Fringe trees for random trees with given vertex degrees

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Fringe Subtrees

Let $T^p$ be the set of all (finite) plane rooted trees (ordered rooted trees).

For $T \in T^p$, $|T|$ = size of $T$ (number of vertices).
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Given $T \in T^p$ and a vertex $v \in T$, let $T_v$ be the subtree rooted at $v$. 
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Example.

$$T_v = \quad \text{These subtrees are called fringe subtrees.}$$

We are interested in $|T_v|$ of a given $T$
Fringe Subtrees

For $T \in T_{pl}$, consider the random fringe subtree $T^*$

$$T^* = T_v \quad \text{where} \quad v \quad \text{is a uniform random vertex in} \ T$$
Fringe Subtrees

For $T \in \mathcal{T}_d$, consider the random fringe subtree $T^*$

Aldous (1991). In general and for many important examples,

- Random recursive trees
- Binary search trees
- Conditioned Galton-Watson trees
  - 
  - 
  -
Subtree Counts

For $T, T' \in \Pi^p$,

$$N_{T'}(T) = |\{v \in T : T_v = T'\}| = \sum_{v \in T} 1_{\{T_v = T'\}},$$

i.e., the number of fringe subtrees of $T$ that are equal (i.e., isomorphic) to $T'$.

Then the distribution of the random fringe subtree $T^*$ is given by

$$P(T^* = T') = \frac{N_{T'}(T)}{|T|}, \quad T' \in \Pi^p.$$

Thus, the study the distribution of $T^*$ is equivalent to study

$$N_{T'}(T).$$
Subtree Counts

In our case, $T$ is a random tree.

- $N_{T'}(T)$ is a random variable for each $T' \in \mathcal{P}_1$
- The distribution of $T^*$ is a random probability distribution on $\mathcal{P}_1$, with

$$P(T^* = T' \mid T) = \frac{N_{T'}(T)}{|T|}, \quad T' \in \mathcal{P}_1$$
Galton-Watson Trees

Consider conditioned G-W-trees to have size \( m \in \mathbb{N} \) with offspring distribution \( \xi \) on \( \mathbb{N}_0 \) such that \( \mathbb{E} [\xi] = 1 \) and \( \sigma^2 := \text{Var}(\xi) \) (and non-zero).

**Theorem.** Aldous (1991). Let \( T_m \) be a conditioned (critical) G-W tree with \( \sigma^2 < \infty \). For every fixed \( T \in \mathcal{T}^n \),

\[
\begin{align*}
\cdot \quad \mathbb{P}(T_m = T | \mathcal{T}_m) &= \frac{N_T(T_m)}{m} \xrightarrow{m \to \infty} \mathbb{P}(T = T) \quad \text{(Quenched Version)} \\
\cdot \quad \mathbb{P}(T_m = T) &= \frac{\mathbb{E} N_T(T_m)}{m} \xrightarrow{m \to \infty} \mathbb{P}(T = T) \quad \text{(Annealed Version)}
\end{align*}
\]

\( \mathcal{T} \) is the corresponding unconditioned G-W tree

\[
\mathbb{P}(T = T) = \prod_{i \geq 0} p_i^{n_T(i)} \quad , \quad n_T(i) = \# \text{ vertices in } T \text{ with out-degree } i.
\]

\( p_i = \mathbb{P}(\xi = i) \).
Galton-Watson Trees

Let $T_m$ be a conditioned (critical) G-W tree with $\sigma^2 < \infty$. Let $T$ be the corresponding unconditioned G-W tree. Let $\pi(T) = \Pr(T = T)$, $T \in \mathbb{T}^p$.

Theorem.

- $\mathbb{E}[N_T(T_m)] = m \pi(T) + o(m)$
- $\text{Var}(N_T(T_m)) = m \left( \pi(T) - (2 \pi - 1 + \sigma^{-2}) \pi(T)^2 \right) + o(m)$
- $\frac{N_T(T_m) - m \pi(T)}{\sqrt{m}} \xrightarrow{d} N \left(0, \pi(T) - (2 \pi - 1 + \sigma^{-2}) \pi(T)^2 \right)$.

