Galled Tree-Child Networks

(joint work with M. Fuchs and G.-R. Yu)

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Department of Mathematical Sciences
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What is a (Binary) Phylogenetic Network?

X . . . a finite set.

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A phylogenetic network (on X) is a rooted simple DAG (directed acyclic network) with the following nodes:

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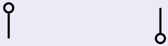

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tree nodes	1	2
reticulations	2	1

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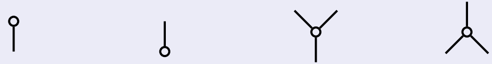

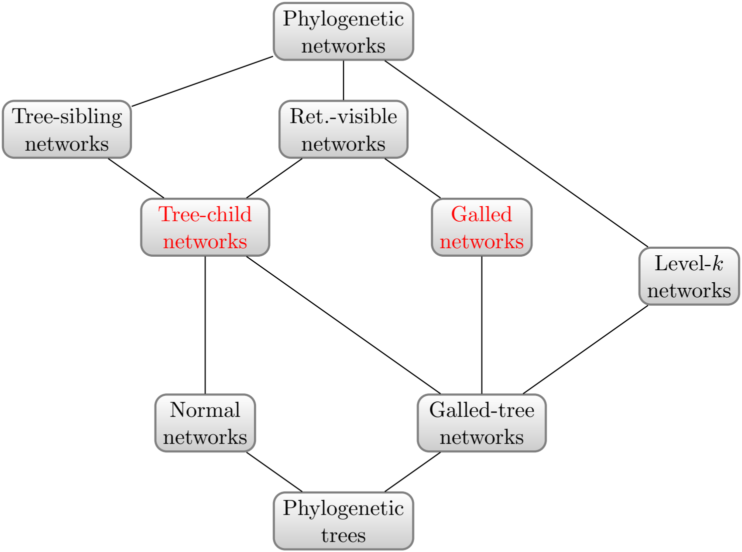
Phylogenetic networks
Tree-sibling networks
Ret.-visible networks
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Galled networks
Level-k networks
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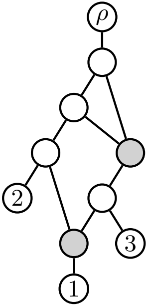
Phylogenetic trees

TC-Networks and Galled Networks

Definition

(a) A phylogenetic network is a tree-child network if every non-leaf node has at least one non-reticulation child.



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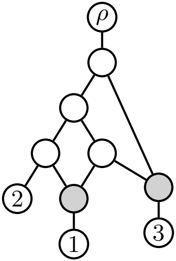

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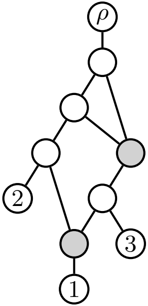
(b) A phylogenetic network is a galled network if every reticulation node is in a tree cycle.



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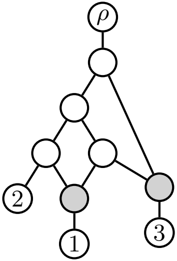

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Enumeration of TC-Networks and Galled Networks

TC_n ... # of tc-networks with n leaves;

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Theorem (Fuchs & Yu & Zhang; 2021)

We have

$$\mathrm{TC}_n = \Theta\left(n^{-2/3}e^{a_1(3n)^{1/3}}\left(\frac{12}{e^2}\right)^n n^{2n}\right),$$

where a_1 is the largest root of the Airy function of first order.

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Theorem (Fuchs & Yu & Zhang; 2022)

As $n \to \infty$,

$$\mathrm{GN}_n \sim \frac{\sqrt{2e\sqrt[4]{e}}}{4} n^{-1} \left(\frac{8}{e^2}\right)^n n^{2n}.$$

Sketch Proofs

TC Networks:

(a) The counting problem for TC_n is reduced to that for maximal reticulated networks, i.e., $\mathrm{TC}_n \sim \sqrt{\mathrm{e}}\,\mathrm{TC}_{n,n-1}$;

(b) Maximal reticulated networks are encoded by certain words;

(c) These words satisfy a recurrence;

(d) A powerful asymptotic method of Elvey Price & Fang & Wallner (2021) can be applied.

