Fringe trees for random trees with given vertex degrees

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Fringe Subtrees

Let $\mathcal{T}^{pl}$ be the set of all (finite) plane rooted trees (ordered rooted trees).

For $T \in \mathcal{T}^{pl}$, $|T|$ = size of T (number of vertices).

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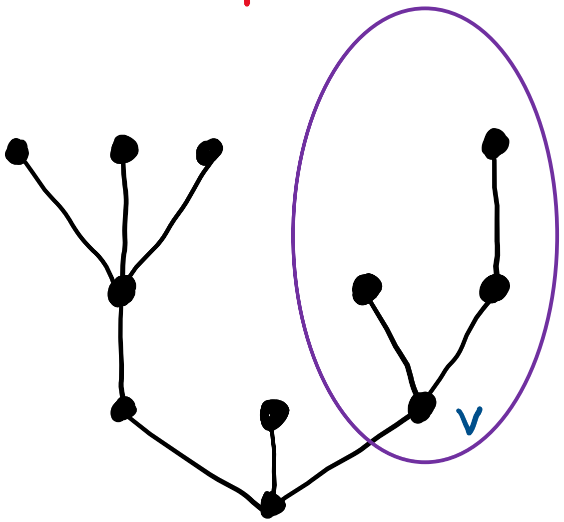
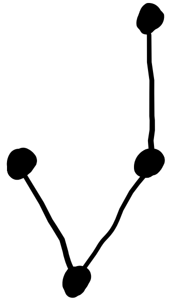
Example.



T

$T_v =$ 

These subtrees are called fringe subtrees.

We are interested in $\{T_v\}$ of a given T

Fringe Subtrees

For $T \in \mathbb{T}^{pl}$, consider the random fringe subtree T^*

$$T^* = T_v \quad \text{where } v \text{ is a uniform random vertex in } T$$

Fringe Subtrees

For $T \in \mathbb{T}^{pl}$, consider the random fringe subtree T^*

Aldous (1991). In general and for many important examples,

- Random recursive trees
- Binary search trees
- Conditioned Galton-Watson trees

$\vdots$

Subtree Counts

For $T, T' \in \mathbb{T}^{pl}$,

$$N_{T'}(T) = |\{v \in T : T_v = T'\}| = \sum_{v \in T} \mathbb{1}\{T_v = T'\},$$

i.e., the number of fringe subtrees of T that are equal (i.e., isomorphic) to T'.

Then the distribution of the random fringe subtree T^* is given by

$$\mathbb{P}(T^* = T') = \frac{N_{T'}(T)}{|T|}, \quad T' \in \mathbb{T}^{pl}.$$

Thus, the study the distribution of T^* is equivalent to study

$$N_{T'}(T).$$

Subtree Counts

In our case, T is a random tree.

- $N_{T'}(T)$ is a random variable for each $T' \in \Pi^{pl}$.
- The distribution of T^* is a random probability distribution on Π^{pl}, with

$$\mathbb{P}(T^* = T' \mid T) = \frac{N_{T'}(T)}{|T|}, \qquad T' \in \Pi^{pl}.$$

Galton - Watson Trees

Consider conditioned GW-trees to have size $m \in \mathbb{N}$ with offspring distribution ξ on $\mathbb{N}_0$ such that $\mathbb{E}[\xi] = 1$ and $\sigma^2 := \mathrm{Var}(\xi)$ (and non-zero)

Theorem. Aldous (1991). Let $\mathcal{T}_m$ be a conditioned (critical) G-W tree with $\sigma^2 < \infty$. For every fixed $T \in \mathbb{T}^{pl}$,

- $\mathbb{P}(\mathcal{T}_m^\bullet = T \mid \mathcal{T}_m) = \dfrac{N_T(\mathcal{T}_m)}{m} \xrightarrow[m \to \infty]{p} \mathbb{P}(\mathcal{T} = T)$ (Quenched Version)
- $\mathbb{P}(\mathcal{T}_m^\bullet = T) = \dfrac{\mathbb{E}\, N_T(\mathcal{T}_m)}{m} \xrightarrow[m \to \infty]{} \mathbb{P}(\mathcal{T} = T)$ (Annealed Version)

$\mathcal{T}$ is the corresponding unconditioned G-W tree

$$\mathbb{P}(\mathcal{T} = T) = \prod_{i \geq 0} p_i^{n_T(i)}, \quad n_T(i) = \#\text{ vertices in } T \text{ with out-degree } i$$

$$p_i = \mathbb{P}(\xi = i).$$

Galton - Watson Trees

Let $\mathcal{T}_m$ be a conditioned (critical) G-W tree with $\sigma^2 < \infty$.

