Asymptotics of unlabeled galled trees with a fixed number of galls

Lily Agranat-Tamir, Michael Fuchs, Bernhard Gittenberger and Noah Rosenberg
AofA 2024

Lecture Outline

Introduction – phylogenetic trees

Definition – galled trees

Previous work – the enumeration of galled trees and galled trees with one gall

Novel work – the enumeration of galled trees with a fixed number of galls

Summary

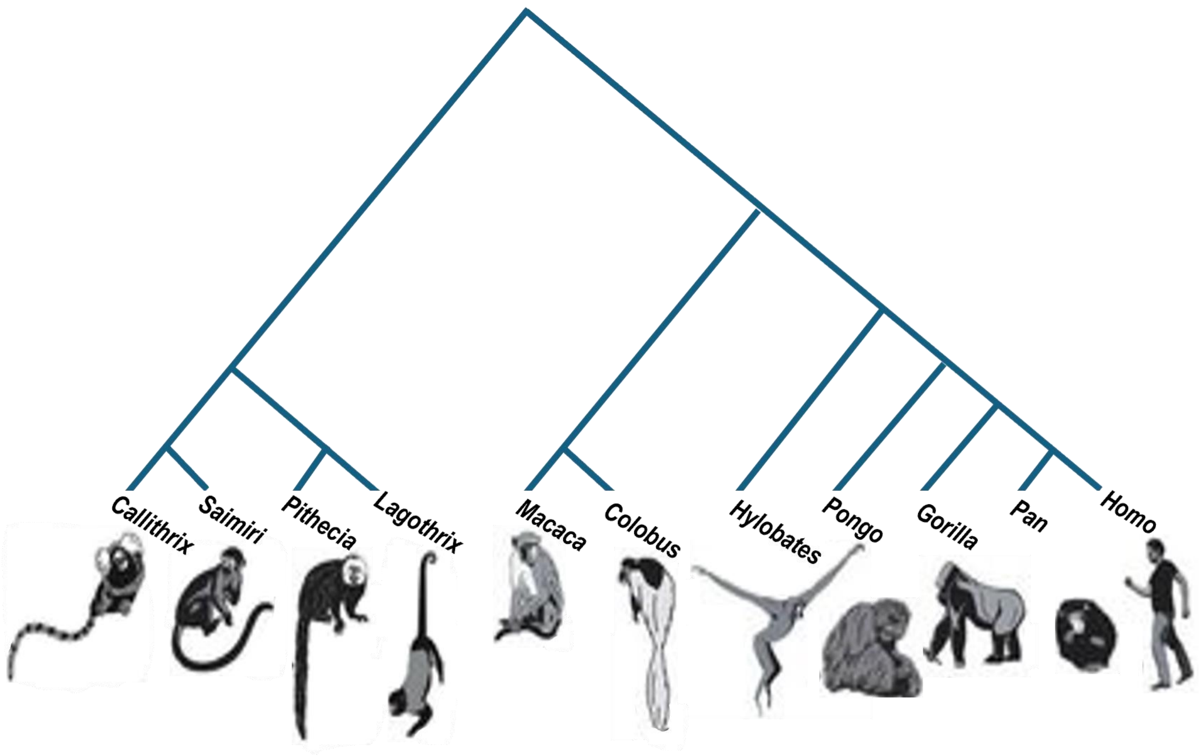
Phylogenetic Trees

Phylogenetic Trees

Species Trees

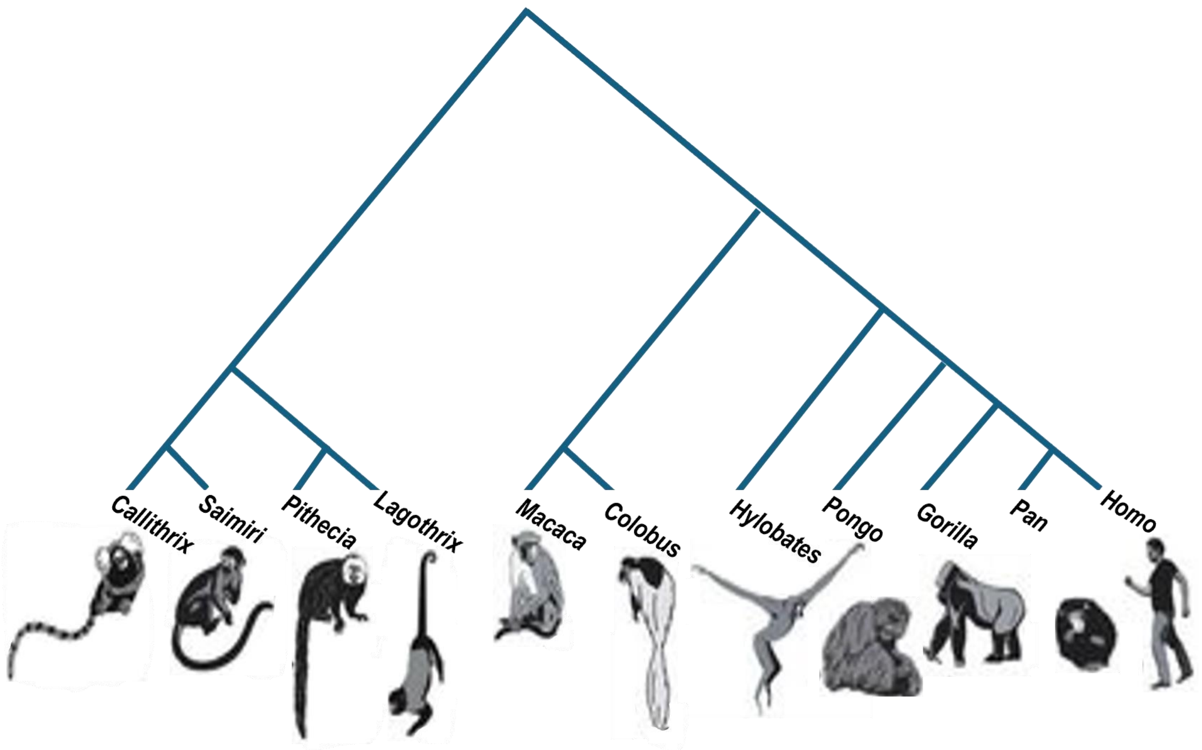
Phylogenetic Trees

Species Trees



Phylogenetic Trees

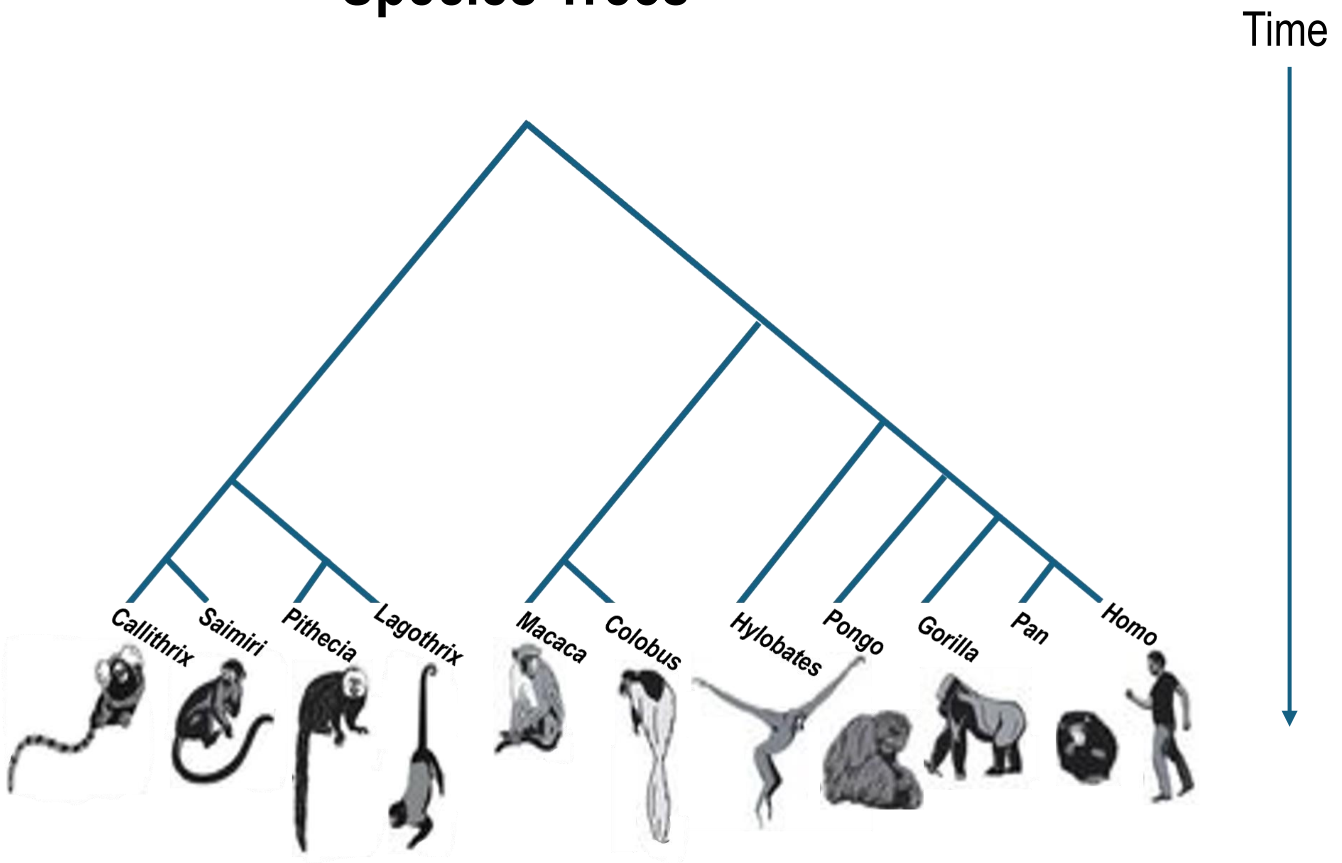
Species Trees



Adapted from http://anthropologyiselemental.ua.edu/osteology.html

Phylogenetic Trees

Species Trees



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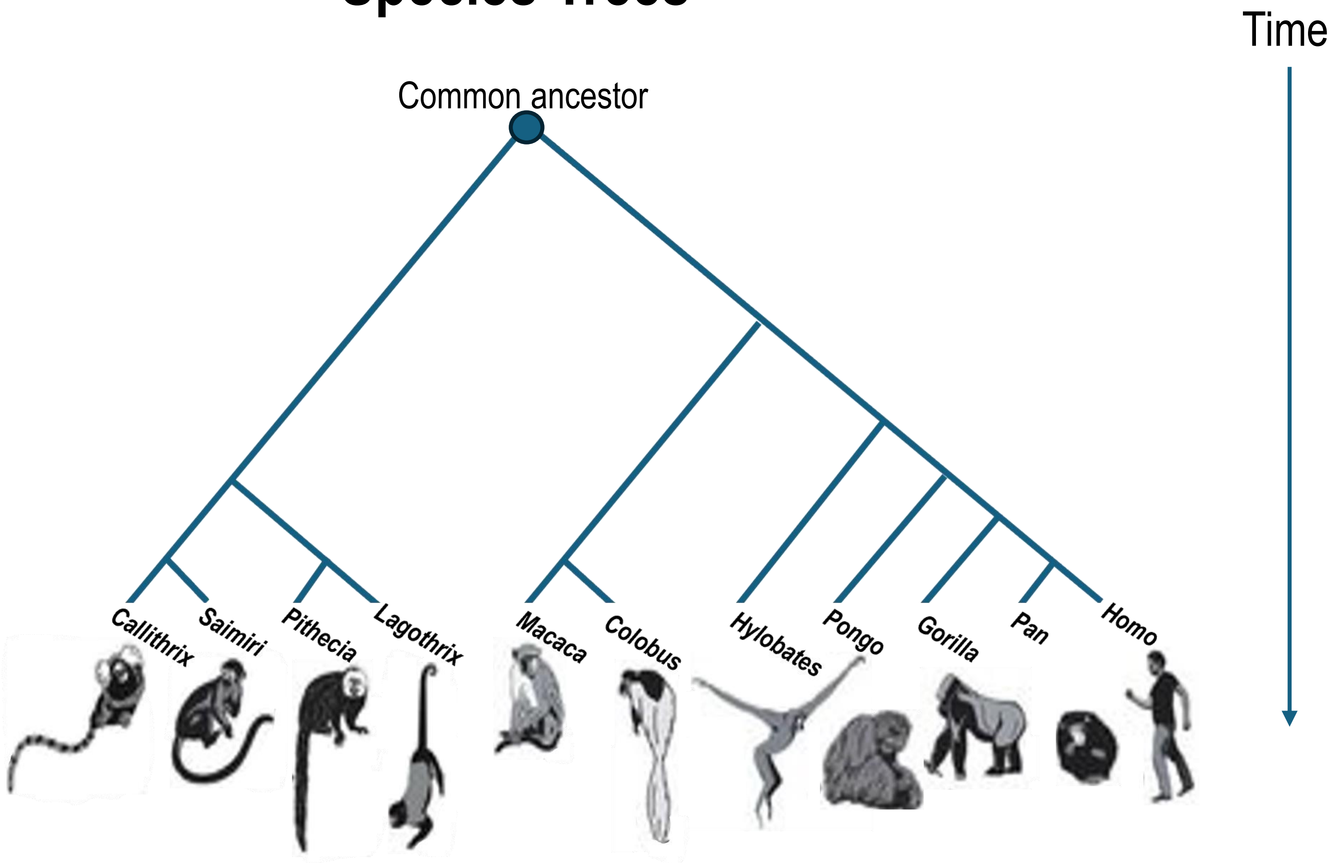

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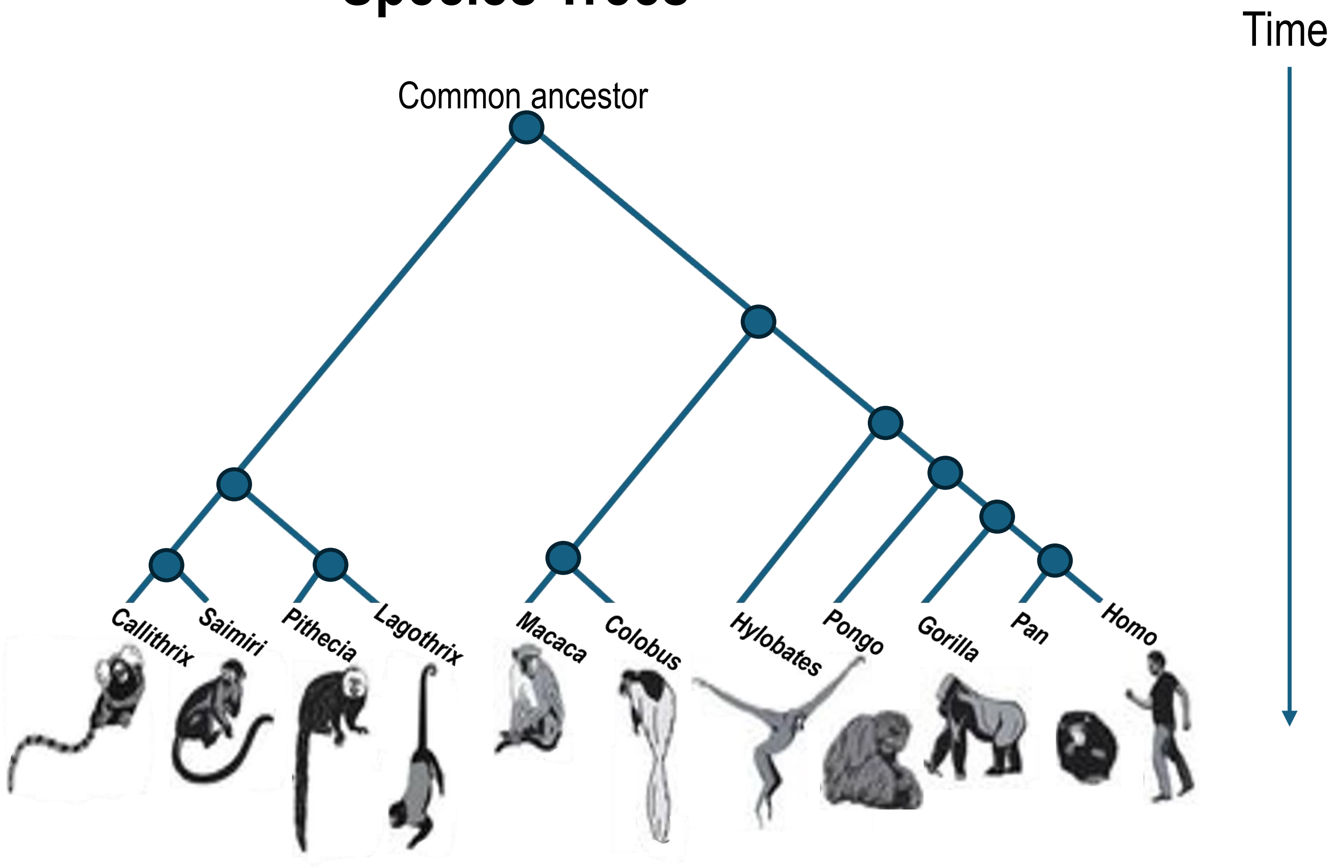

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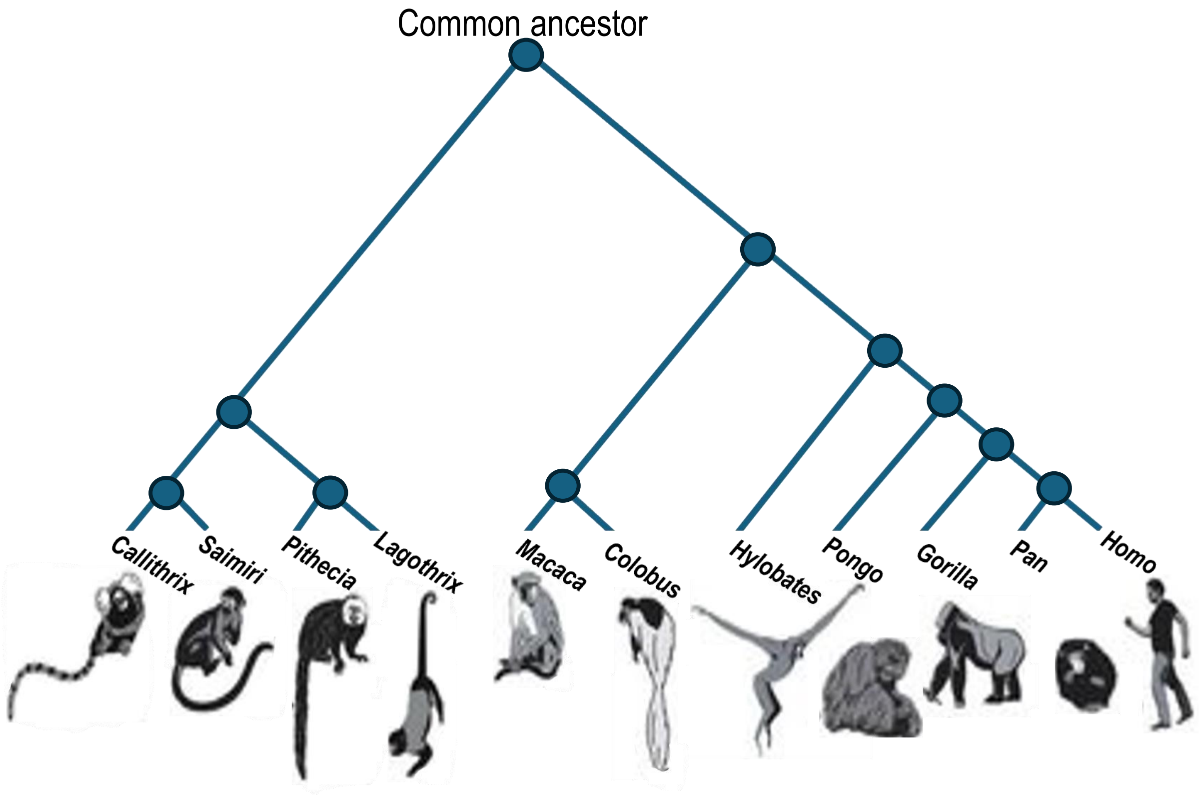

