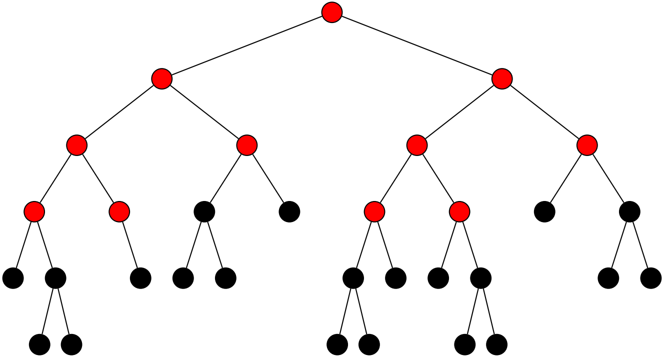

Identify subtree τ (in red) of all nodes whose parents are the roots of large subtrees.

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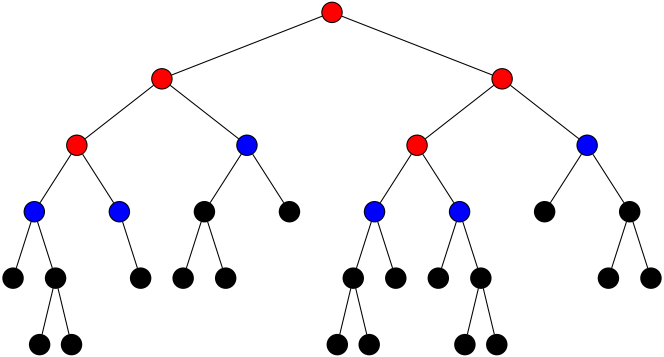

Name the shapes of subtrees rooted at leaves (in blue) of τ (in red/blue)

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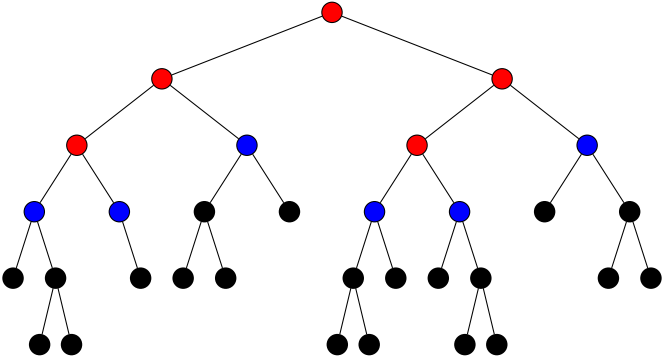
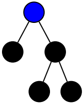
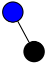
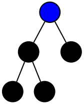


00 = 

01 = 

10 = 

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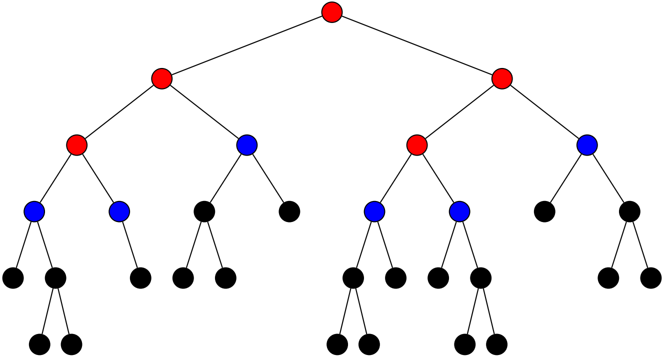
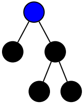
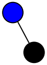
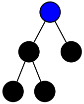


$00 =$ 

$01 =$ 

$10 =$ 

Write τ, codeword mapping, and names of leaf trees in leaf order.

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4. Store τ, the lookup table, and the codewords in order.

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3. The names of the codewords are asymptotically optimal in bits per node used.

Generalizing the Encoding

Definition

A class $\mathcal{T}$ of trees is *weakly tame*[2] if

1. All subtrees of a tree in the class are also in the class
2. The subtree τ defined previously satisfies $|\tau| = o(n)$
3. $\log \mathcal{T}_n = c \cdot n + o(n)$ for some constant c.

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Theorem (C., Melczer, Munro, Pun, 2023)

There exists a static succinct encoding for any weakly tame class of trees.

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Conclusion

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Conclusion

Theorem (C., Melczer, Munro, Pun, 2023)

If f has positive fixed point B and β_h is the positive root of $B = F_h(z)$ then the limit $\beta = \lim_{h\to\infty} \beta_h$ exists[3] *and there are $\beta^{-n}\,\theta(n)$ objects in $\mathcal{F}$ of size n where $\theta(n)$ is sub-exponential.*

Theorem (C., Melczer, Munro, Pun, 2023)

There exists a static succinct encoding for any weakly tame class of trees.

[3] f must satisfy an additional technical condition

Future Work

1. Construct dynamic succinct encodings

Future Work

1. Construct dynamic succinct encodings
2. Apply enumeration tools to other balanced data structures.

Thank you!