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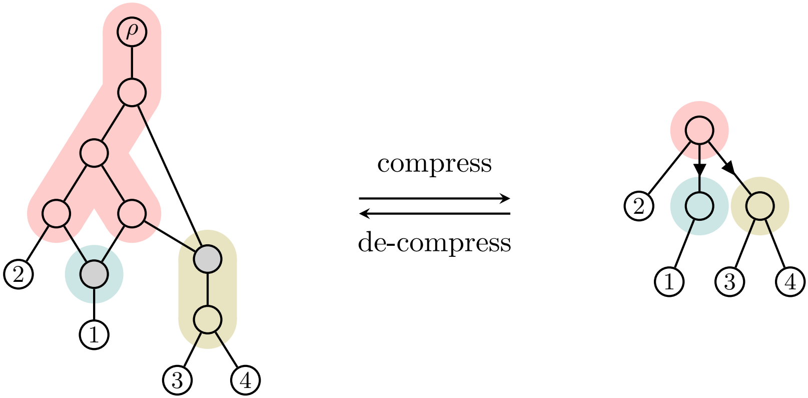
Galled Networks:

(a) The proof uses the component graph method;

(b) The Laplace method and an old result of Bender & Richmond (1984) play an important role.

Component Graph Method (on Galled Networks)



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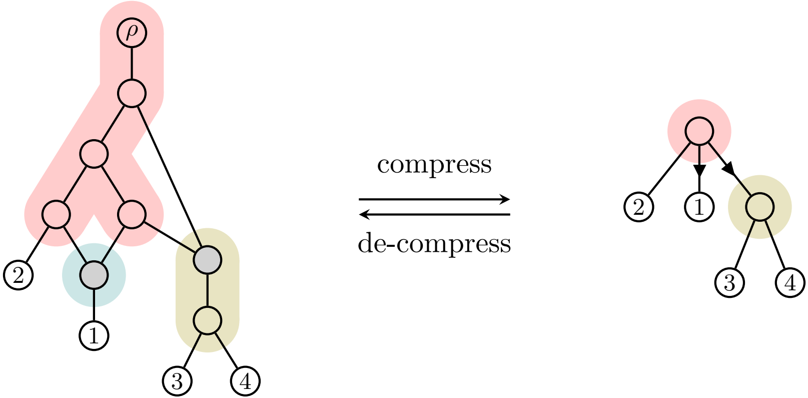

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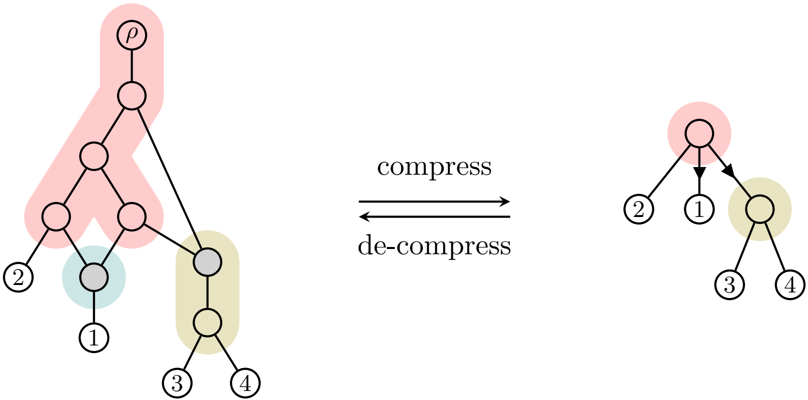

Lemma (Gunawan & Rathin & Zhang; 2020)

The component graph of a galled network (without arrows) is a (possibly non-binary) phylogenetic tree.