- Janson (2001) (assuming a third moment),
- Minami (2005) and Drmota (2009) (assuming an exponential moment)
- Janson (2016) (in general, but $\sigma^2 < \infty$)
Galton-Watson Trees

Let $T_m$ be a conditioned (critical) G-W tree with $\sigma^2 < \infty$. Let $T$ be the corresponding unconditioned G-W tree. Let $\pi(T) := P(T = T)$, $T \in \mathcal{T}$. 

Theorem. 
- $E[N_T(T_m)] = m\pi(T) + o(m)$
- $\text{Var}(N_T(T_m)) = m \left( \pi(T) - (2\pi - 1 + \sigma^{-2})\pi(T)^2 \right) + o(m)$
- \[ \frac{N_T(T_m) - m\pi(T)}{\sqrt{m}} \xrightarrow{m \to \infty} N\left(0, \pi(T) - (2\pi - 1 + \sigma^{-2})\pi(T)^2 \right). \]

Remark. Janson (2016) ($\sigma^2 < \infty$). Extended it to a Multivariate Version, i.e.
\[ (N_{T_1}(T_m), \ldots, N_{T_k}(T_m)) \text{ is asymptotically normal (up to renormalization)} \]
Trees with given vertex degrees

For $T \in \mathcal{P}$ and $v \in T$, let $d_T(v)$ be the out-degree of $v$ (number of children).

The degree statistic of $T$ is the sequence $n_T = (n_{T,i})_{i \geq 0}$ where

$$n_{T,i} = |\{v \in T : d_T(v) = i\}|$$

is the number of vertices of $T$ with $i$ children. We have

$$|T| = \sum_{i \geq 0} n_{T,i} = 1 + \sum_{i \geq 0} i n_{T,i}.$$
Trees with given vertex degrees

A sequence of non-negative integers $n = (n(i))_{i \geq 0}$ is the degree statistic of some tree if and only if

$$\sum_{i \geq 0} n(i) = 1 + \sum_{i \geq 0} i \cdot n(i).$$
Trees with given vertex degrees

A sequence of non-negative integers \( n = (n_c(i))_{i \geq 0} \) is the degree statistic of some tree if and only if

\[
\sum_{i \geq 0} n(i) = 1 + \sum_{i \geq 0} i \cdot n(i).
\]

Let \( |n| = \sum_{i \geq 0} n(i) \) be the size of \( n = (n_c(i))_{i \geq 0} \).
Trees with given vertex degrees

A sequence of non-negative integers $n = (n(i))_{i=0} \text{ is the degree statistic of some tree if and only if } \sum_{i=0}^{\infty} n(i) = 1 + \sum_{i=0}^{\infty} i \cdot n(i)$.

Let $|n| = \sum_{i=0}^{\infty} n(i)$ be the size of $n = (n(i))_{i=0}$

Let $T_n$ be the set of plane rooted trees with degree statistic $n = (n(i))_{i=0}$
Trees with given vertex degrees

A sequence of non-negative integers $n = (n(i))_{i \geq 0}$ is the degree statistic of some tree if and only if $\sum_{i \geq 0} n(i) = 1 + \sum_{i \geq 0} i \cdot n(i)$.

Let $|n| = \sum_{i \geq 0} n(i)$ be the size of $n = (n(i))_{i \geq 0}$.

Let $\mathbb{T}_n^d$ be the set of plane rooted trees with degree statistic $n = (n(i))_{i \geq 0}$.

Let $\mathbb{T}_n \sim \text{Unif}(\mathbb{T}_n^d)$ (i.e. a uniform tree with given degree statistic $n = (n(i))_{i \geq 0}$).