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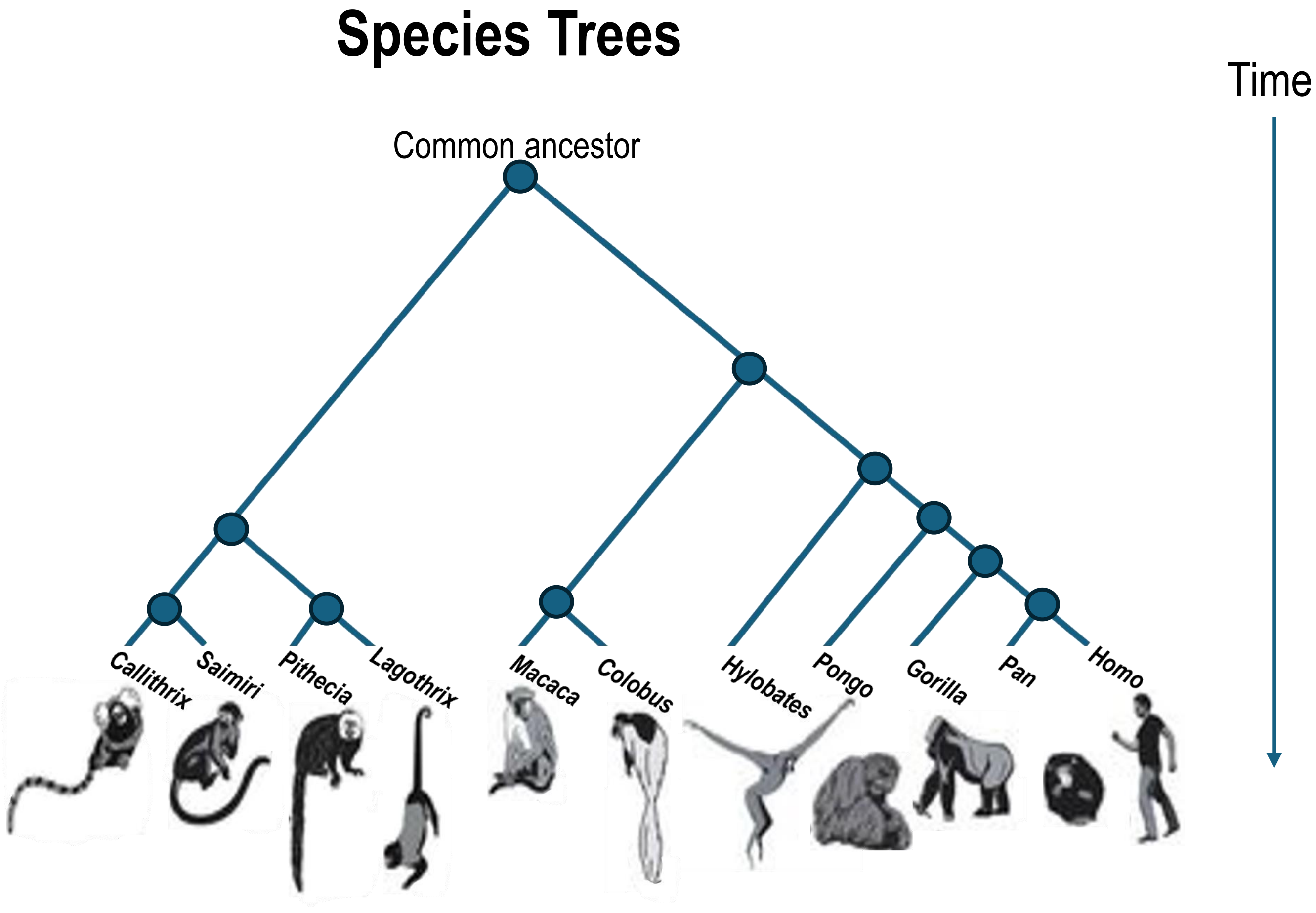
Time

DNA based Trees

Phylogenetic Trees



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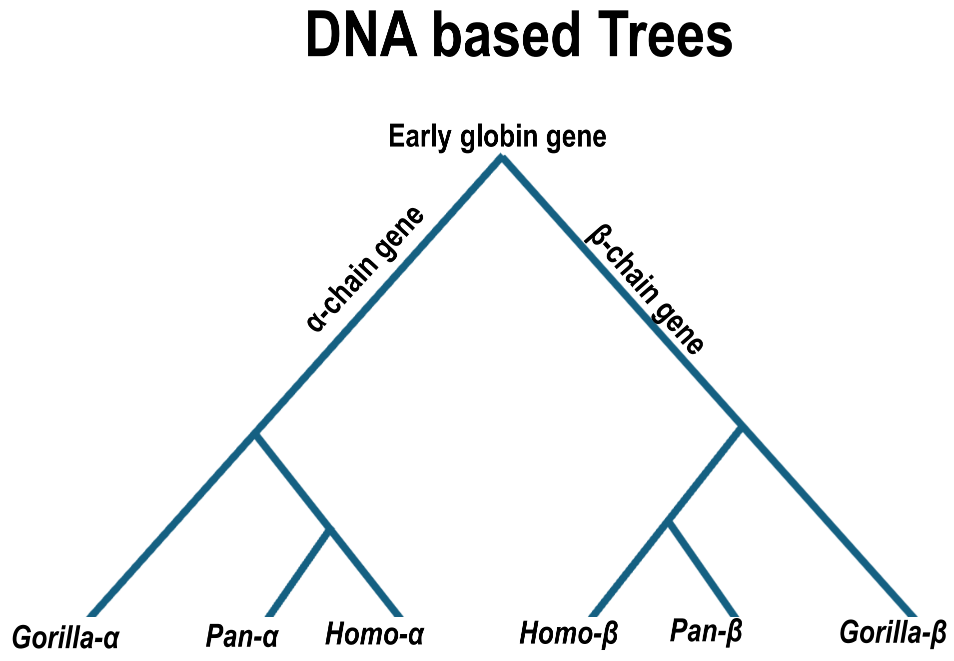

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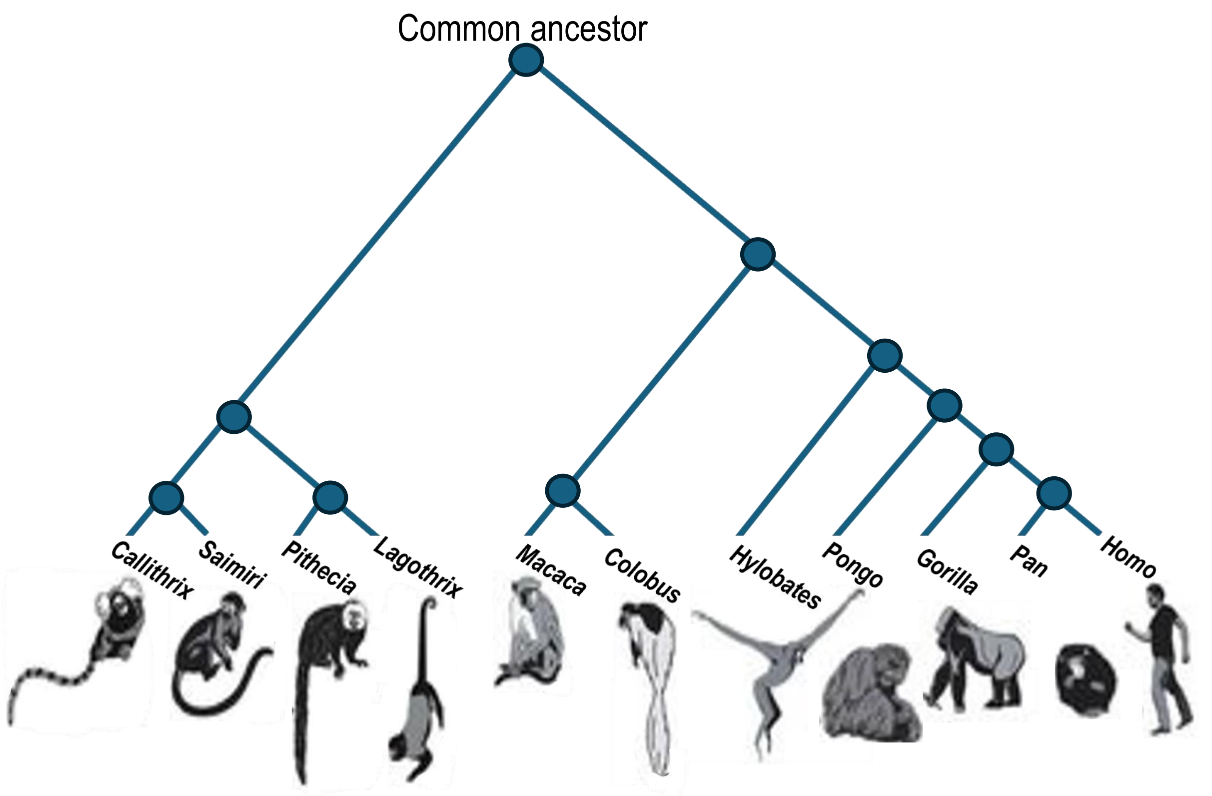

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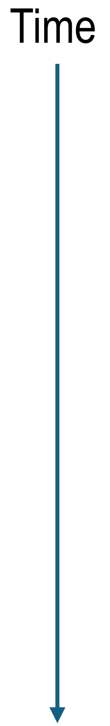

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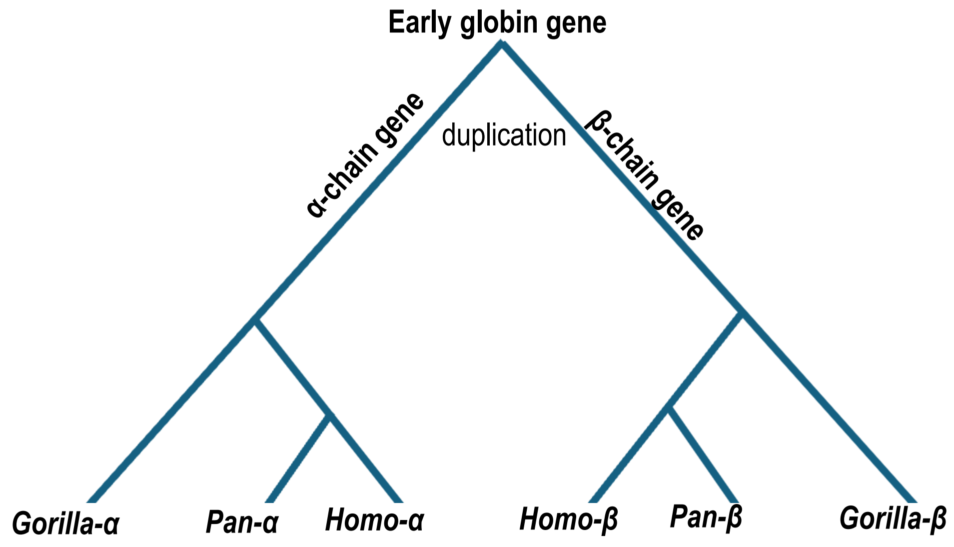

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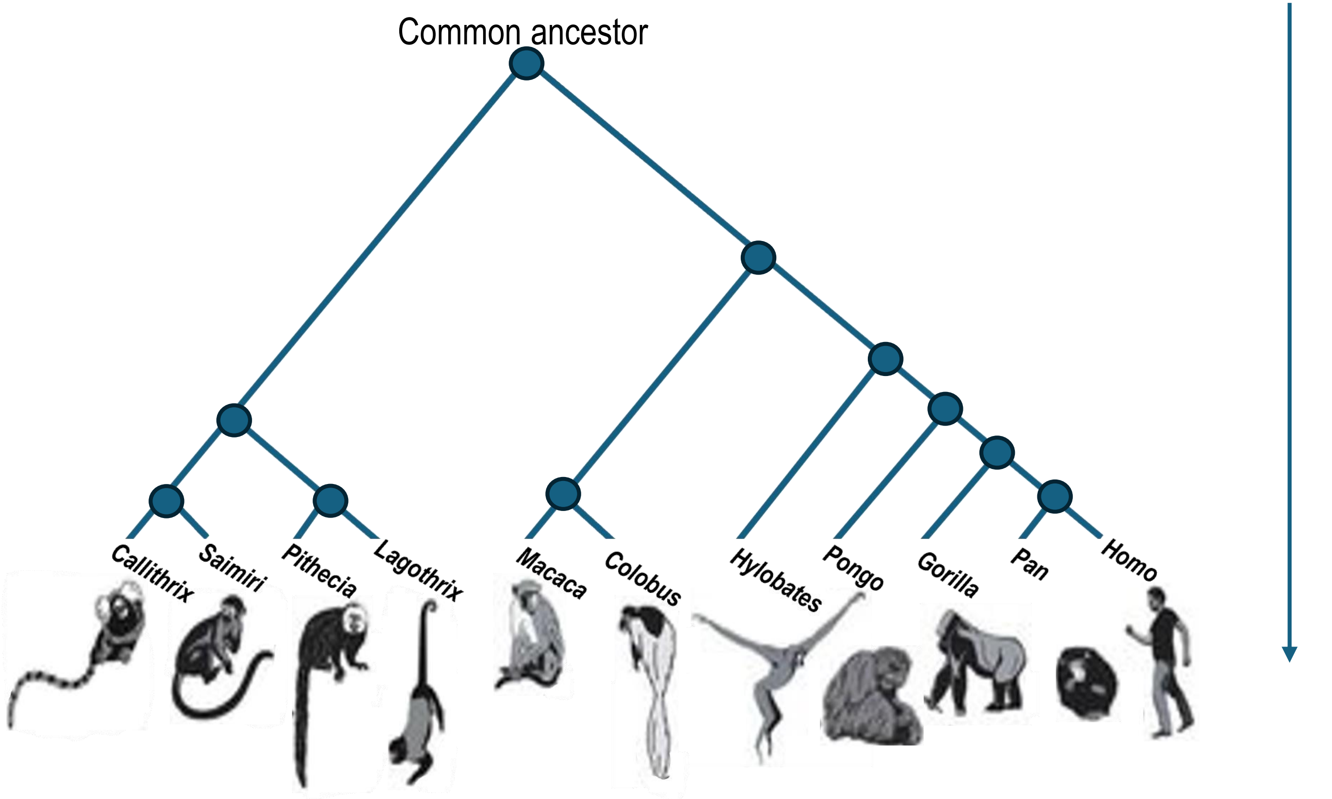

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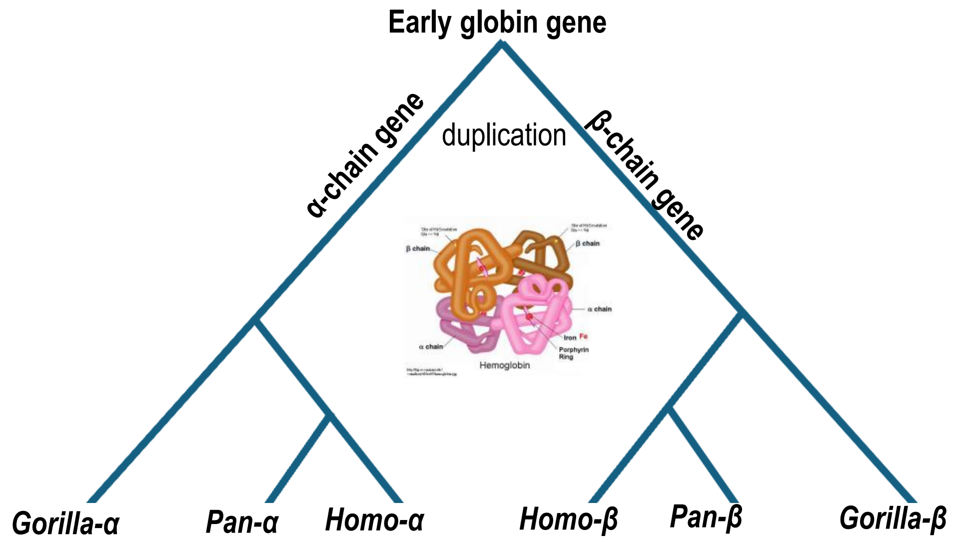

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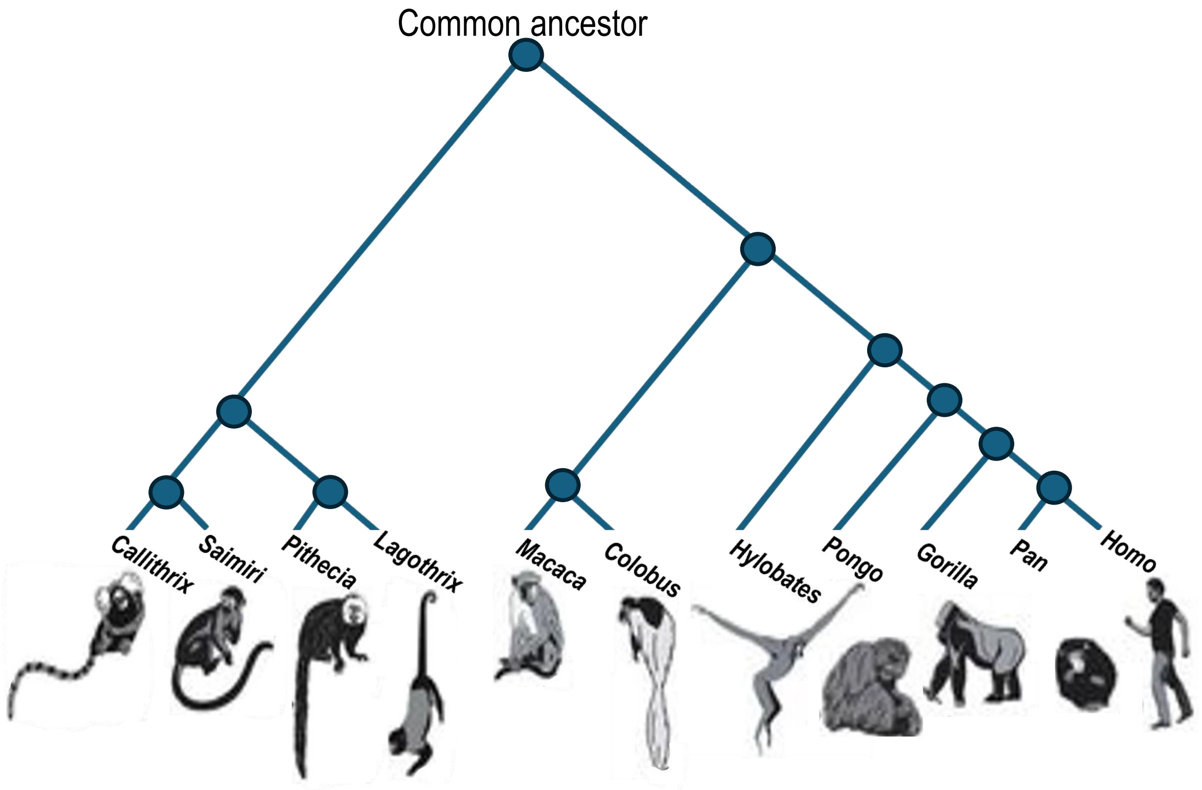