Let $\mathcal{T}$ be the corresponding unconditioned G-W tree. Let $\pi(T) := \mathbb{P}(\mathcal{T} = T)$, $T \in \mathbb{T}^{\mathrm{pl}}$.

Theorem.

- $\mathbb{E}[N_T(\mathcal{T}_m)] = m\pi(T) + o(\sqrt{m})$
- $\mathrm{Var}(N_T(\mathcal{T}_m)) = m\left(\pi(T) - (2|T| - 1 + \sigma^{-2})\pi(T)^2\right) + o(m)$
- $$\frac{N_T(\mathcal{T}_m) - m\pi(T)}{\sqrt{m}} \xrightarrow[m \to \infty]{d} N\left(0, \pi(T) - (2|T| - 1 + \sigma^{-2})\pi(T)^2\right).$$

- Janson (2001) (assuming a third moment),
- Minami (2005) and Drmota (2009) (assuming an exponential moment)
- Janson (2016) (in general, but $\sigma^2 < \infty$)

Galton - Watson Trees

Let $\mathcal{T}_n$ be a conditioned (critical) G-W tree with $\sigma^2 < \infty$.

Let $\mathcal{T}$ be the corresponding unconditioned G-W tree. Let $\pi(T) := \mathbb{P}(\mathcal{T} = T)$, $T \in \mathbb{T}^{PI}$.

Theorem.

- $\mathbb{E}[N_T(\mathcal{T}_n)] = n\pi(T) + o(\sqrt{n})$
- $\mathrm{Var}(N_T(\mathcal{T}_n)) = n\left(\pi(T) - (2|T| - 1 + \sigma^{-2})\pi(T)^2\right) + o(n)$
- $$\frac{N_T(\mathcal{T}_n) - n\pi(T)}{\sqrt{n}} \xrightarrow[n\to\infty]{d} N\left(0, \pi(T) - (2|T| - 1 + \sigma^{-2})\pi(T)^2\right).$$

Remark. Janson (2016) ($\sigma^2 < \infty$). Extended it to a Multivariate Version, i.e.

$(N_{T_1}(\mathcal{T}_n), \dots, N_{T_k}(\mathcal{T}_n))$ is asymptotically normal (up to renormalization)

Trees with given vertex degrees

For $T \in \mathbb{T}^{\mathrm{Pl}}$ and $v \in T$, Let $d_T(v)$ be the out-degree of v (number of children)

The degree statistic of T is the sequence $n_T = (n_T(i))_{i \geq 0}$ where

$$n_T(i) = |\{v \in T : d_T(v) = i\}|$$

is the number of vertices of T with i children. We have

$$|T| = \sum_{i \geq 0} n_T(i) = 1 + \sum_{i \geq 0} i\, n_T(i).$$

Trees with given vertex degrees

A sequence of non-negative integers $n = (n(i))_{i \geq 0}$ is the degree statistic of some tree if and only if $\sum_{i \geq 0} n(i) = 1 + \sum_{i \geq 0} i\, n(i)$.

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Let $|n| = \sum_{i \geq 0} n(i)$ be the size of $n = (n(i))_{i \geq 0}$.

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A sequence of non-negative integers $\mathbf{n} = (n(i))_{i \geq 0}$ is the degree statistic of some tree if and only if $\sum_{i \geq 0} n(i) = 1 + \sum_{i \geq 0} i\, n(i)$.

Let $|\mathbf{n}| = \sum_{i \geq 0} n(i)$ be the size of $\mathbf{n} = (n(i))_{i \geq 0}$

Let $\Pi_{\mathbf{n}}^{d}$ be the set of plane rooted trees with degree statistic $\mathbf{n} = (n(i))_{i \geq 0}$

Trees with given vertex degrees

A sequence of non-negative integers $\mathbf{n}=(n(i))_{i\geq 0}$ is the degree statistic of some tree if and only if $\sum_{i\geq 0} n(i) = 1 + \sum_{i\geq 0} i\, n(i)$.