Pitman (2002): $|\mathbb{T}_n^d| = \frac{|n|!}{\prod_{i \geq 0} n(i)!}$. 
Trees with given vertex degrees

A sequence of non-negative integers \( n = (n(c_i))_{i \geq 0} \) is the degree statistic of some tree if and only if \( \sum_{i \geq 0} n(c_i) = 1 + \sum_{i \geq 0} i \cdot n(c_i) \).

Let \( |n| = \sum_{i \geq 0} n(c_i) \) be the size of \( n = (n(c_i))_{i \geq 0} \).

Let \( \mathcal{T}_n^d \) be the set of plane rooted trees with degree statistic \( n = (n(c_i))_{i \geq 0} \).

Let \( \tilde{T}_n \sim \text{Unif}(\mathcal{T}_n^d) \) (i.e. a uniform tree with given degree statistic).

Remark. A G-W tree, conditioned on having degree statistic \( n = (n(c_i))_{i \geq 0} \), is uniformly distributed on \( \mathcal{T}_n^d \).
Trees with given vertex degrees

For a degree statistic $n = (n_c(i))_{i \geq 0}$, denote by $p(n) = (p_i(n))_{i \geq 0}$ its (empirical) degree distribution, i.e.,

$$p_i(n) = \frac{n_c(i)}{n}, \quad i \geq 0$$

Condition. $n_k = (n_k(i))_{i \geq 0}, \quad k \geq 1,$ are degree statistics s.t. as $k \to \infty$,

$lnk \to \infty$ and $p_i(n_k) \to p_i$, where $p = (p_i)_{i \geq 0}$ is a probability distribution on $\mathbb{N}_0$. 
Trees with given vertex degrees

Condition. $n_k = (n_k(i))_{i \geq 0}$, $k \geq 1$, are degree statistics s.t. as $k \to \infty$,

$|n_k| \to \infty$ and $p_i(n_k) \to p_i$, where $p = (p_i)_{i \geq 0}$ is a probability distribution on $\mathbb{N}_0$.

Theorem 1. Let $\mathcal{T}_{n_k} \sim \text{Uniform}(\Pi_{n_k}^d)$

- $\text{IP}(\mathcal{T}_{n_k}^* = T \mid \mathcal{T}_{n_k}) = \frac{N_T(\mathcal{T}_{n_k})}{|n_k|} \xrightarrow{k \to \infty} \Pi_p(T)$.

- $\text{IP}(\mathcal{T}_{n_k}^* = T) = \frac{\text{E} N_T(\mathcal{T}_{n_k})}{|n_k|} \xrightarrow{k \to \infty} \Pi_p(T)$.

where $\Pi_p(T) = \prod_{i \geq 0} p_i^{n_{T(i)}}$, $T \in \Pi$, (with $0^0 = 1$)

Remark. $\Pi_p(T)$ is the distribution of an (unconditioned) G-NW tree with offspring distribution $p = (p_i)_{i \geq 0}$.
Trees with given vertex degrees

Condition. \( n_k = (n_{k(i)})_{i \geq 0}, k \geq 1, \) are degree statistics such as \( k \to \infty, \)
\( n_k \to \infty \) and \( P_i(n_k) \to P_i, \) where \( P = (P_i)_{i \geq 0} \) is a probability distribution on \( \mathbb{N}. \)

Theorem 2. Let \( T_{n_k} \sim \text{Uniform}(\mathcal{T}_{n_k}^d) \)

\[ \mathbb{E} N_T(T_{n_k}) = \pi P(T) n_k | + o(n_k) \]

\[ \text{Var} \left( N_T(T_{n_k}) \right) = \left( \pi P(T) + \pi P(T) \pi P(T)^2 \right) n_k | + o(n_k) \]

\[ \frac{N_T(T_{n_k}) - \pi P(n_k) (T) | n_k |}{\sqrt{n_k}} \xrightarrow{d} \mathcal{N} \left( 0, \pi P(T) + \pi P(T) \pi P(T)^2 \right). \]

where \( \pi P(T) = (|T| - 1)^2 - \sum_{i \geq 0} \frac{n_T(i)}{P_i}^2 \) \( \left( \text{here } 0/0 = 0 \right). \)
Trees with given vertex degrees