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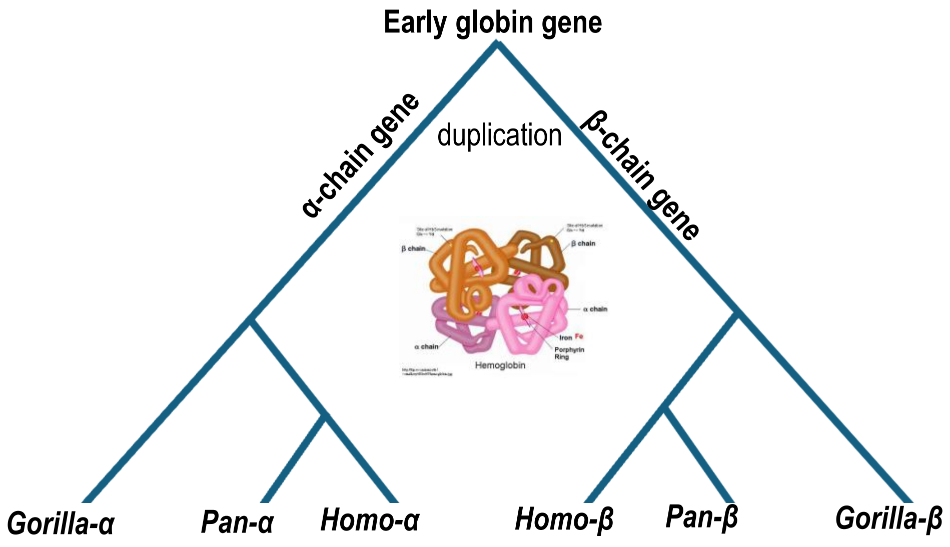

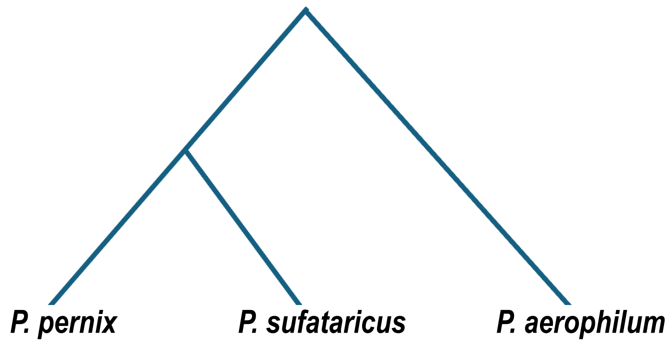

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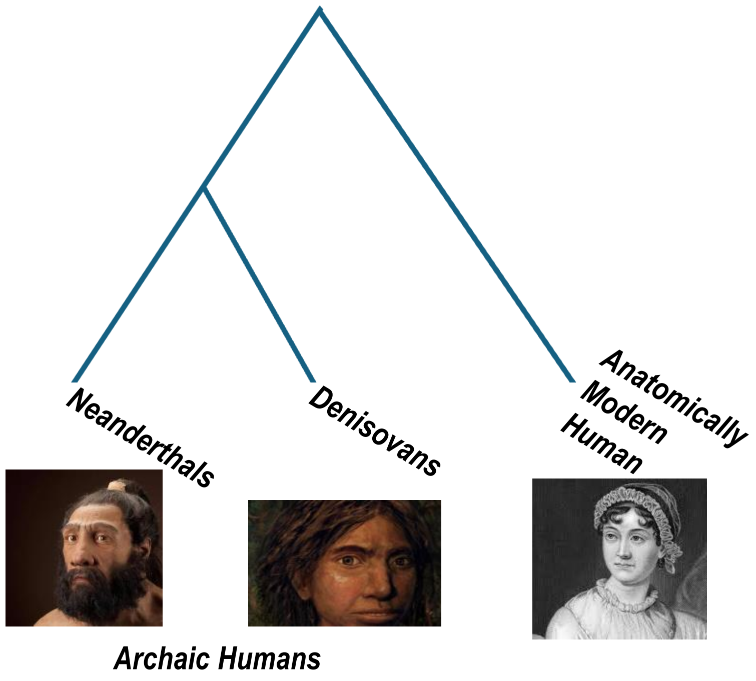

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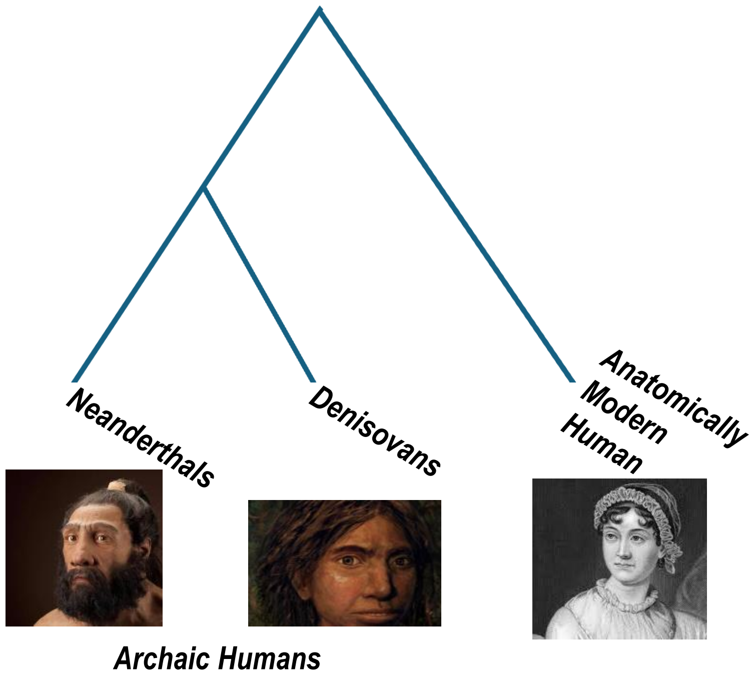

https://humanorigins.si.edu/evidence/human-fossils/species/homo-neanderthalensis

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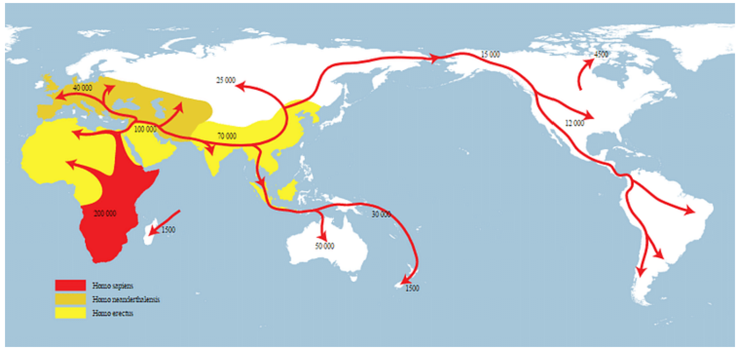
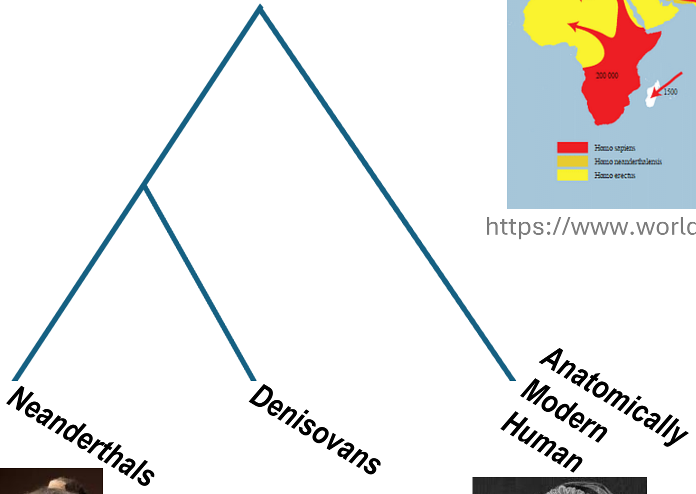
Phylogenetic Trees



https://www.worldhistory.org/article/1070/early-human-migration/



Archaic Humans

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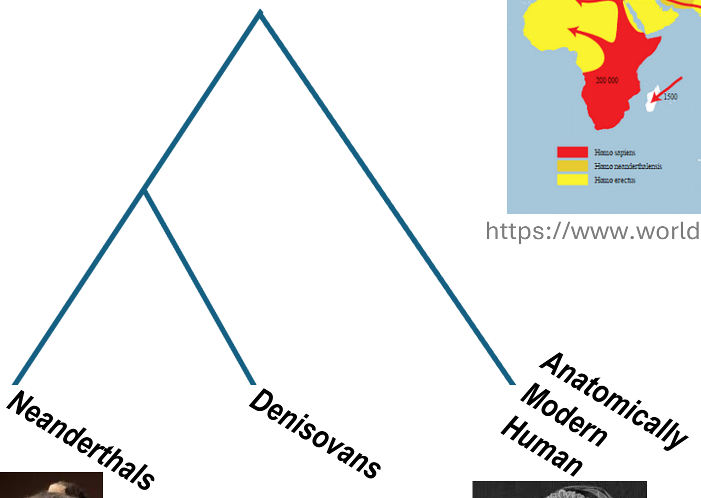

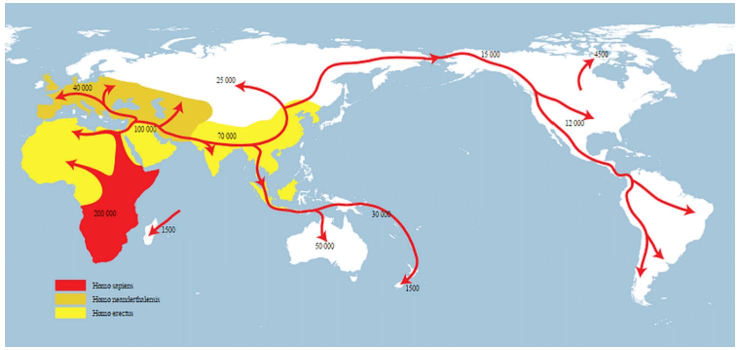

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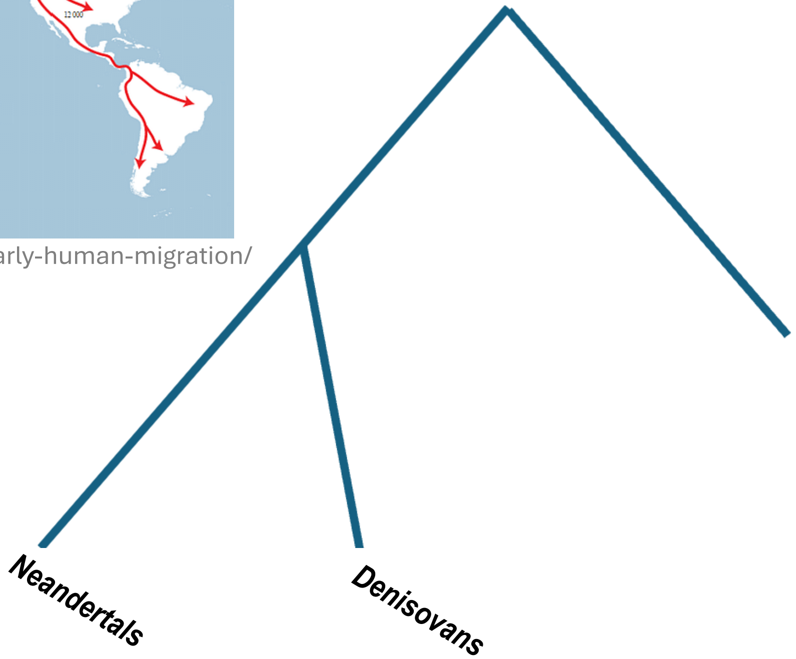

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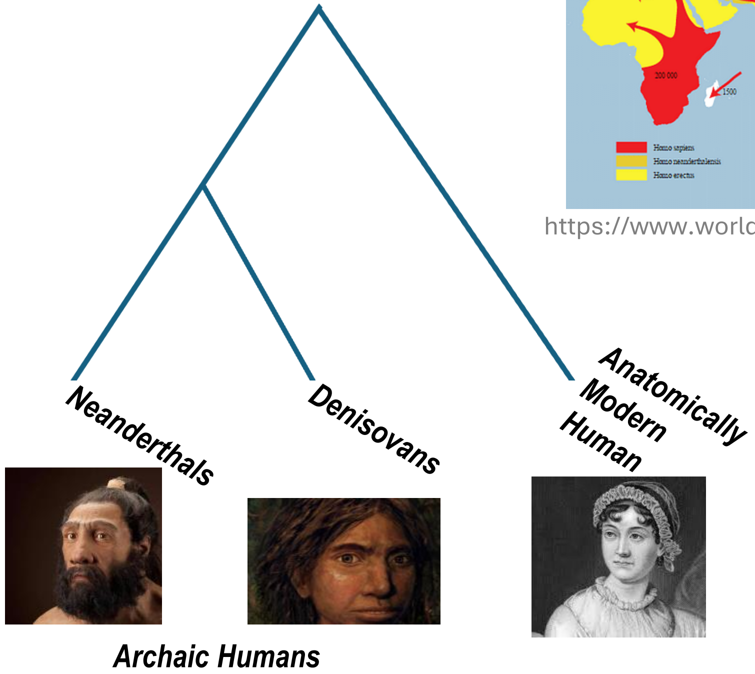

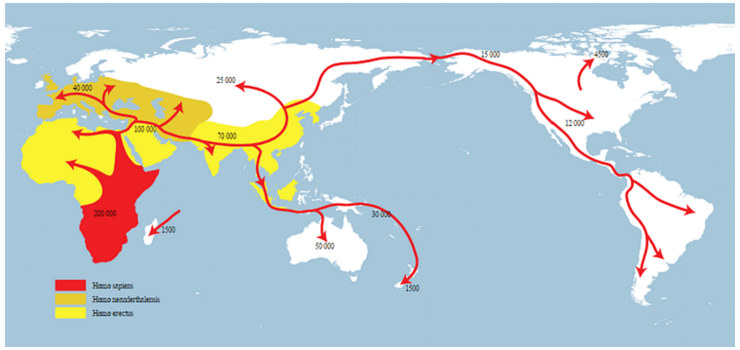

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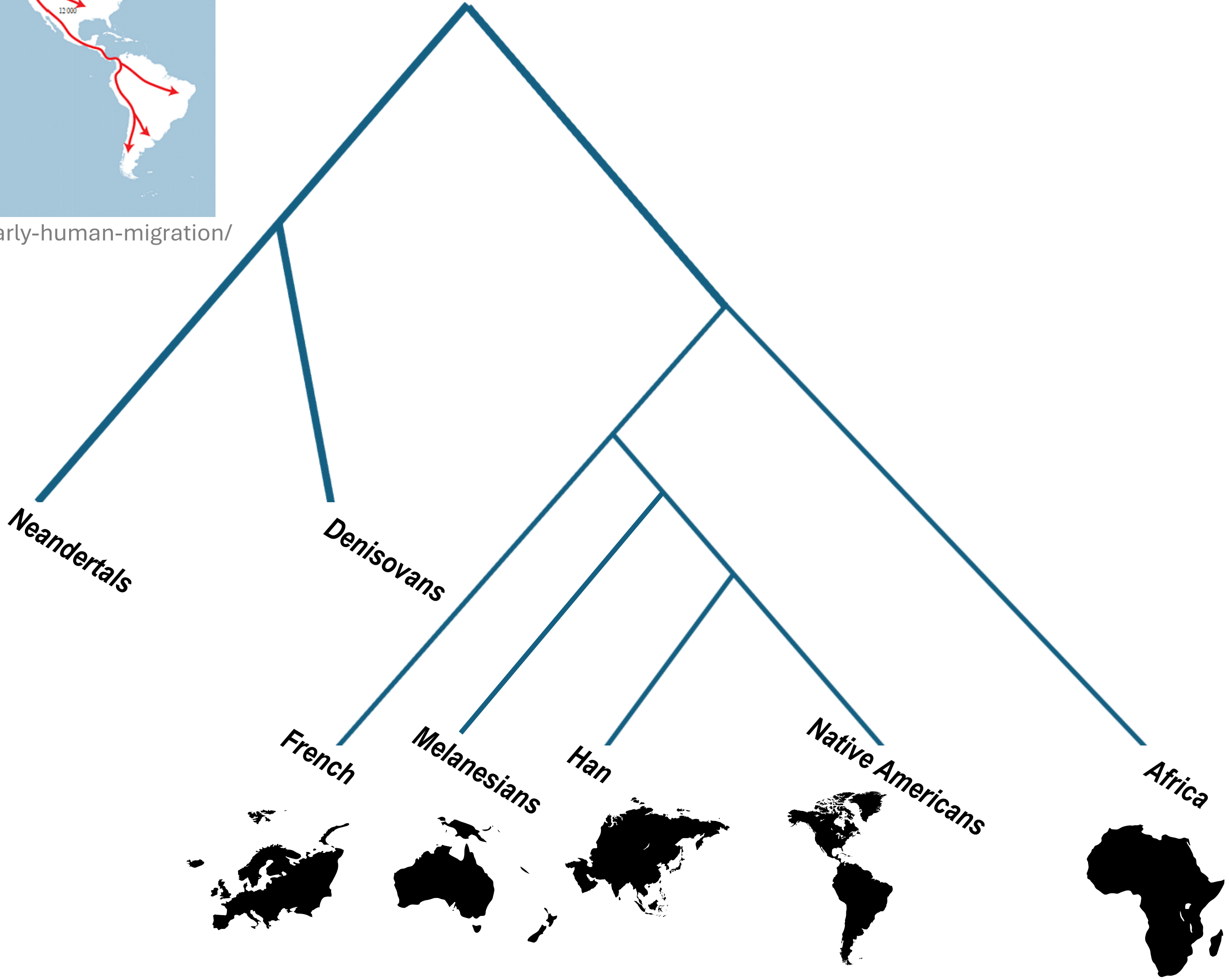