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A network is one-component if

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Theorem (Gunawan & Rathin & Zhang; 2020)

We have

$$\mathrm{GN}_n = \sum_{\mathcal{T}} \prod_{v} \sum_{j=0}^{c_{\mathrm{lf}}(v)} \binom{c_{\mathrm{lf}}(v)}{j} \mathrm{M}_{c(v),c(v)-c_{\mathrm{lf}}(v)+j}.$$

Counting Galled Networks (ii)

For $2 \le k \le n$,

$$\begin{aligned} \mathrm{M}_{n,k} &= (n+k-2)\mathrm{M}_{n,k-1} + (k-1)\mathrm{M}_{n,k-2} \\ &\quad + \frac{1}{2} \sum_{1 \le d \le k-1} \binom{k-1}{d} (2d-1)!! \left(\mathrm{M}_{n-d,k-1-d} - \mathrm{M}_{n+1-d,k-1-d}\right). \end{aligned}$$

with initial values $\mathrm{M}_{n,0} = (2n-3)!!$ and $\mathrm{M}_{n,1} = (n-1)(2n-3)!!$.

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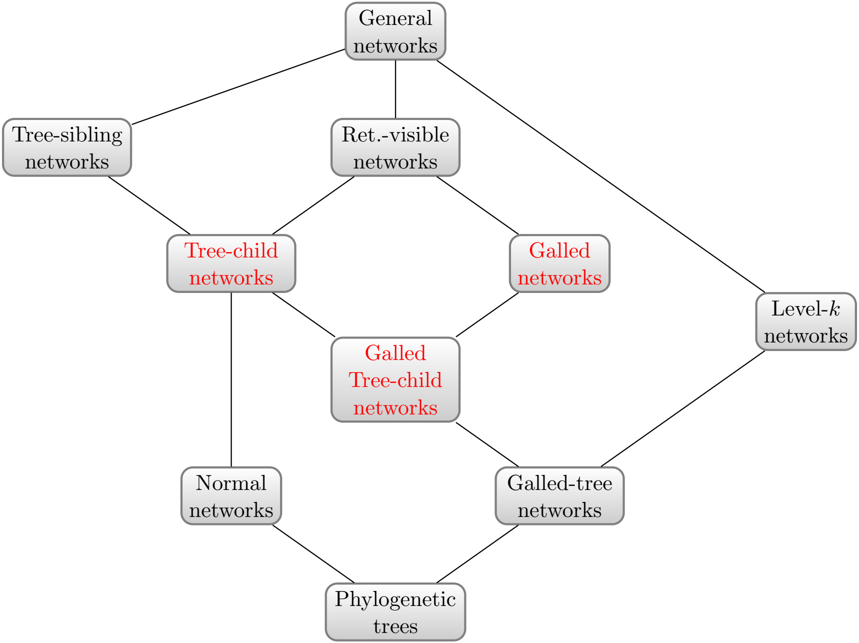
Two Steps:

(i) Obtain an asymptotic result for $\mathrm{M}_{n,k}$ from the above recurrence;

(ii) Use the formula for GN_n to derive the asymptotic result. For this, proof (asymptotic) matching upper and lower bound results.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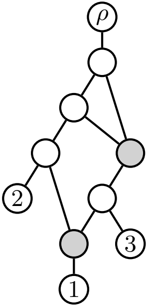
A galled tree-child network is a network which is both a galled network and a tree-child network.



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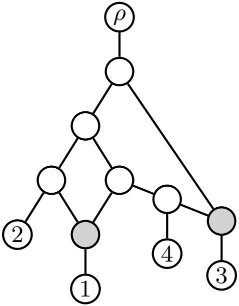
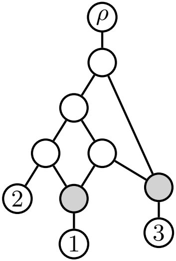


$\in \mathcal{GTC}.$



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Proposition

We have,

$$\mathrm{GTC}_n = \sum_{\mathcal{T}} \prod_{v} \sum_{j=0}^{c_{\mathrm{lf}}(v)} \binom{c_{\mathrm{lf}}(v)}{j} \mathrm{L}_{c(v), c(v) - c_{\mathrm{lf}}(v) + j}.$$

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Theorem (Cardona & Zhang; 2020; Fuchs & Yu & Zhang; 2021)