Let $|\mathbf{n}| = \sum_{i\geq 0} n(i)$ be the size of $\mathbf{n}=(n(i))_{i\geq 0}$

Let $\mathbb{T}_{\mathbf{n}}^{d}$ be the set of plane rooted trees with degree statistic $\mathbf{n}=(n(i))_{i\geq 0}$

Let $T_{\mathbf{n}} \sim \mathrm{Unif}(\mathbb{T}_{\mathbf{n}}^{d})$ (i.e. a uniform tree with given degree statistic $\mathbf{n}=(n(i))_{i\geq 0}$)

Pitman (2002): $|\mathbb{T}_{\mathbf{n}}^{d}| = \frac{1}{|\mathbf{n}|} \frac{|\mathbf{n}|!}{\prod_{i\geq 0} n(i)!}$.

Trees with given vertex degrees

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Let $|n| = \sum_{i \geq 0} n(i)$ be the size of $n = (n(i))_{i \geq 0}$

Let $\mathbb{T}_n^d$ be the set of plane rooted trees with degree statistic $n = (n(i))_{i \geq 0}$

Let $T_n \sim \mathrm{Unif}(\mathbb{T}_n^d)$ (i.e. a uniform tree with given degree statistic)

Remark. A G-W tree, conditioned on having degree statistic $n = (n(i))_{i \geq 0}$, is uniformly distributed on $\mathbb{T}_n^d$.

Trees with given vertex degrees

For a degree statistic $n = (n(i))_{i \geq 0}$, denote by $p(n) = (p_i(n))_{i \geq 0}$ its (empirical) degree distribution, i.e.,

$$p_i(n) = \frac{n(i)}{|n|}, \quad i \geq 0$$

Condition. $n_k = (n_k(i))_{i \geq 0}$, $k \geq 1$, are degree statistics s.t. as $k \to \infty$,

$|n_k| \to \infty$ and $p_i(n_k) \to p_i$, where $p = (p_i)_{i \geq 0}$ is a probility distribution on $\mathbb{N}_0$.

Trees with given vertex degrees

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Theorem 1. Let $T_{n_k} \sim \text{Uniform}(\Pi^d_{n_k})$

- $\mathbb{P}(T^\bullet_{n_k} = T \mid T_{n_k}) = \dfrac{N_T(T_{n_k})}{|n_k|} \xrightarrow[k \to \infty]{P} \pi_p(T).$
- $\mathbb{P}(T^\bullet_{n_k} = T) = \dfrac{\mathbb{E}\, N_T(T_{n_k})}{|n_k|} \xrightarrow[k\to\infty]{} \pi_p(T).$

where $\pi_p(T) = \prod_{i\geq 0} p_i^{n_T(i)}$, $T \in \Pi$. (with $0^0 = 1$)

Remark. $\pi_p(\cdot)$ is the distribution of a (uncondioned) G-W tree

with offspring distribution $p = (p_i)_{i\geq 0}$.

Trees with given vertex degrees

Condition. $\mathbf{n}_k = (n_k(i))_{i\geq 0}$, $k\geq 1$, are degree statistics s.t. as $k\to\infty$,

$|\mathbf{n}_k|\to\infty$ and $p_i(\mathbf{n}_k)\to p_i$, where $\mathbf{p}=(p_i)_{i\geq 0}$ is a probability distribution on $\mathbb{N}_0$.

Theorem 2. Let $T_{\mathbf{n}_k}\sim \text{Uniform}(\mathbb{T}^d_{\mathbf{n}_k})$

- $\mathbb{E}\, N_T(T_{\mathbf{n}_k}) = \pi_{\mathbf{p}}(T)\,|\mathbf{n}_k| + o(|\mathbf{n}_k|)$
- $\text{Var}(N_T(T_{\mathbf{n}_k})) = \left(\pi_{\mathbf{p}}(T) + \eta_{\mathbf{p}}(T)\,\pi_{\mathbf{p}}(T)^2\right)|\mathbf{n}_k| + o(|\mathbf{n}_k|)$
- $$\frac{N_T(T_{\mathbf{n}_k}) - \pi_{\mathbf{p}(\mathbf{n}_k)}(T)\,|\mathbf{n}_k|}{\sqrt{|\mathbf{n}_k|}} \xrightarrow[k\to\infty]{d} N\left(0,\ \pi_{\mathbf{p}}(T) + \eta_{\mathbf{p}}(T)\,\pi_{\mathbf{p}}(T)^2\right).$$

where $\eta_{\mathbf{p}}(T) = (|T|-1)^2 - \sum_{i\geq 0}\frac{n_T(i)^2}{p_i}$ (here $0/0=0$).