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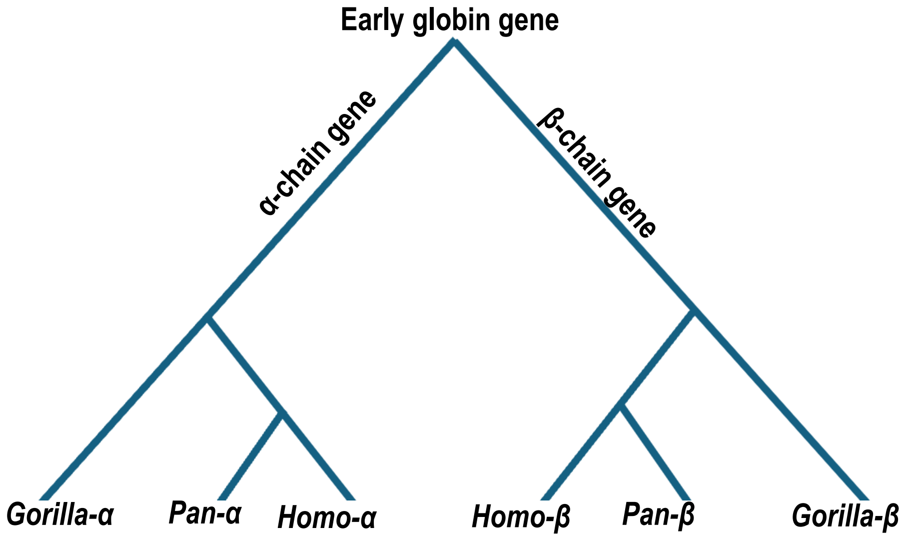

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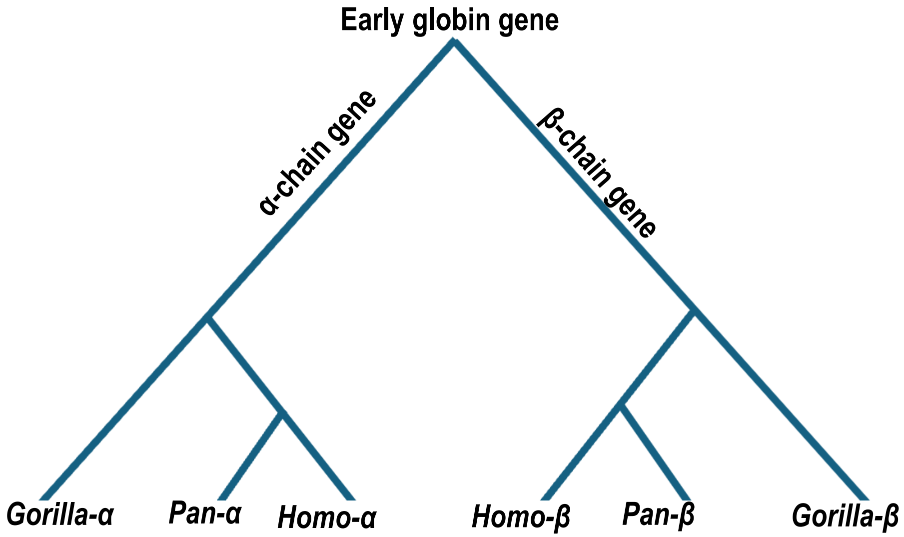
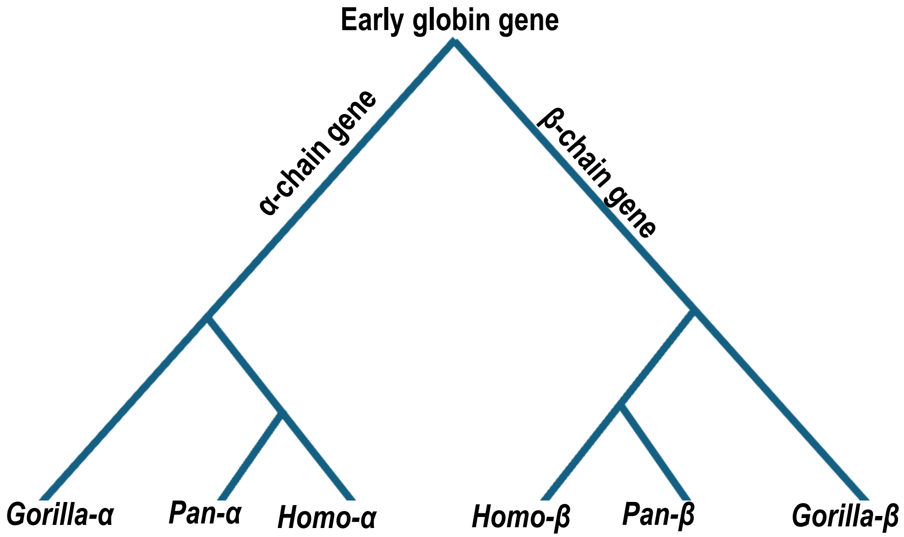


Recombination

Phylogenetic Trees



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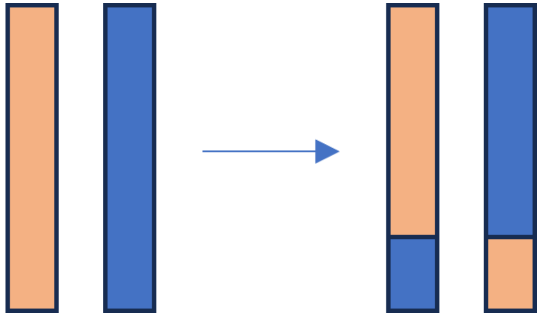

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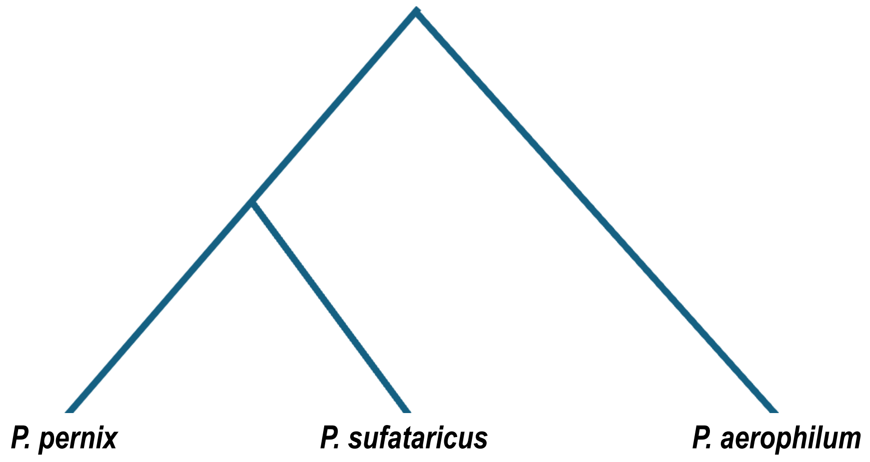
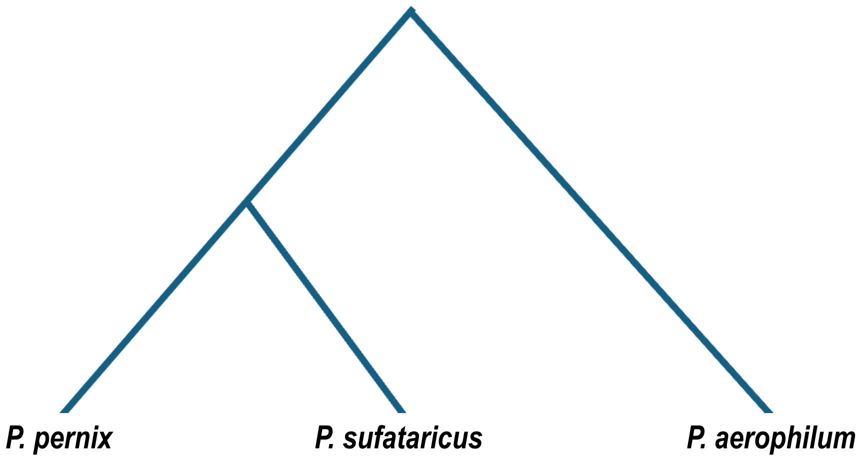


Phylogenetic Trees

Horizontal Gene Transfer



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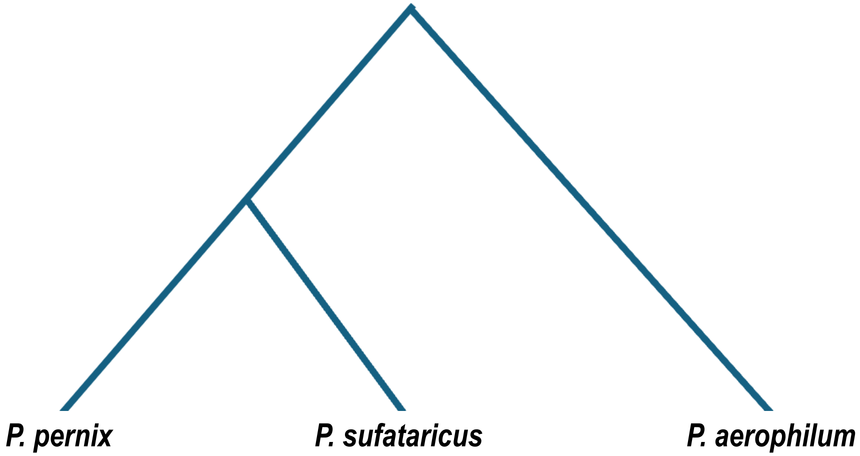

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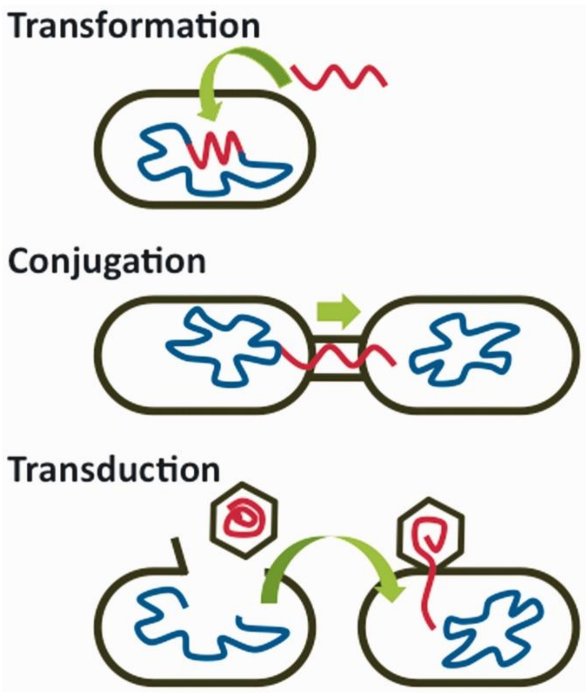

Burmeister, 2015, *Evol Med Public Health*

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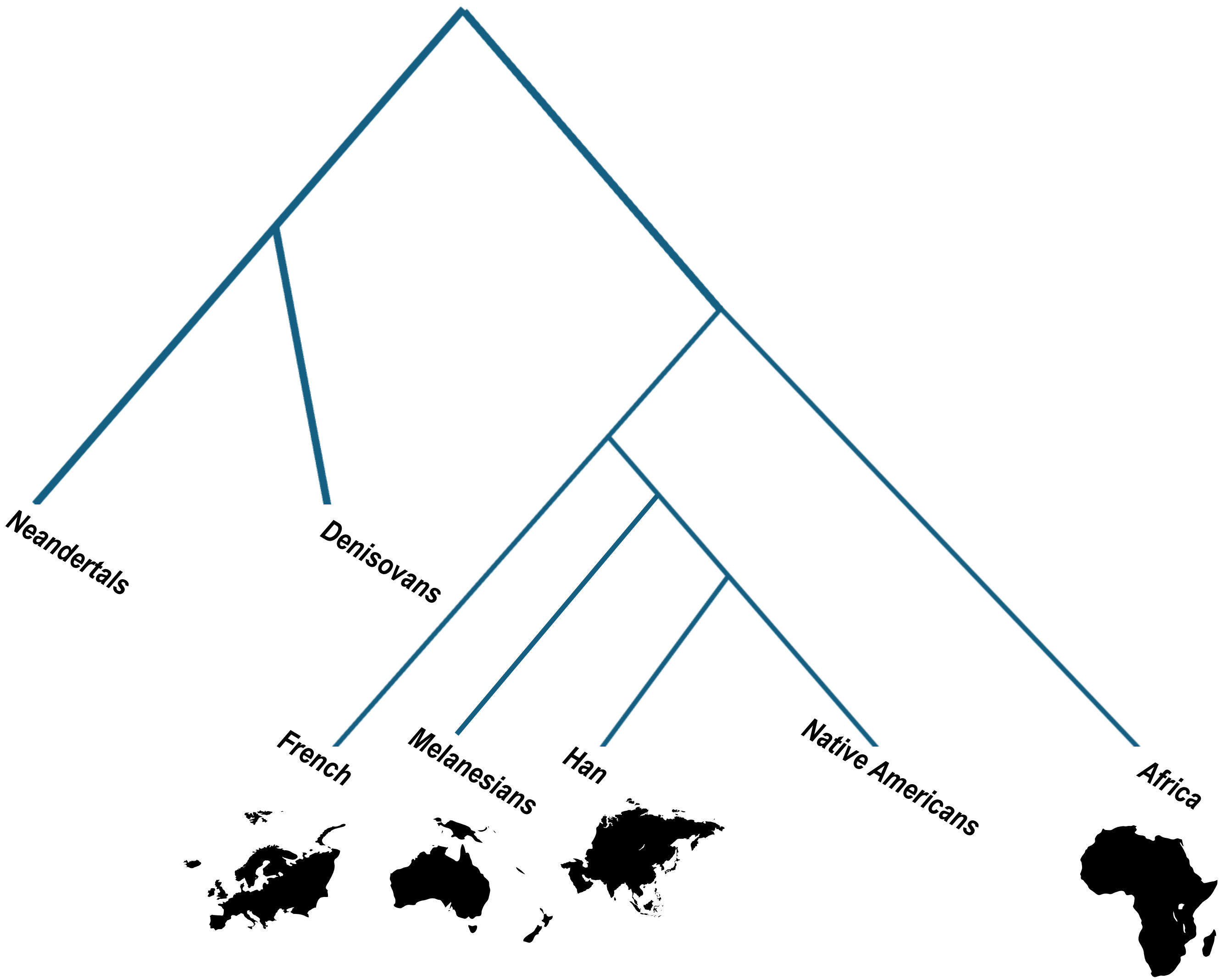

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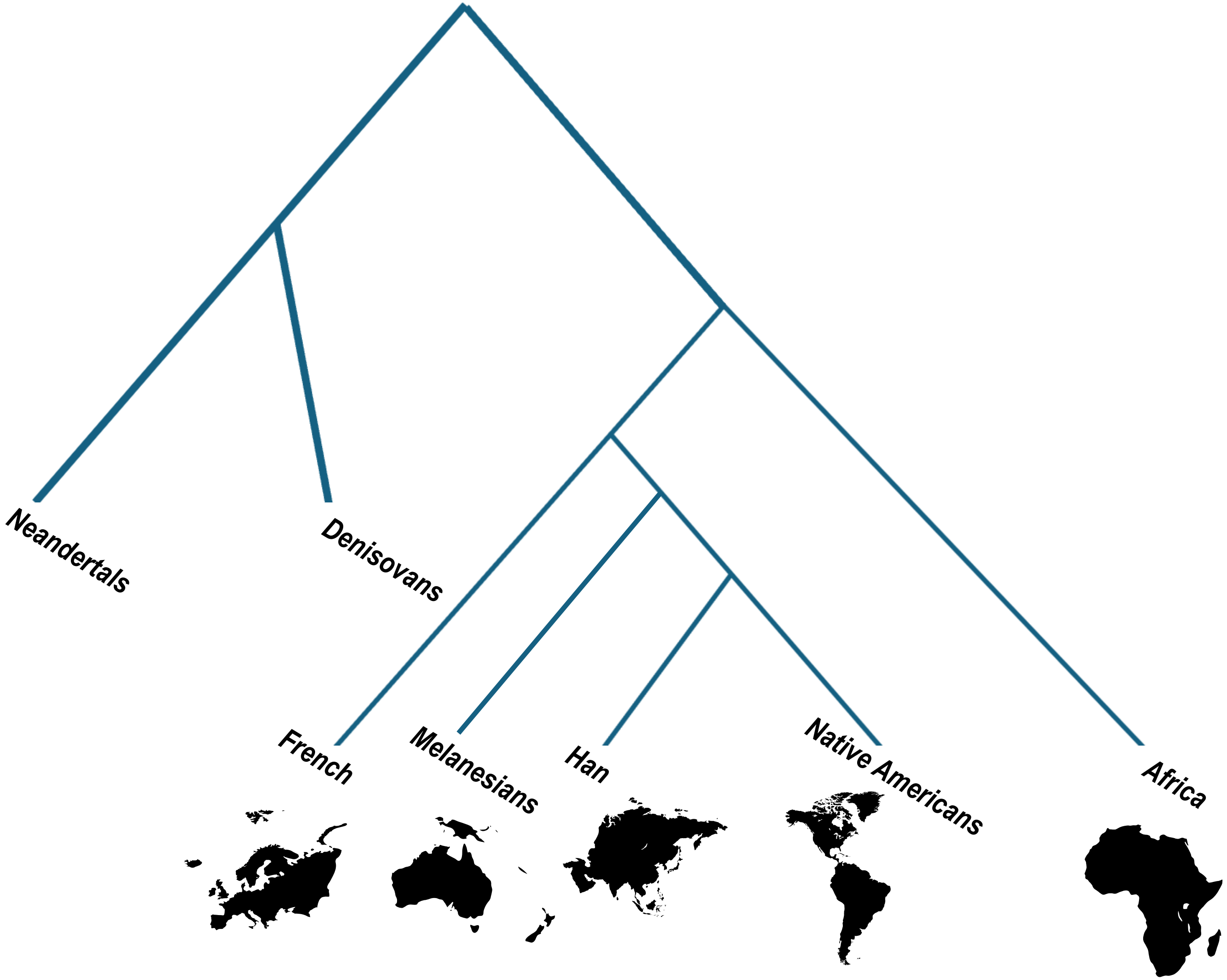