(i)

$$\mathrm{OTC}_{n,k} = \binom{n}{k} \frac{(2n-2)!}{2^{n-1}(n-k-1)!}.$$

(ii) As $n \to \infty$,

$$\mathrm{OTC}_n := \sum_{k=0}^{n-1} \mathrm{OTC}_{n,k} \sim \frac{1}{2\sqrt{e}} n^{-5/4} e^{2\sqrt{n}} \left(\frac{2}{e^2}\right)^n n^{2n}.$$

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Set

$$\mathrm{A}(z) = \sum_{n\geq 1} \mathrm{GTC}_{n,n-1}\frac{z^n}{n!} \qquad \text{and} \qquad L(z) = \sum_{n\geq 1} \mathrm{L}_{n,n-1}\frac{z^n}{n!}.$$

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Then,

$$\begin{aligned} \mathrm{A}(z) &= z + \sum_{n} \mathrm{L}_{n,n-1}\, z \frac{\left(\mathrm{GTC}_{n,n-1}\right)^{n-1}}{(n-1)!} \\ &= z + zL'(\mathrm{A}(z)). \end{aligned}$$

Maximal Reticulated Galled TC-Networks (ii)

Theorem (Lagrange Inversion Formula)

Let $A(z) = \sum_{n\geq 0} a_n z^n$ and $\Phi(z) = \sum_{n\geq 0} c_n z^n$ be formal power series with

$$A(z) = z\Phi(A(z)).$$

Then,

$$[z^n]A(z) = \frac{1}{n}[\omega^{n-1}]\Phi^n(\omega).$$

For a beautiful (recent) proof see:

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Applying it gives:

$$\mathrm{GTC}_{n,n-1} = n![z^n]\mathrm{A}(z) = (n-1)![\omega^{n-1}](1 + L'(\omega))^n.$$

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Theorem (Bender & Richmond; 1984)

Let $S(z)$ be a formal power series with $s_0 = 0, s_1 \neq 0$ and $ns_{n-1} \sim \gamma s_n$. Then,

$$[z^n](1 + S(z))^{\alpha n+\beta} \sim \alpha e^{\alpha s_1 \gamma} n s_n.$$

Note that $[z^n]L'(z) \sim \sqrt{2}(2/e)^n n^n$.

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Theorem (C. & Fuchs & Yu; 2024)

As $n \to \infty$,

$$\mathrm{GTC}_{n,n-1} \sim \sqrt{e\pi}\, n^{-1/2} \left(\frac{2}{e^2}\right)^n n^{2n}.$$

Counting Galled TC-Networks (ii)

Recall

$$\mathrm{GTC}_n = \sum_{\mathcal{T}} \prod_{v} \sum_{j=0}^{c_{\mathrm{lf}}(v)} \binom{c_{\mathrm{lf}}(v)}{j} \mathrm{L}_{c(v),c(v)-c_{\mathrm{lf}}(v)+j},$$

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The upper bound is given by

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Lemma

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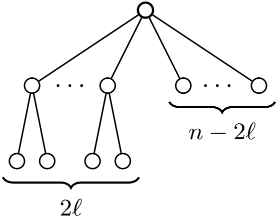
The following component graphs in $\mathcal{T}' \subset \mathcal{T}$, are dominating:



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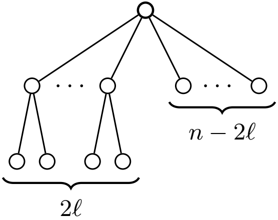

$$\mathrm{L}_n := \sum_{\mathcal{T}'} \prod_v \sum_j (\cdots) \quad \Longrightarrow \quad \mathrm{L}_n \leq \mathrm{GTC}_n \text{ and } \mathrm{L}_n \sim \mathrm{U}_n$$