Trees with given vertex degrees

Theorem 2. Let $T_{n_k} \sim \mathrm{Uniform}(\Pi^{d}_{n_k})$

- $\mathbb{E}\, N_T(T_{n_k}) = \pi_p(T)\,|n_k| + o(|n_k|)$
- $\mathrm{Var}\,(N_T(T_{n_k})) = \left(\pi_p(T) + \eta_p(T)\,\pi_p(T)^2\right)|n_k| + o(|n_k|)$
- $\dfrac{N_T(T_{n_k}) - \pi_{p(n_k)}(T)\,|n_k|}{\sqrt{|n_k|}} \xrightarrow[k\to\infty]{d} N\left(0, \pi_p(T) + \eta_p(T)\,\pi_p(T)^2\right)$

where $\eta_p(T) = (|T|-1)^2 - \sum_{i\geq 0} \frac{n_T(i)^2}{p_i}$ (here $0/0 = 0$)

Remark. We also proved a Multivariate Version, i.e.

$(N_{T_1}(T_{n_k}), \dots, N_{T_q}(T_{n_k}))$ is asymptotically normal (up to renormalization)

Galton - Watson Trees

Let $\mathcal{T}_m$ be a conditioned (critical) G-W tree with offspring distribution in the domain of attraction of a stable law of index $\alpha \in (1,2)$. $(\sigma^2 = \infty)$

Theorem.

$$\frac{N_T(\mathcal{T}_m) - m\pi(T)}{\sqrt{m}} \xrightarrow[m\to\infty]{d} N\left(0,\ \pi(T) - (2|T|-1)\,\pi(T)^2\right)$$

where $\mathcal{T}$ is the corresponding unconditioned G-W tree and $\pi(T) := \mathbb{P}(\mathcal{T} = T)$, $T \in \mathbb{T}^{Pl}$.

Sketch of Proofs (Theorem 2)

For $x \in \mathbb{R}$ and $q \in \mathbb{N}_0$, $(x)_q = x(x-1)\cdots(x-q+1)$ the q-th falling factorial of x

(Here $(x)_0 = 1$. Note that $(x)_q = 0$ whenever $x \in \mathbb{N}_0$ and $x-q+1 \leq 0$).

Sketch of Proofs (Main ingredient)

Theorem (Gao and Wormald (2004)). Let $(X_m)_{m\geq 1}$ be a sequence of non-negative r.v. Supose that μ_m and σ_m are positive real numbers s.t., as $m\to\infty$

$$\sigma_m \ll \mu_m \ll \sigma_m^3$$

Let $\gamma > 0$ be a fixed real number. Let $c > 0$ be a constant, and suppose further that, as $m\to\infty$, uniformly for all integer sequences $(k_m)_{m\geq 1}$ with $0 \leq k_m \leq c\,\mu_m/\sigma_m$,

$$\mathbb{E}(X_m)_{k_m} = \mu_m^{k_m} \cdot \exp\left(\frac{1}{2}\,\frac{\gamma\sigma_m^2 - \mu_m}{\mu_m^2}\,k_m^2 + o(1)\right).$$

Then,

$$\frac{X_m - \mu_m}{\sigma_m} \xrightarrow[m\to\infty]{d} N(0,\gamma).$$

Sketch of Proofs (Main ingredient)