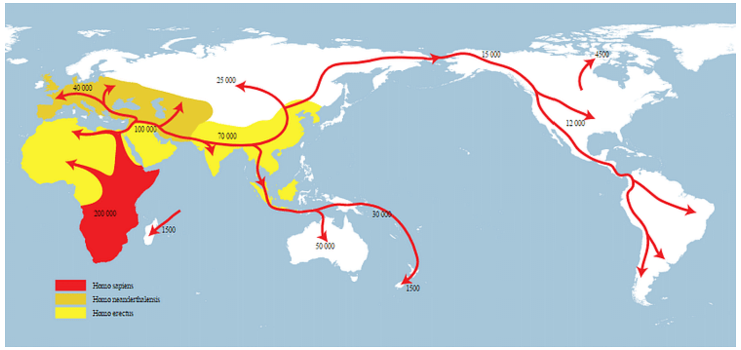

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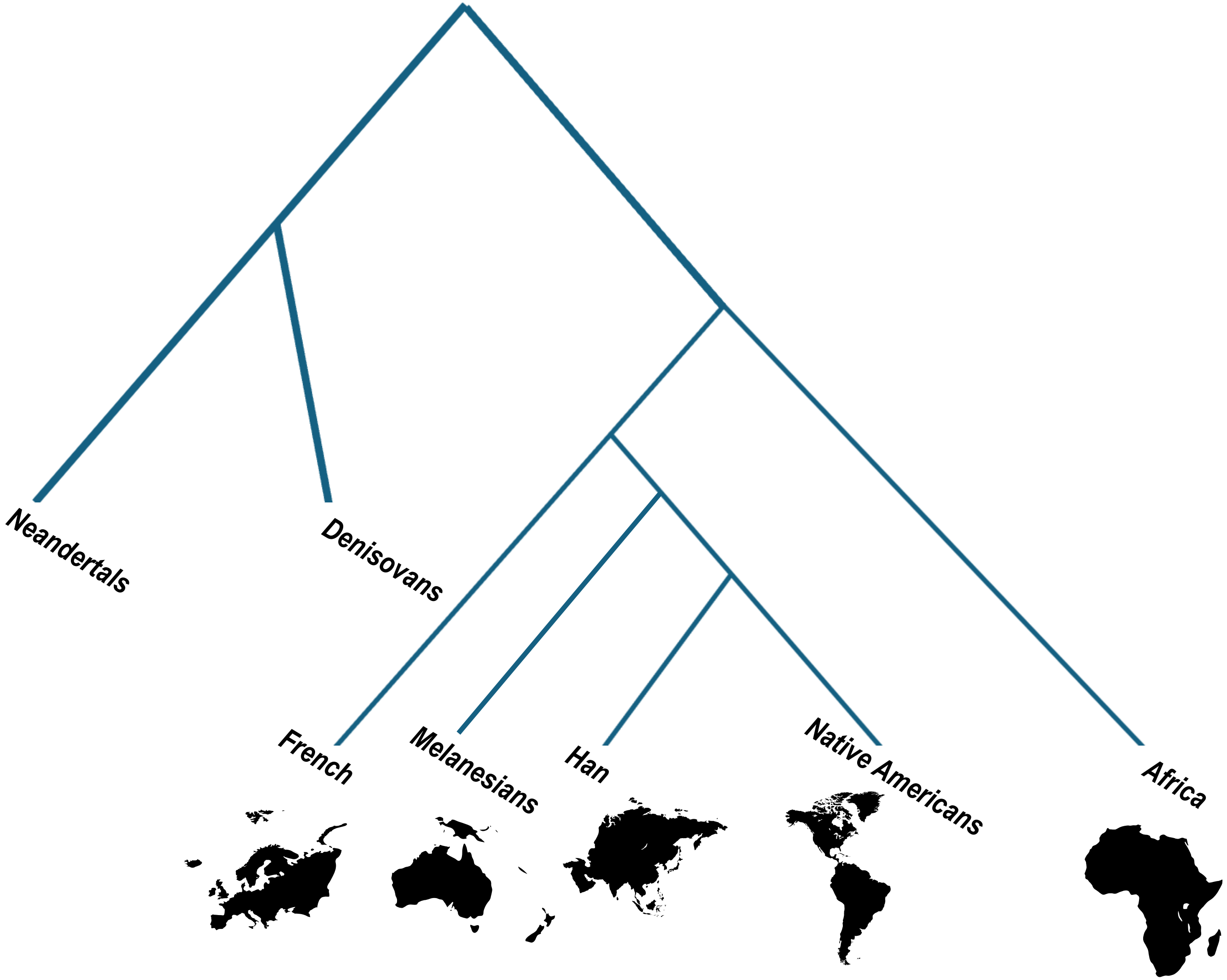

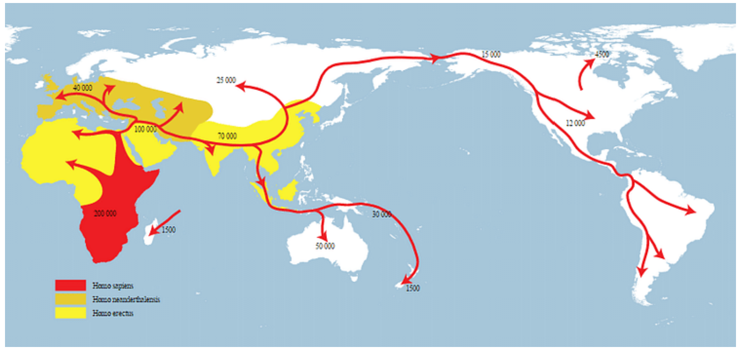

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Admixture

Lecture Outline

Introduction – phylogenetic trees

Definition – galled trees

Previous work – the enumeration of galled trees and galled trees with one gall

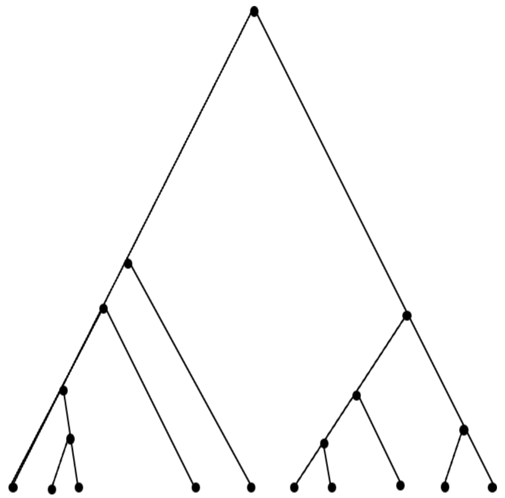
Novel work – the enumeration of galled trees with a fixed number of galls

Summary

Galled Trees

Galled Trees

A rooted binary tree



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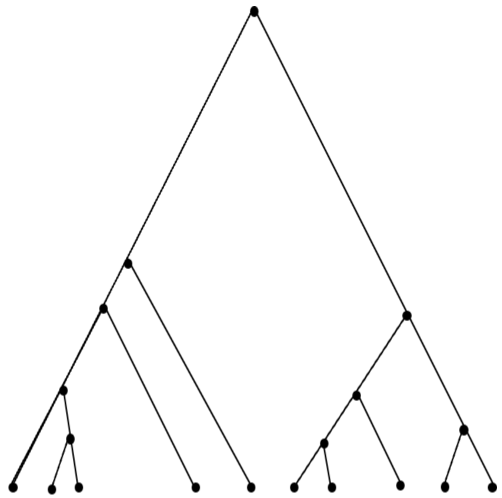
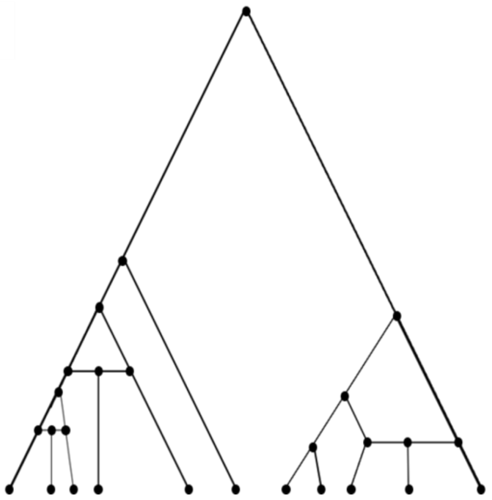


A galled tree



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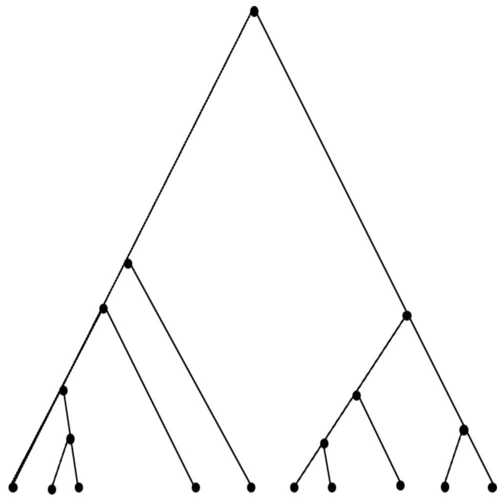

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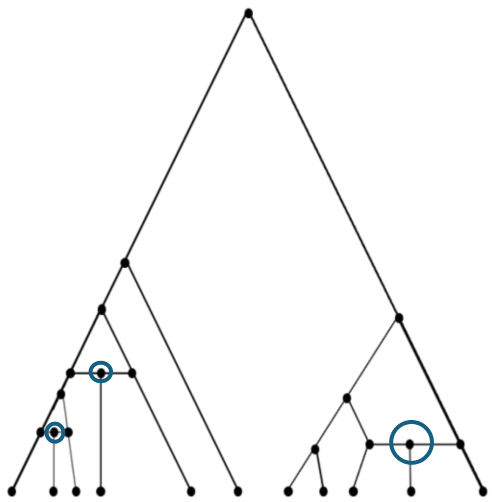

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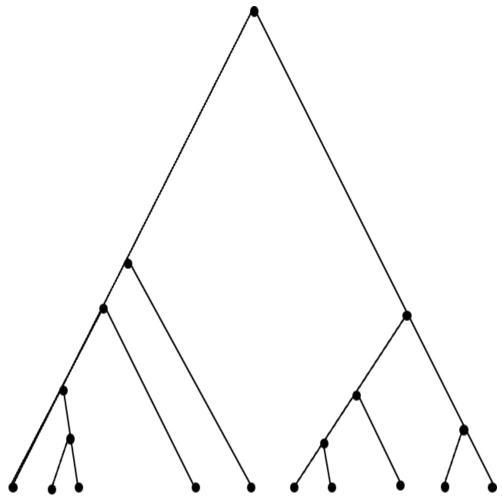

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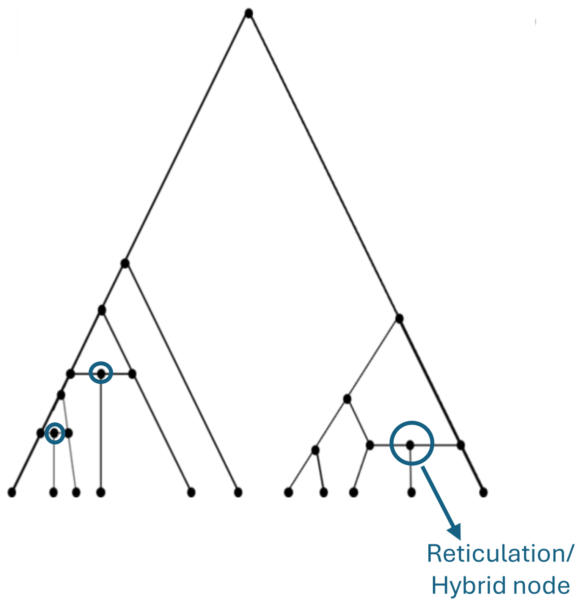

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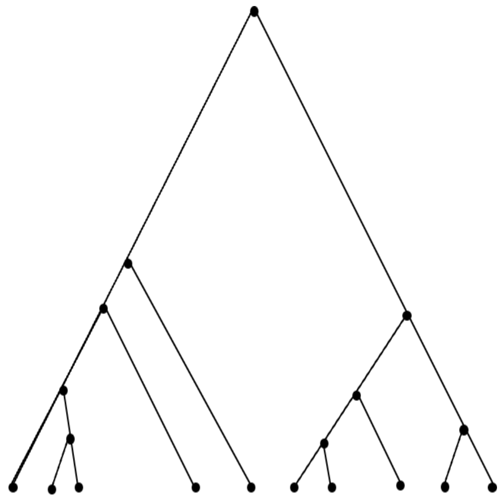

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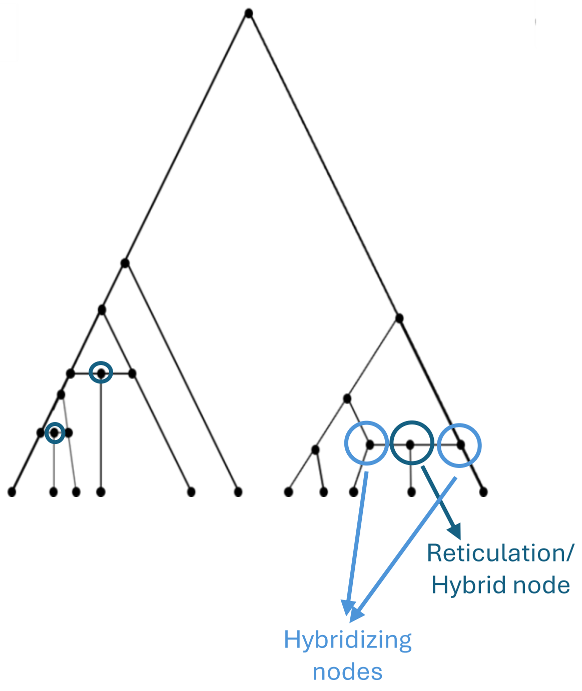

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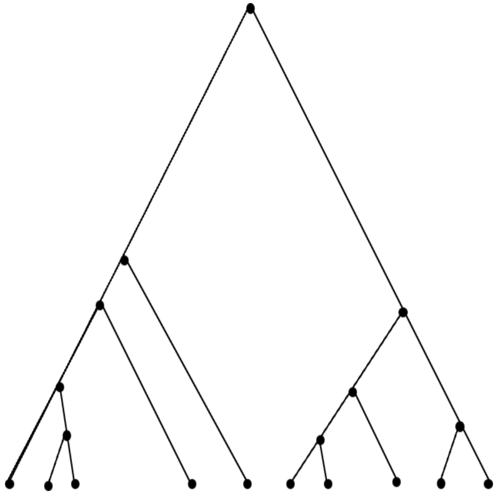

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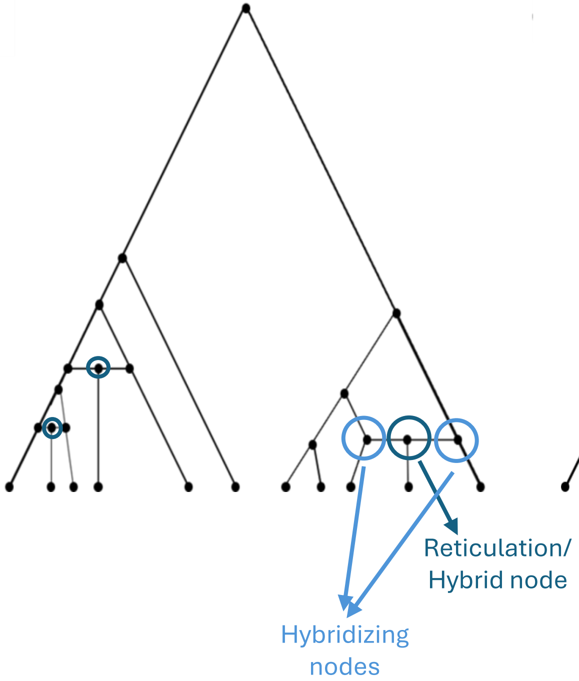
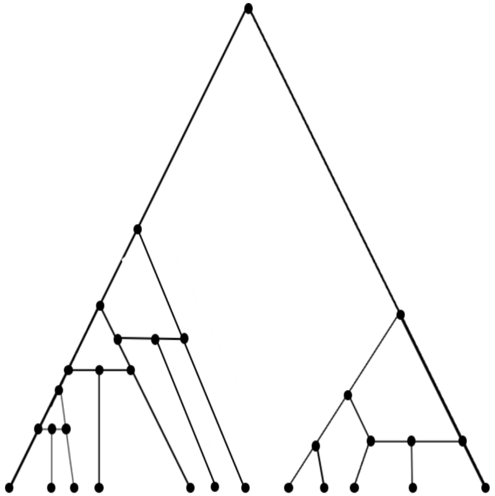


Not a galled tree



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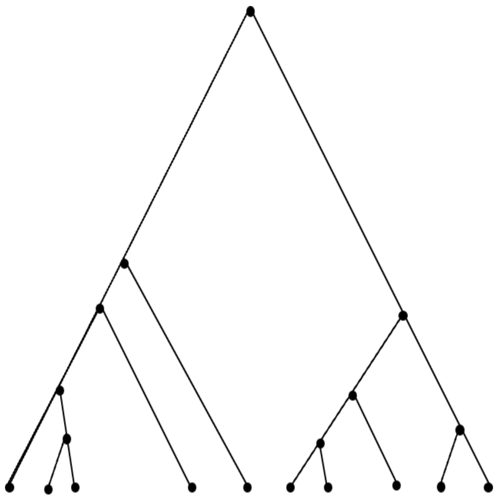

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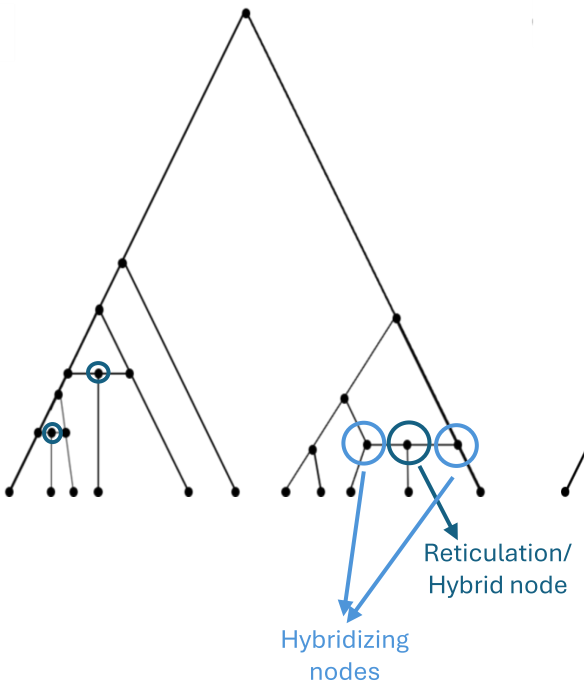

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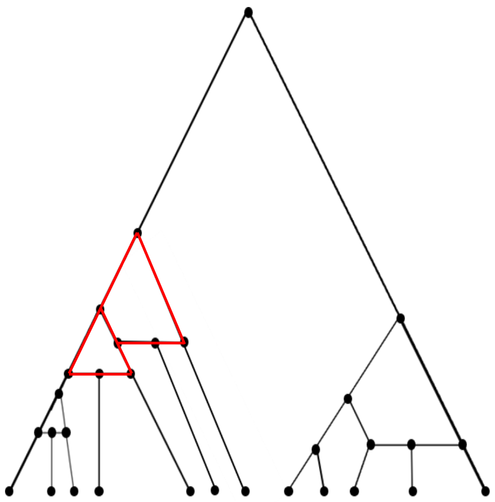