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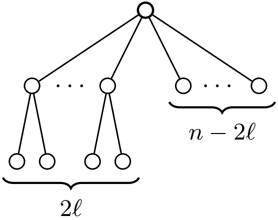

$$\mathrm{L}_n := \sum_{\mathcal{T}'} \prod_v \sum_j (\cdots) \quad \Longrightarrow \quad \mathrm{L}_n \leq \mathrm{GTC}_n \text{ and } \mathrm{L}_n \sim \mathrm{U}_n$$

Theorem (C. & Fuchs & Yu; 2024)

As $n \to \infty$,

$$\mathrm{GTC}_n \sim \frac{1}{2\sqrt[4]{e}} n^{-5/4} e^{2\sqrt{n}} \left(\frac{2}{e^2}\right)^n n^{2n}.$$

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Theorem (C. & Fuchs & Yu; 2024)

For fixed k, as $n \to \infty$,

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Theorem (Fuchs & Yu & Zhang; 2022)

For galled networks,

$$(I_n, n-R_n) \xrightarrow{d} (I, R),$$

where for $j \geq 0$ and $k \geq -j$,

$$\mathbb{P}(I=j, R=k) = \frac{e^{-7/8}}{16^j\, j!}[z^{j-k}]e^{1/(2z)}\left(1+2z+3z^2\right)^j.$$

Number of Reticulation Vertices (ii)

Theorem (C. & Fuchs & Yu; 2024)

For galled tc-networks,

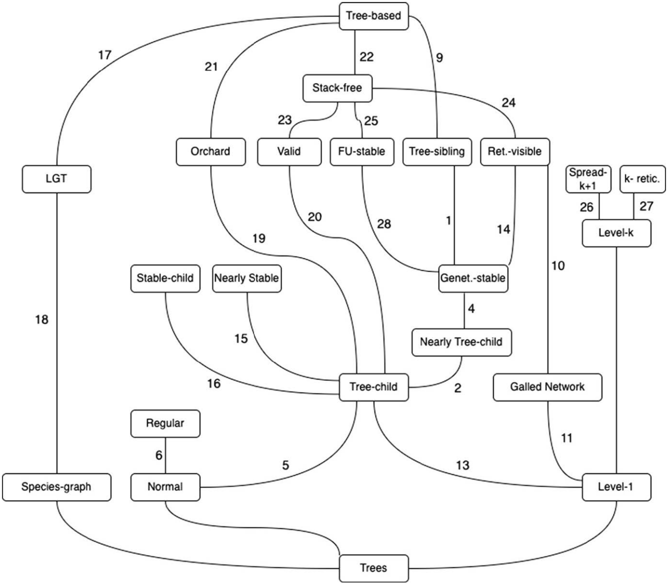
$$\left(I_n, \frac{R_n - \mathbb{E}(R_n)}{\sqrt{\mathrm{Var}(R_n)}}\right) \xrightarrow{d} (I, R),$$

where I and R are independent with

$$I \stackrel{d}{=} \mathrm{Poisson}(3/8) \qquad \text{and} \qquad R \stackrel{d}{=} N(0,1).$$

In addition,

$$\mathbb{E}(R_n) = n - \sqrt{n} + o(\sqrt{n}) \qquad \text{and} \qquad \mathrm{Var}(R_n) = \sqrt{n/4} + o(\sqrt{n}).$$



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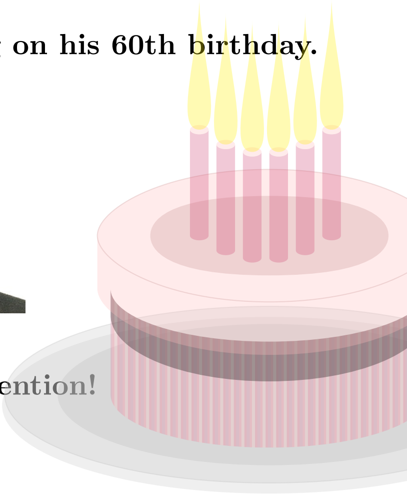
Reference and Dedication

Y.-S. Chang, M. Fuchs, G.-R. Yu. Galled Tree-Child Networks.

Dedicated to Hsien-Kuei Hwang on his 60th birthday.



Thanks for the attention!