Theorem 2. Let $T_{n_k} \sim \text{Uniform} (\Pi_{n_k}^d)$

- $E N_T (T_{n_k}) = \Pi_{p} (T) \ln n_k + o (\ln n_k)$
- $\text{Var} (N_T (T_{n_k})) = (\Pi_{p} (T) + \eta_p (T) \Pi_{p} (T)^2) \ln n_k + o (\ln n_k)$
- $\frac{N_T (T_{n_k}) - \Pi_{p} (n_k) (T) \ln n_k}{\sqrt{\ln n_k}} \xrightarrow{d} N \left( 0, \Pi_{p} (T) + \eta_p (T) \Pi_{p} (T)^2 \right)$

where $\eta_p (T) = (|T| - 1)^2 - \sum_{i \leq 0} \frac{n_T (c_i)}{p_i}^2$ (here $0/0 = 0$)

Remark. We also proved a Multivariate Version, i.e.

$(N_{T_1} (T_{n_k}), \ldots, N_{T_q} (T_{n_k}))$ is asymptotically normal (up to renormalization)
Galton-Watson Trees

Let $T_m$ be a conditioned (critical) G-W tree with offspring distribution in the domain of attraction of a stable law of index $\alpha \in (1,2)$. ($\sigma^2 = \infty$)

Theorem.

$$\frac{N(T_m) - mn\pi(T)}{\sqrt{m}} \xrightarrow{d} N\left(0, \pi(T) - (2\pi-1)\pi(T)^2\right)$$

where $T$ is the corresponding unconditioned G-W tree and $\pi_T := P(T=T)$, $Te T$.\(\text{pl}\)
Sketch of Proof (Theorem 2)

For $x \in \mathbb{N}$ and $q \in \mathbb{N}_0$, $(x)_q = x(x-1) \cdots (x-q+1)$ the $q$-th falling factorial of $x$.

(Here $(x)_0 = 1$. Note that $(x)_q = 0$ whenever $x \in \mathbb{N}_0$ and $x-q+1 \leq 0$.)
Sketch of Proofs (Main ingredient)

Theorem (Gao and Wormald (2004)). Let \((X_m)_{m \geq 1}\) be a sequence of non-negative r.v.

Suppose that \(\mu_m\) and \(\sigma_m\) are positive real numbers s.t., as \(m \to \infty\)

\[ \sigma_m \ll \mu_m \ll \sigma_m^3 \]

Let \(\gamma > 0\) be a fixed real number. Let \(c > 0\) be a constant, and suppose further that, as \(m \to \infty\), uniformly for all integer sequences \((k_m)_{m \geq 1}\) with \(0 \leq k_m \leq cM_m/\sigma_m\),

\[ \mathbb{E} (X_m)^{k_m} = \mu_m^{k_m} \cdot \exp \left( \frac{1}{2} \frac{\gamma \sigma_m^2 - \mu_m}{\mu_m^2} k_m^2 + o(1) \right). \]

Then,

\[ \frac{X_m - \mu_m}{\sigma_m} \xrightarrow{m \to \infty} N(0, \gamma). \]
Theorem (Multivariate G-W Theorem). For \( m \in \mathbb{N} \), let \((x_1, \ldots, x_m)\) be vectors of nonnegative r.v. Suppose that \( \min \) and \( \sigma_{in} \) are positive real numbers s.t. for each \( 1 \leq i \leq m \), as \( n \to \infty \),

\[
\sigma_{in} \ll \min \ll \sigma_{in}^3
\]

Let \( \Gamma = (\gamma_{ij})_{ij=1}^m \) be a fixed matrix. Let \( c > 0 \) be a constant, and suppose further that, as \( m \to \infty \), uniformly for all integer sequences \((k_i)_{i=1}^m\) with \( 0 \leq k_i \leq c \min \sigma_{in} \)

\[
\mathbb{E} \prod_{i=1}^m (x_{ki})_{k_i} = \prod_{i=1}^m \min \cdot \exp \left( \frac{1}{2} \sum_{ij=1}^m \gamma_{ij} \sigma_{ij} \sigma_{in} - \delta_{ij} \min \right) k_i k_j + o(1) \]