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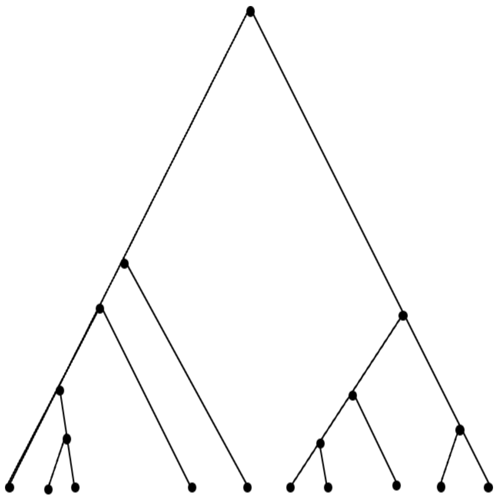

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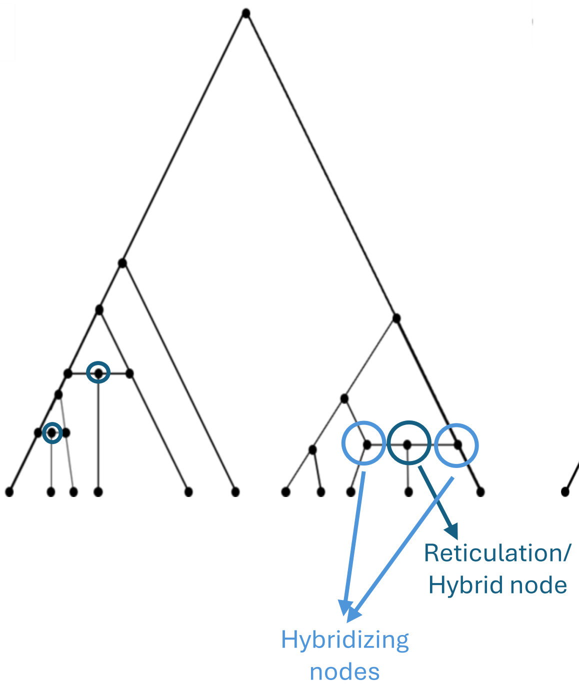

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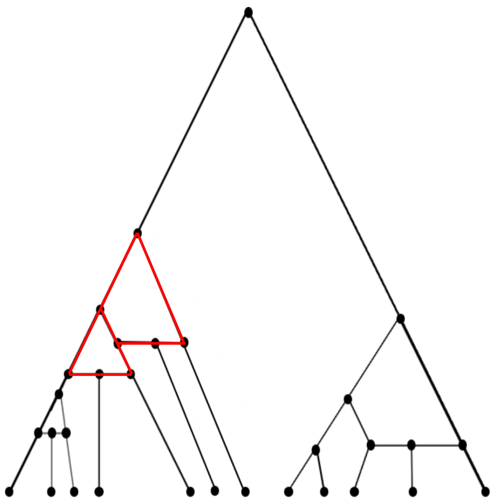

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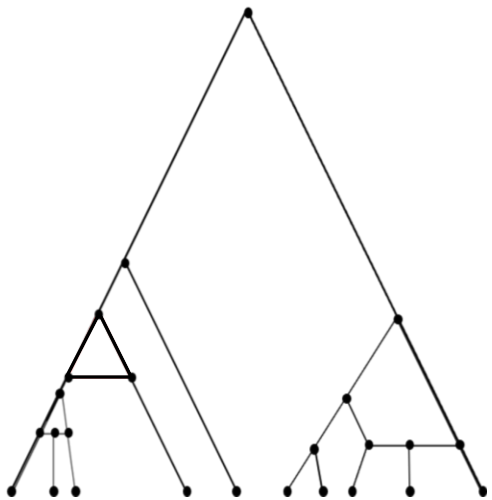

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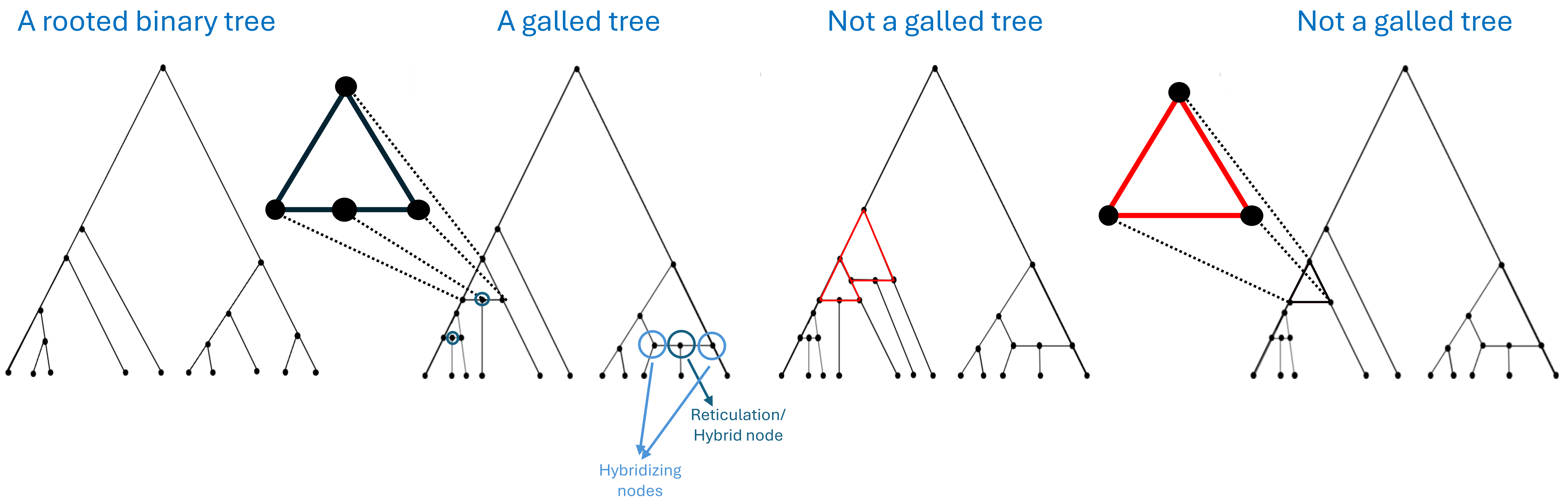

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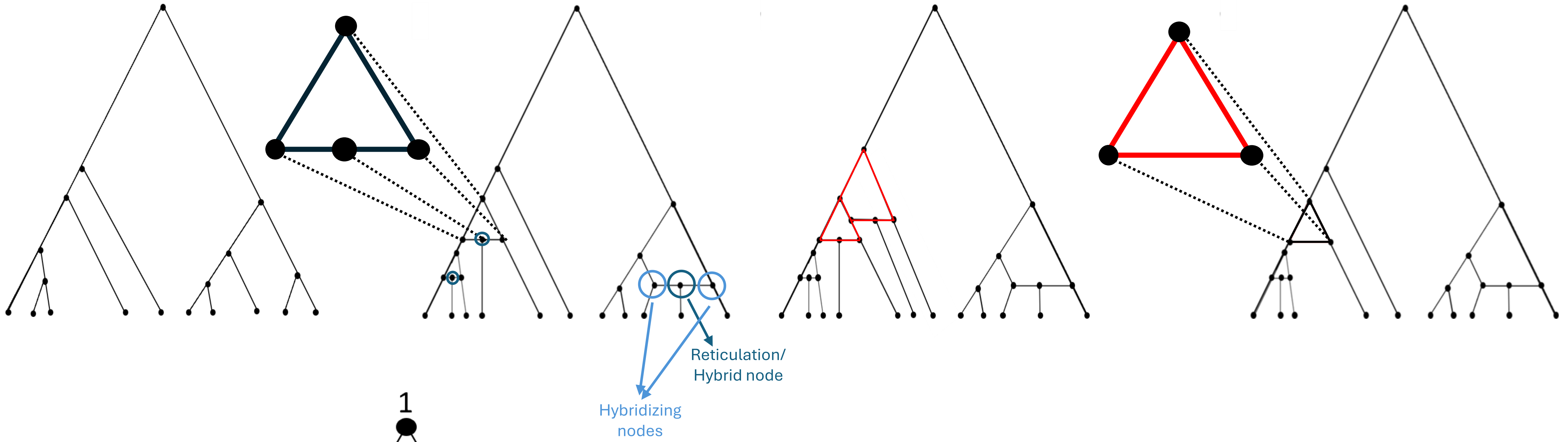
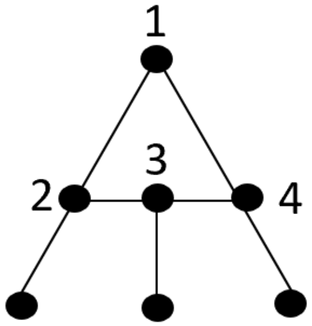
Galled Trees

A rooted binary tree | A galled tree | Not a galled tree | Not a galled tree



The smallest galled tree



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Galled Trees

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Definitions

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Ignoring the direction of edges, the two paths connecting r and a_r form a cycle C_r, known as a gall.

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Ignoring the direction of edges, the two paths connecting r and a_r form a cycle C_r, known as a gall.

Second, the set of nodes in the gall C_r, associated with reticulation node a_r, and the set of nodes in the gall C_s, associated with reticulation node a_s, are disjoint.

Galled Trees

Definitions

A rooted galled tree is a rooted binary phylogenetic network in which three properties hold:

First, each reticulation node a_r has a unique ancestor node r such that exactly two non-overlapping paths of edges exist from r to a_r

Ignoring the direction of edges, the two paths connecting r and a_r form a cycle C_r, known as a gall.

Second, the set of nodes in the gall C_r, associated with reticulation node a_r, and the set of nodes in the gall C_s, associated with reticulation node a_s, are disjoint.

Third (time consistency, normality), the ancestor node r must be separated from a_r by at least two edges.

Galled Trees

Definitions

A rooted galled tree is a rooted binary phylogenetic network in which three properties hold:

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https://discoverandshare.org/2021/06/24/all-about-galls/

Lecture Outline

Introduction – phylogenetic trees

Definition – galled trees

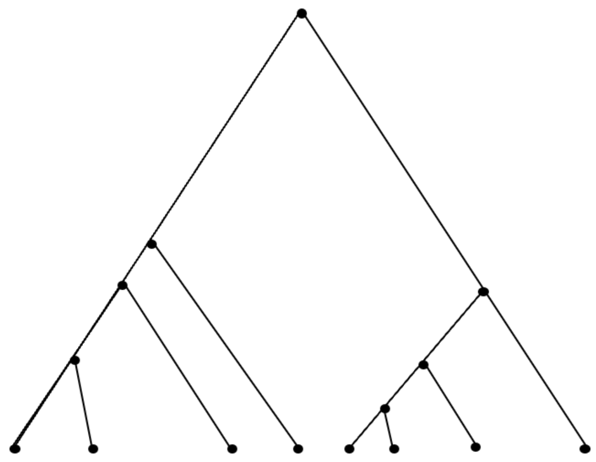
Previous work – the enumeration of galled trees and galled trees with one gall

Novel work – the enumeration of galled trees with a fixed number of galls

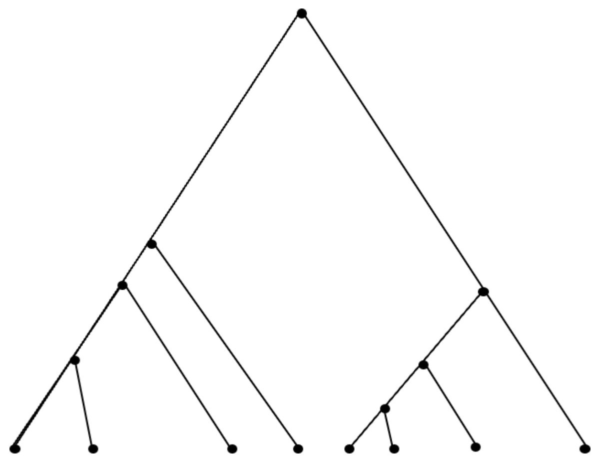
Summary

Rooted Unlabeled Binary Non-plane Trees

Rooted Unlabeled Binary Non-plane Trees



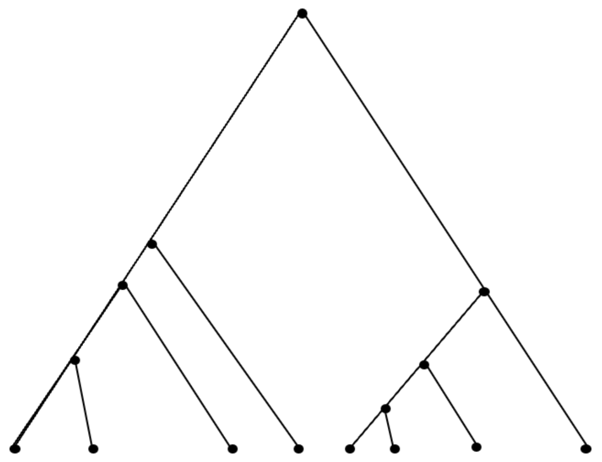
Rooted Unlabeled Binary Non-plane Trees



$$\mathcal{U}(t) = t + \frac{1}{2}\mathcal{U}^2(t) + \frac{1}{2}\mathcal{U}(t^2)$$

Otter 1948, Comtet, 1974

Rooted Unlabeled Binary Non-plane Trees



$$\mathcal{U}(t) = t + \frac{1}{2}\mathcal{U}^2(t) + \frac{1}{2}\mathcal{U}(t^2)$$

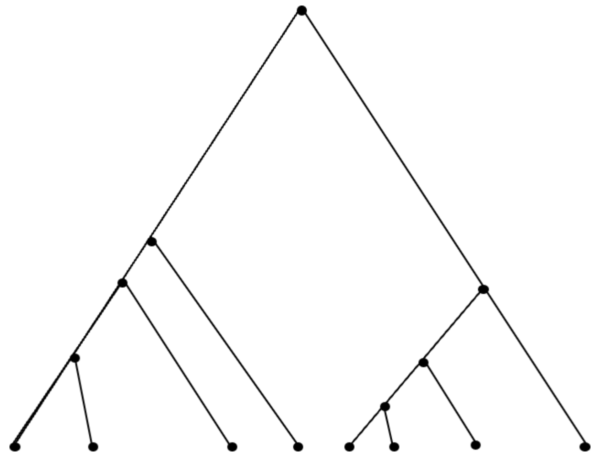
Otter 1948, Comtet, 1974

$$\mathcal{U}(t) \sim 1 - \gamma\sqrt{1 - \frac{t}{\rho}}$$

$\rho \approx 0.4027$
$\gamma \approx 1.1301$

Landau, 1977; Flajolet and Sedgewick, 2009

Rooted Unlabeled Binary Non-plane Trees



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Otter 1948, Comtet, 1974

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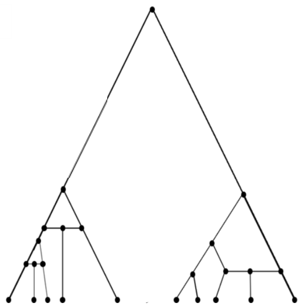
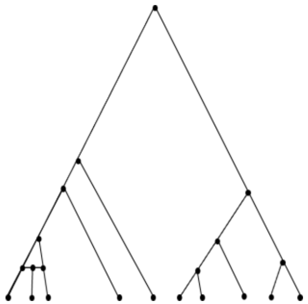
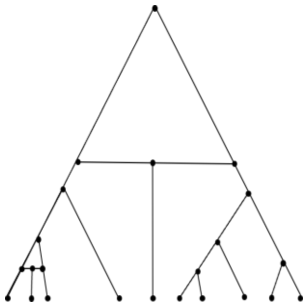
Rooted Unlabeled Binary Non-plane Galled Trees

Rooted Unlabeled Binary Non-plane Galled Trees







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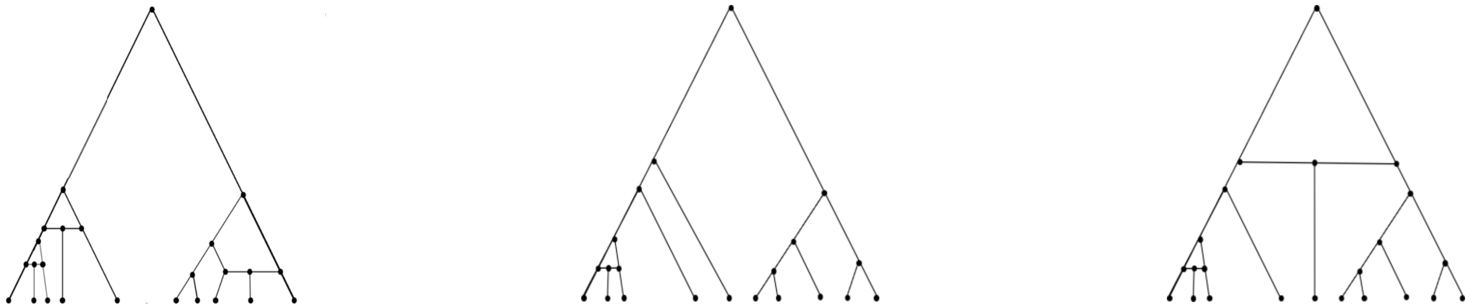
Rooted Unlabeled Binary Non-plane Galled Trees



$$\mathcal{A}(t) = 1 + t + \frac{1}{2}\mathcal{A}^2(t) + \frac{1}{2}\mathcal{A}(t^2) - \frac{1}{1-\mathcal{A}(t)} + \frac{\mathcal{A}(t)}{2[1-\mathcal{A}(t)]^2} + \frac{\mathcal{A}(t)}{2[1-\mathcal{A}(t^2)]}$$

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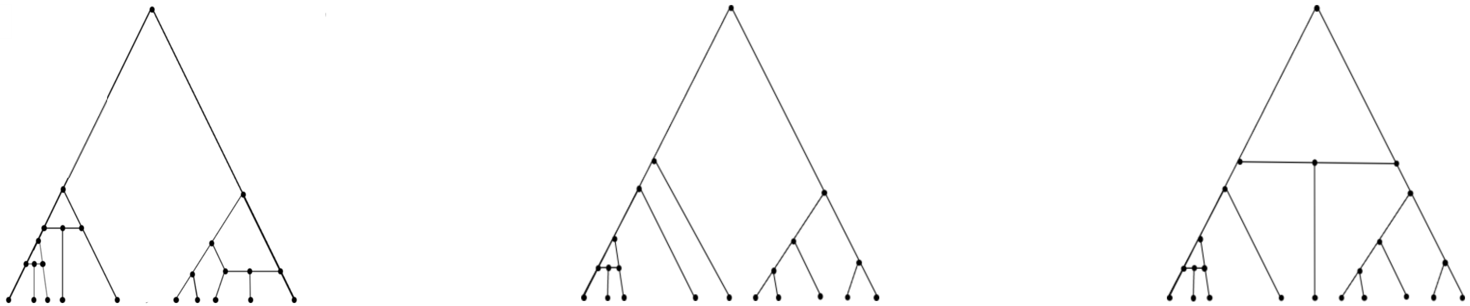
Rooted Unlabeled Binary Non-plane Galled Trees