Theorem (Multivariate G-W Theorem). For $m, n \in \mathbb{N}$, let $(X_{1n}, \dots, X_{mn})$ be vectors of nonnegative r.v. Suppose that μ_{in} and σ_{in} are positive real numbers s.t for each $1 \le i \le m$, as $n \to \infty$,

$$\sigma_{in} \ll \mu_{in} \ll \sigma_{in}^3$$

Let $\Gamma = (\gamma_{ij})_{i,j=1}^{m}$ be a fixed matrix. Let $c > 0$ be a constant, and suppose further that, as $m \to \infty$, uniformly for all integer sequences $(k_{in})_{i=1}^{m}$ with $0 \le k_{in} \le c\mu_{in}/\sigma_{in}$

$$\mathbb{E} \prod_{i=1}^{m} (X_{in})_{k_{in}} = \prod_{i=1}^{m} \mu_{in}^{k_{in}} \cdot \exp\left(\frac{1}{2} \sum_{i,j=1}^{m} \frac{\gamma_{ij}\sigma_{in}\sigma_{jn} - \delta_{ij}\mu_{in}}{\mu_{in}\mu_{jn}} k_{in}k_{jn} + o(1) \right)$$

Then,

$$\left(\frac{X_{1n} - \mu_{1n}}{\sigma_{1n}}, \dots \frac{X_{mn} - \mu_{mn}}{\sigma_{mn}} \right) \xrightarrow[n \to \infty]{d} N(0, \Gamma)$$

* δ_{ij} is Kronecker's delta

Sketch of Proofs

Lemma 1. Let $n=(n(i))_{i\geq 0}$ be a degree statistic and let $T_n \sim \mathrm{Uniform}(\Pi_n^d)$

For $q \in \mathbb{N}$ and $T \in \Pi^{pl}$ s.t. $|n| \geq q|T|-q+1$,

$$\mathbb{E}(N_T(T_n))_q = \frac{|n|}{(|n|)_{q|T|-q+1}} \prod_{i\geq 0} (n(i))_{q n_T(i)}$$

Recall $|n| = \sum_{i\geq 0} n(i)$ and $n_T(i) = |\{v \in T : d_T(v) = i\}|$, $i \geq 0$.

Recall:

Condition. $n_k = (n_k(i))_{i\geq 0}$, $k \geq 1$, are degree statistics s.t. as $k \to \infty$,

$|n_k| \to \infty$ and $p_i(n_k) \to p_i$, where $p = (p_i)_{i\geq 0}$ is a probability distribution on $\mathbb{N}_0$

Theorem 2. Let $T_{n_k} \sim \text{Uniform}(\Pi^d_{n_k})$

- $\mathbb{E}\, N_T(T_{n_k}) = \Pi_p(T)\,|n_k| + o(|n_k|)$
- $\mathrm{Var}\,(N_T(T_{n_k})) = \left(\Pi_p(T) + \eta_p(T)\,\Pi_p(T)^2\right)|n_k| + o(|n_k|)$
- $\dfrac{N_T(T_{n_k}) - \Pi_{p(n_k)}(T)\,|n_k|}{\sqrt{|n_k|}} \xrightarrow[k\to\infty]{d} N\left(0, \Pi_p(T) + \eta_p(T)\,\Pi_p(T)^2\right)$

where $\eta_p(T) = (|T|-1)^2 - \sum_{i\geq 0} \frac{n_T(i)^2}{p_i}$ (here $0/0 = 0$)

Sketch of Proofs (Theorem 2)

By using the estimation, $(x)_k = x^k \exp\left(-\frac{k(k-1)}{2x} + O\left(\frac{k^3}{x^2}\right)\right)$ $x \geq 1$ a real number, $0 \leq k \leq x/2$ an integer,

We see that, for $q = q_k = O(|n_k|^{1/2})$,

$$\mathbb{E}(N_T(\Upsilon_{n_k}))_{q_k} = \frac{|n_k|}{(|n_k|)_{q|T|-q+1}} \prod_{i\geq 0} (n_k(i))_{q n_T(i)}$$

$$= \mu_{n_k}(T)^{q_k} \exp\left(\frac{\gamma_p(T)|n_k| - \mu_{n_k}(T)}{2\mu_{n_k}(T)} q_k^2 + o(1)\right),$$

where

$\mu_{n_k}(T) = |n_k| \pi_{p(n_k)}(T) = |n_k| \prod_{i\geq 0} p_i(n_k)^{n_T(i)}$ and $\gamma_p(T) = \pi_p(T) + \eta_p(T)\pi_p(T)^2$

Finally, we apply Gao-Wormald Theorem.

Thanks