Then,

\[
\left( \frac{x_1 - \min}{\sigma_{1n}}, \ldots, \frac{x_m - \min}{\sigma_{mn}} \right) \xrightarrow{d} N(0, \Gamma)
\]

* \( \delta_{ij} \) is Kronecker's delta
Sketch of Proofs

Lemma 1. Let \( n=(n_{ci})_{i \geq 0} \) be a degree statistic and let \( T_n \sim \text{Uniform}(\mathcal{T}^d_n) \)

For \( q \in \mathbb{N} \) and \( T \in \mathcal{T}^q \) s.t. \( 1_{n} \geq q \cdot |T|-q+1 \),

\[
\mathbb{E}(N_T(T_n))_q = \frac{1_{n}}{(1_{n})_{q \cdot |T|-q+1}} \prod_{i \geq 0} (n_{ci})_{q \cdot n_{T}(ci)}
\]

Recall \( 1_{n} = \sum_{i \geq 0} n_{ci} \) and \( n_{T}(ci) = |\{v \in T : d_T(v) = i\}| \), \( i \geq 0 \).
Recall:

Condition. \( n_k = (n_k(i))_{i \geq 0}, k \geq 1 \), are degree statistics s.t. as \( k \to \infty \),
\[ \ln k \to \infty \] and \( \pi_k(i) \to \pi_i \), where \( \pi = (\pi_i)_{i \geq 0} \) is a probability distribution on \( \mathbb{N} \).

**Theorem 2.** Let \( T_{n_k} \sim \text{Uniform}(\mathbb{T}^d_{n_k}) \)

- \( \mathbb{E} \left[ N_T(T_{n_k}) \right] = \pi_T(T) \ln k + o(\ln k) \)

- \( \text{Var} \left( N_T(T_{n_k}) \right) = \left( \pi_T(T) + \eta_T(T) \pi_T(T)^2 \right) \ln k + o(\ln k) \)

- \( \frac{N_T(T_{n_k}) - \pi_T(T) n_k}{\sqrt{\ln k}} \xrightarrow{d} N\left(0, \pi_T(T) + \eta_T(T) \pi_T(T)^2 \right) \)

where \( \eta_T(T) = (|T| - 1)^2 - \sum_{i \geq 0} \frac{n_T(i)^2}{\pi_i} \quad \text{(here 0/0 = 0)} \).
Sketch of Proofs (Theorem 2)

By using the estimation, \((x)_k = x^k \exp\left(-\frac{k(k-1)}{2x} + O\left(\frac{k^3}{x^2}\right)\right)\) \(x \geq 1\) a real number \(0 \leq k \leq x/2\) an integer,

we see that, for \(q = q_k = O\left(\ln k^{1/2}\right)\),

\[
E(N_T(\Gamma_{\text{n}}))_{q_k} = \frac{\ln k!}{(\ln k)^q \prod_{q+1} \prod_{i=0}} (\Gamma_{\text{n}}(i))_{q \prod_{i=0} T(i)}
\]

\[
= M_{n_k}(T)^{q_k} \exp\left(\frac{\gamma_\rho(T) \ln k! - M_{n_k}(T)}{2 M_{n_k}(T)} q_k^2 + O(1)\right),
\]

where

\[
M_{n_k}(T) = \ln k! \prod \rho_{k!(n_k)}(T) = \ln k! \prod_i \rho_i(n_k) n_{T(i)}^{\gamma_{\rho}(T)} \quad \text{and} \quad \gamma_\rho(T) = \prod \rho(T) + \eta_\rho(T) \prod \rho(T)^2
\]

Finally, we apply Gao-Wormald Theorem.
Thanks