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$$[t^n]\mathcal{A}(t) \sim \frac{\delta}{2\sqrt{\pi}} n^{-\frac{3}{2}} r^{-n}$$

$$r \approx 0.2073;\ \delta \approx 0.2793$$

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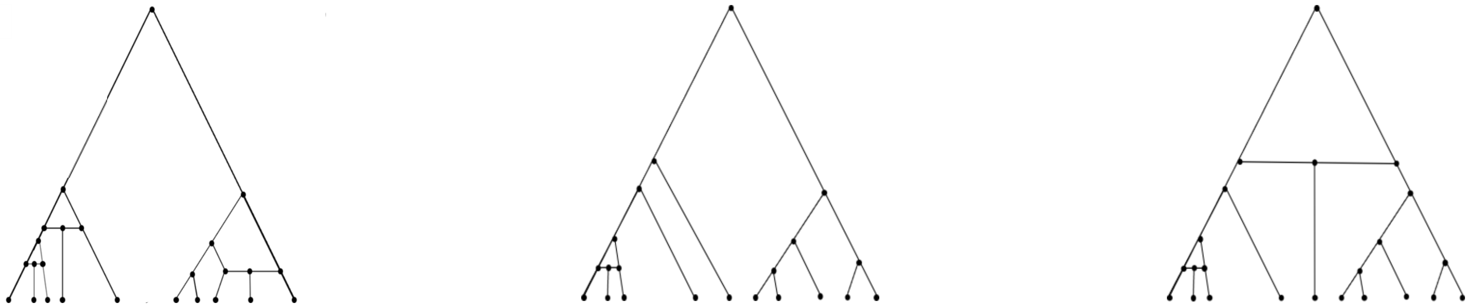
Rooted Unlabeled Binary Non-plane Galled Trees



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$$r \approx 0.2073;\ \delta \approx 0.2793$$

$$\frac{1}{r} \approx 4.82;\ \frac{1}{\rho} \approx 2.48$$

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Galled Trees with Exactly One Gall

Galled Trees with Exactly One Gall

g is the number of galls; n is the number of leaves

Galled Trees with Exactly One Gall

$g = 1$

g is the number of galls; n is the number of leaves

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Galled Trees with Exactly One Gall

$n = 1$

$g = 1$

g is the number of galls; n is the number of leaves

Galled Trees with Exactly One Gall

$n = 1$

No trees

$g = 1$

g is the number of galls; n is the number of leaves

Galled Trees with Exactly One Gall

$n = 1$ $n = 2$

No trees

$g = 1$

g is the number of galls; n is the number of leaves

Galled Trees with Exactly One Gall

	$n = 1$	$n = 2$
$g = 1$	No trees	No trees

g is the number of galls; n is the number of leaves

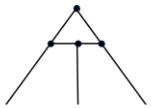
Galled Trees with Exactly One Gall

$n = 1$ | $n = 2$ | $n = 3$

$g = 1$

No trees

No trees



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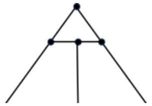
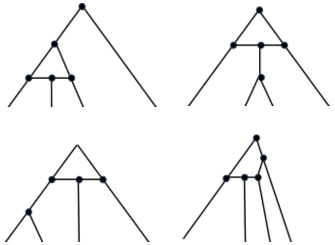
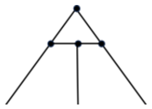
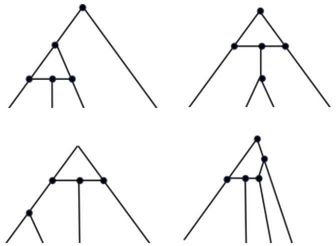
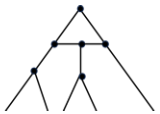
Galled Trees with Exactly One Gall

	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$g = 1$	No trees	No trees		

g is the number of galls; n is the number of leaves

Galled Trees with Exactly One Gall

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$g = 1$	No trees	No trees			 +14 more

g is the number of galls; n is the number of leaves

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Galled Trees with Exactly One Gall

$g = 1$

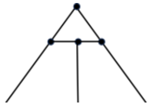
$n = 1$

No trees

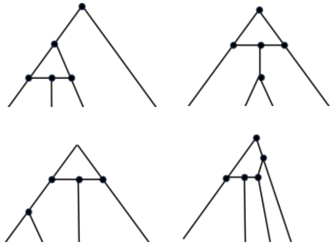
$n = 2$

No trees

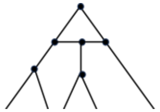
$n = 3$



$n = 4$

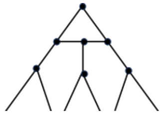


$n = 5$



+14 more

$n = 6$



+47 more

g is the number of galls; n is the number of leaves

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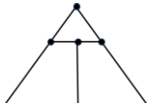
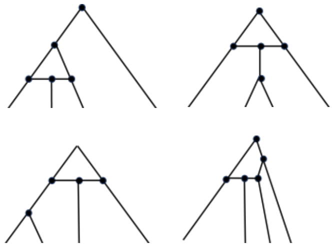
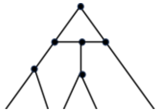
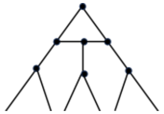
Galled Trees with Exactly One Gall

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$g = 1$	No trees	No trees			 +14 more	 +47 more

g is the number of galls; n is the number of leaves

$$\mathcal{E}_1(t) = \frac{1}{1-\mathcal{U}(t)} - \frac{1}{[1-\mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1-\mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1-\mathcal{U}(t)][1-\mathcal{U}(t^2)]}$$

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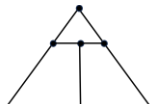
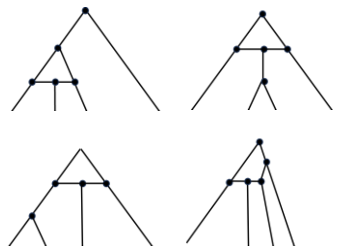
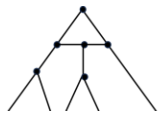
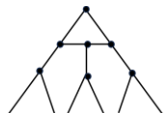
Galled Trees with Exactly One Gall

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$g = 1$	No trees	No trees			 +14 more	 +47 more

g is the number of galls; n is the number of leaves

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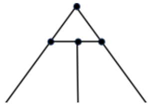
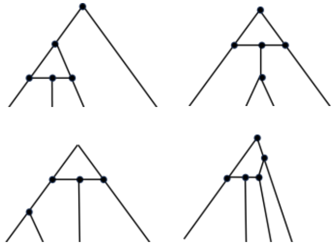
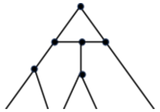
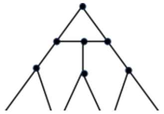
Agranat-Tamir *et al*, 2024

Galled Trees with Exactly One Gall

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$g = 1$	No trees	No trees			 +14 more	 +47 more

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$$\mathcal{E}_1(t) \sim \frac{1}{2\gamma^3 (1 - \frac{t}{\rho})^{\frac{3}{2}}} \qquad [t^n]\mathcal{E}_1(t) \sim \frac{1}{\gamma^3 \sqrt{\pi}} n^{\frac{1}{2}} \rho^{-n}$$

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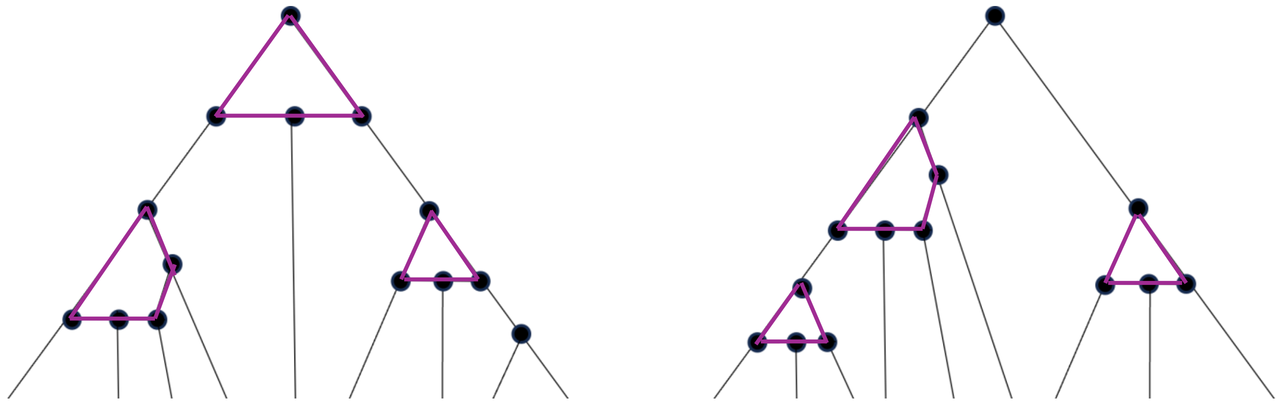
Goal

Asymptotic enumeration of unlabeled galled trees
with a fixed number of galls



Lecture Outline

Introduction – phylogenetic trees

Definition – galled trees

Previous work – the enumeration of galled trees and galled trees with one gall

Novel work – the enumeration of galled trees with a fixed number of galls

Summary

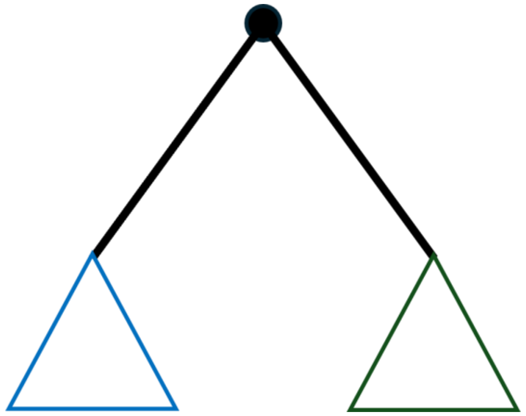
Recursion

No root gall

With a root gall

Recursion

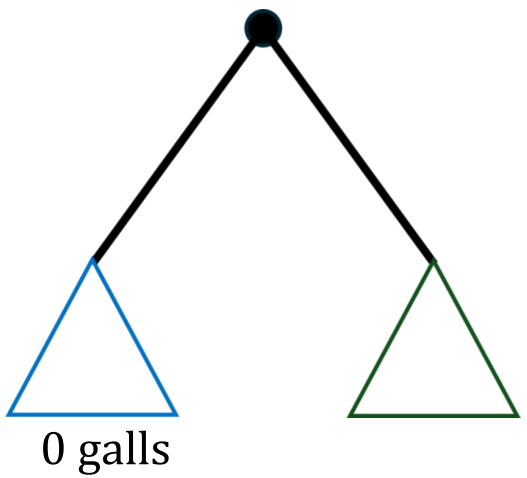
No root gall



With a root gall

Recursion

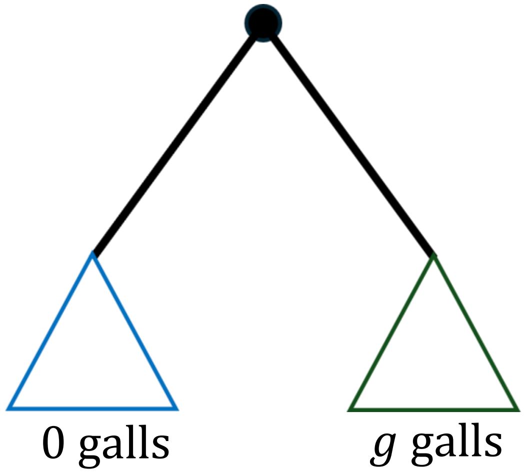
No root gall



With a root gall

Recursion

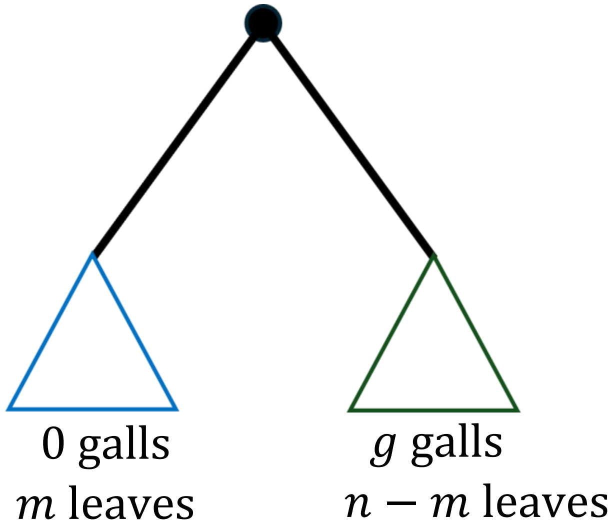
No root gall



With a root gall

Recursion

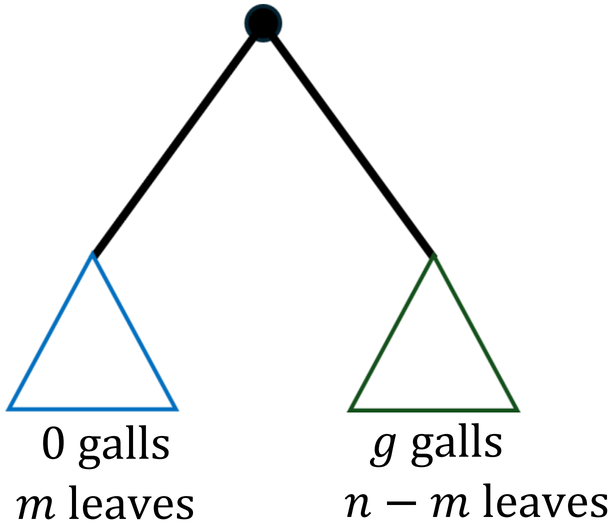
No root gall

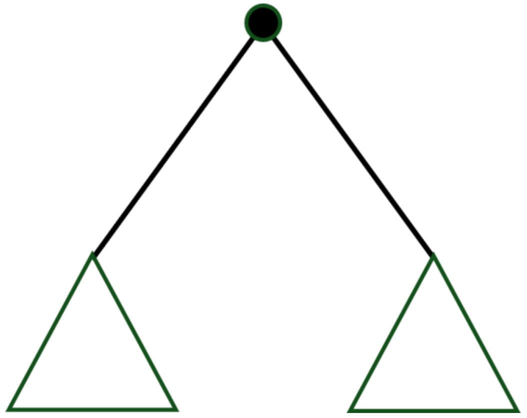


With a root gall

Recursion

No root gall

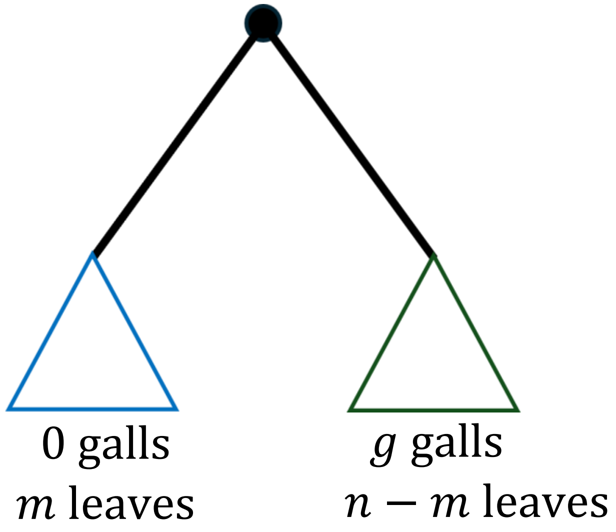


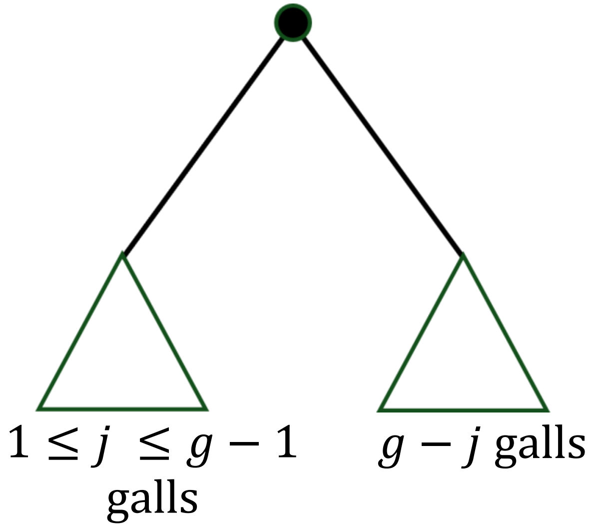


With a root gall

Recursion

No root gall

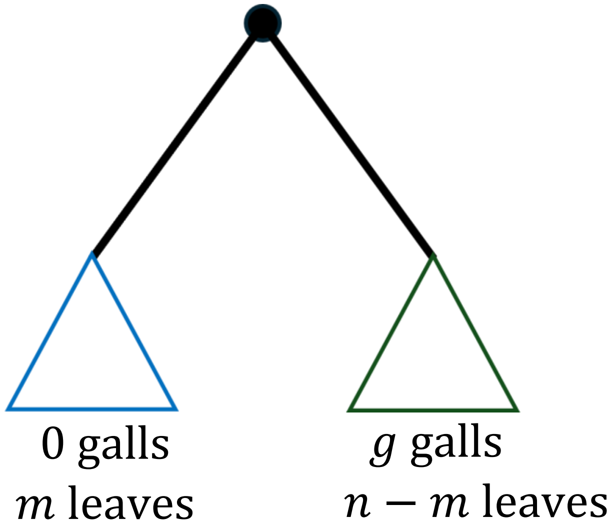


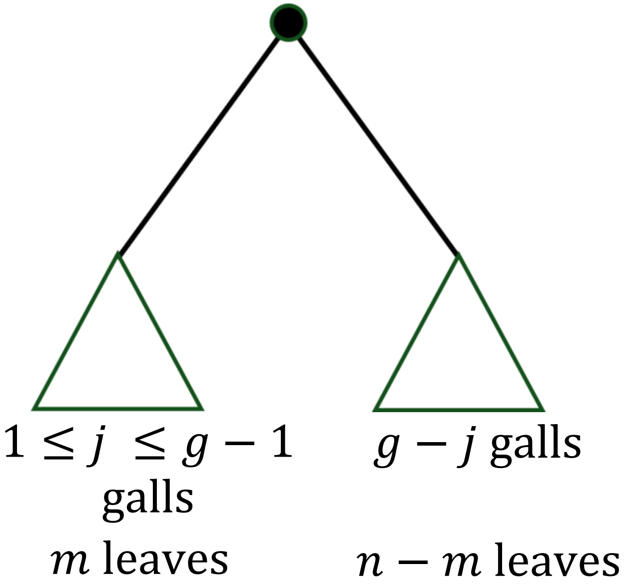


With a root gall

Recursion

No root gall

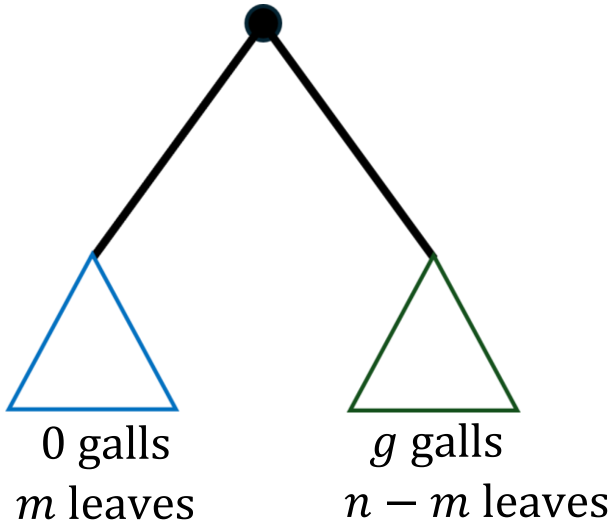


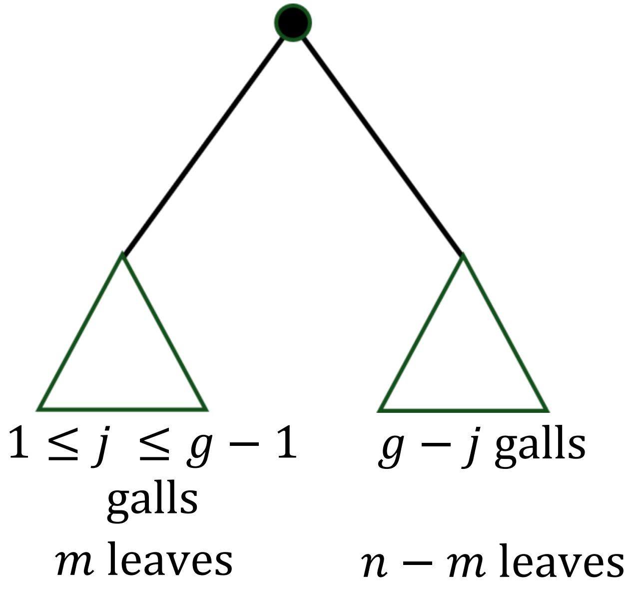


With a root gall

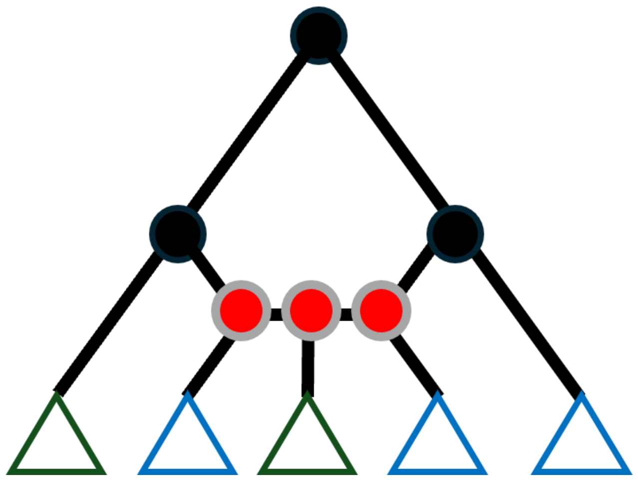
Recursion

No root gall



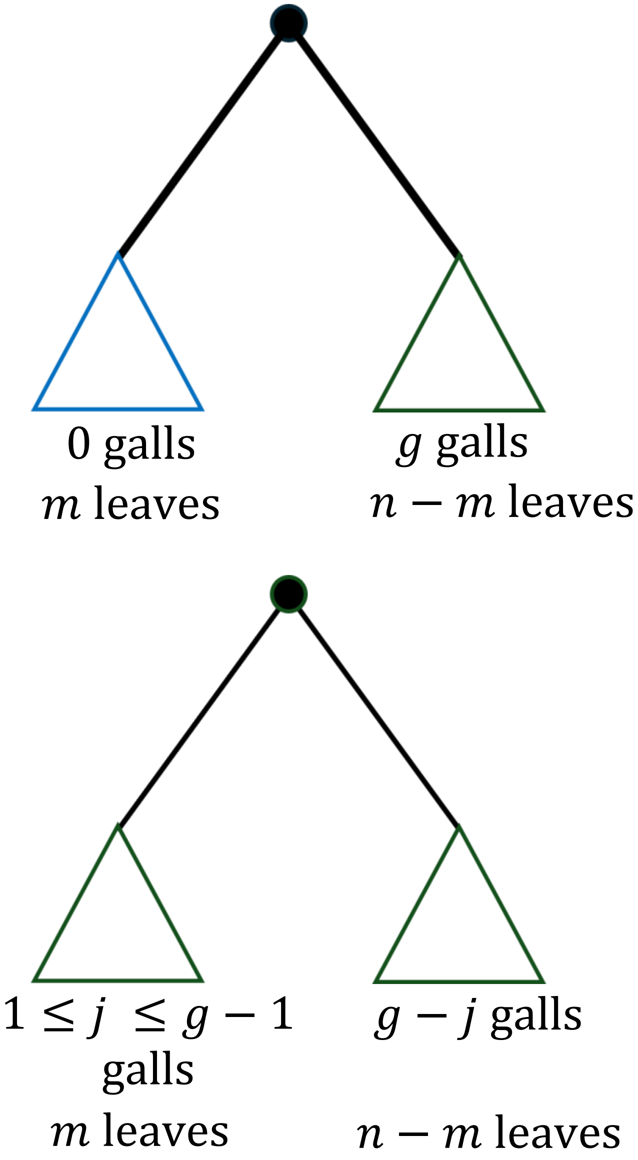


With a root gall



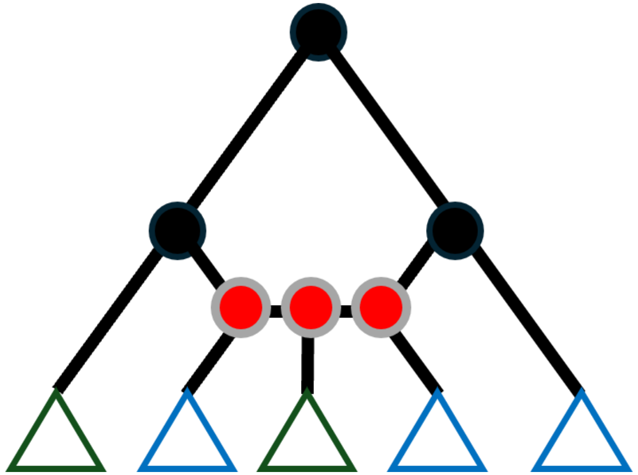
Recursion

No root gall



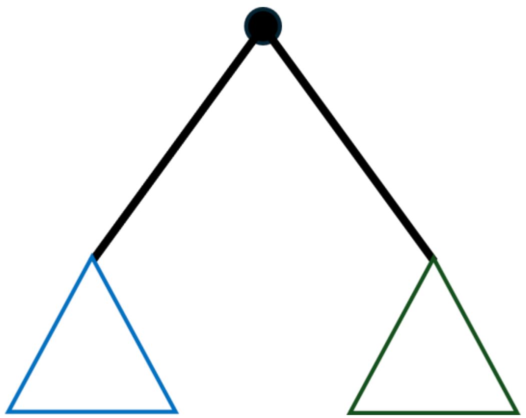
With a root gall

$3 \leq k \leq n$ main subtrees



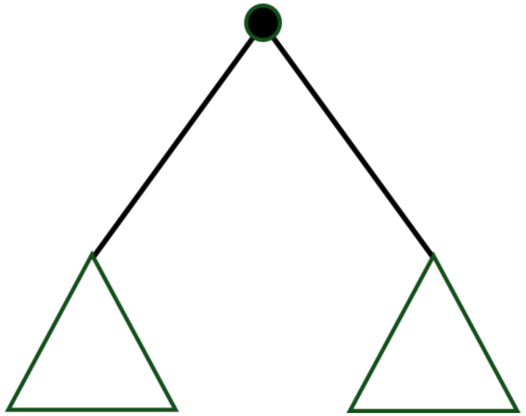
Recursion

No root gall



0 galls
m leaves

g galls
$n-m$ leaves

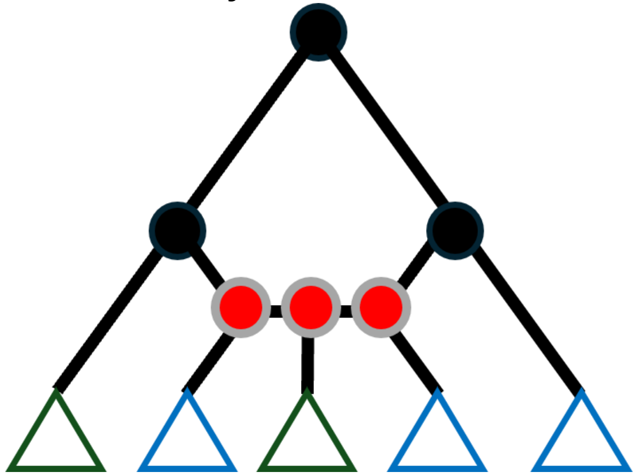


$1 \leq j \leq g-1$
galls
m leaves

$g-j$ galls
$n-m$ leaves

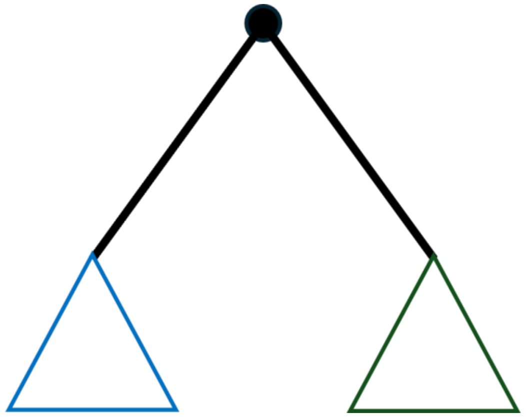
With a root gall

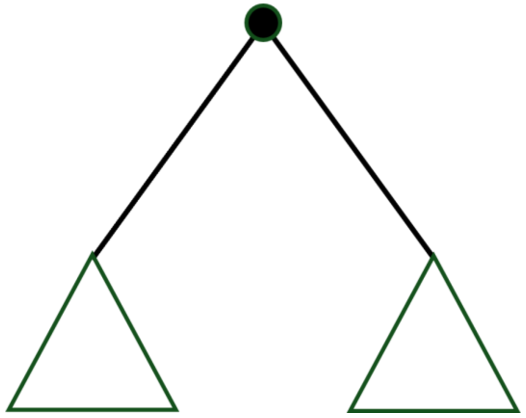
$3 \leq k \leq n$ main subtrees
$k-2$ possibilites
to place the
hybrid node



Recursion

No root gall





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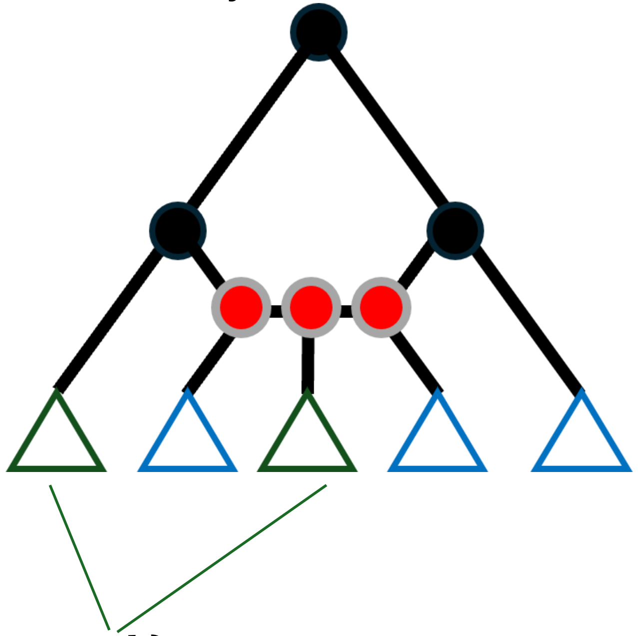

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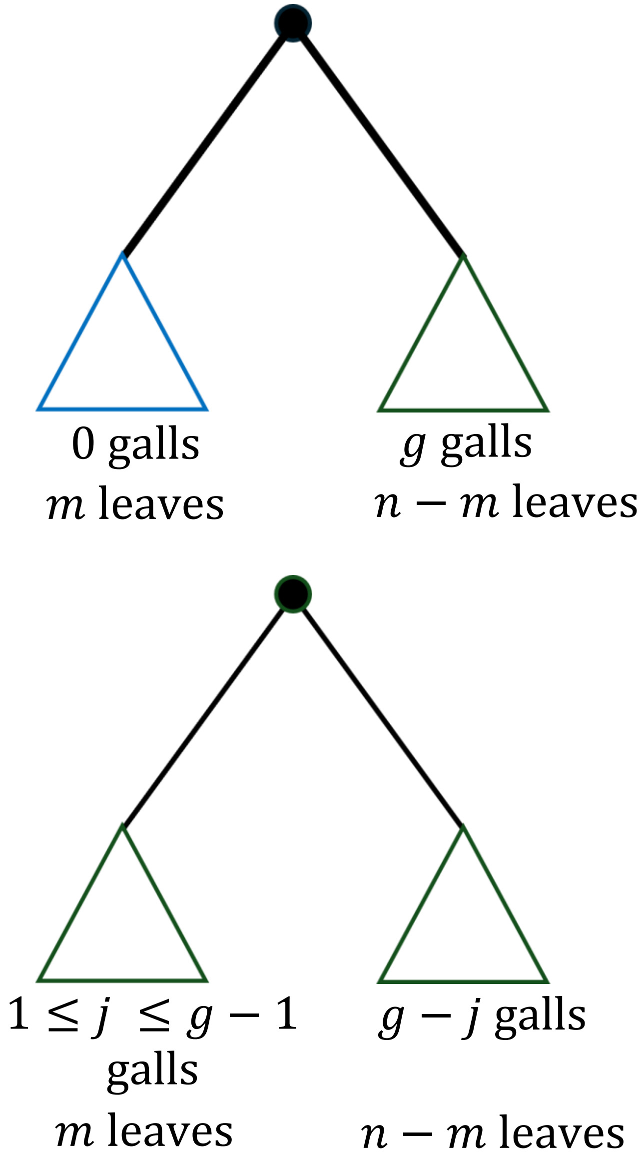
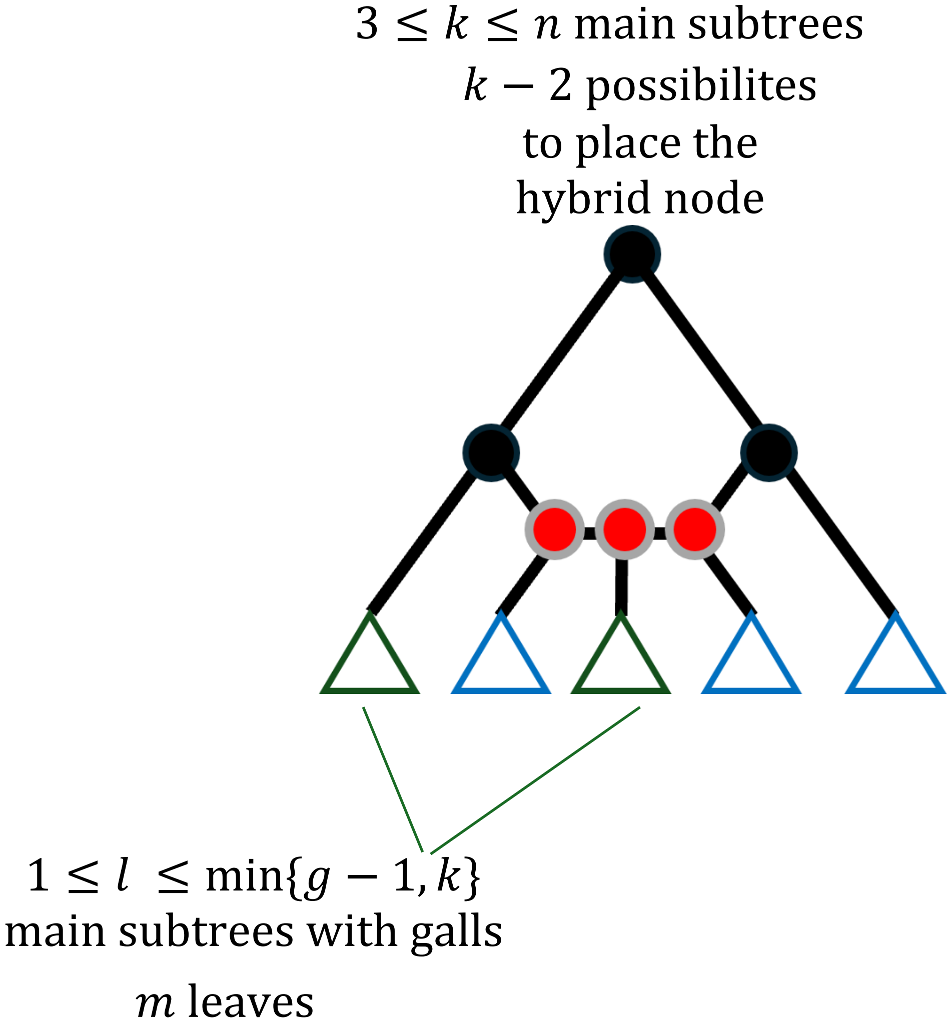


With a root gall



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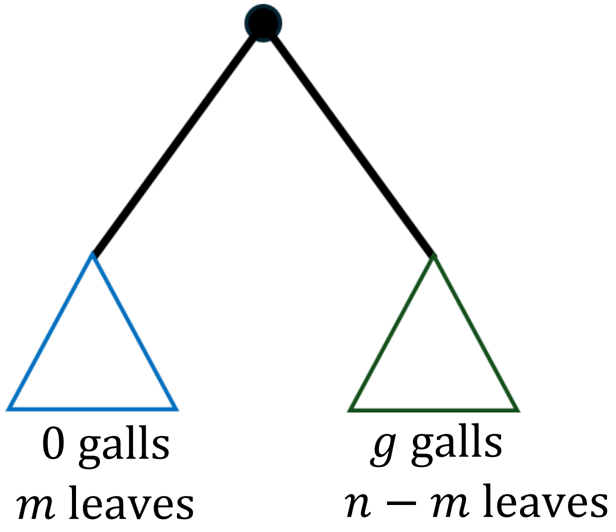
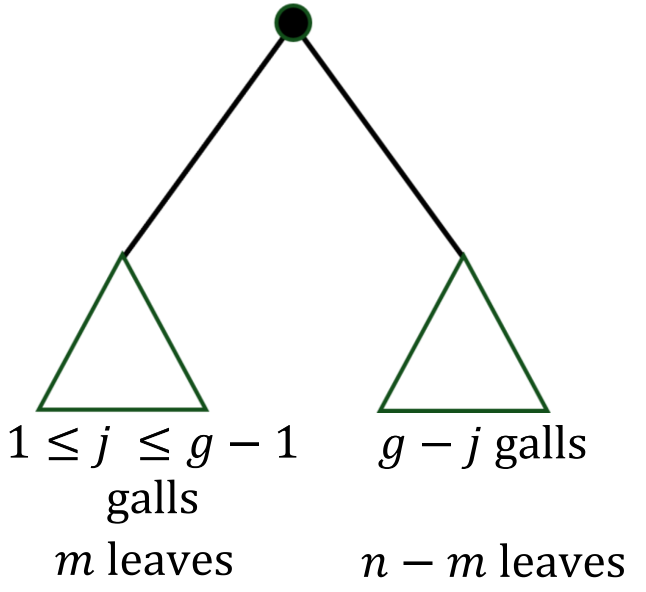
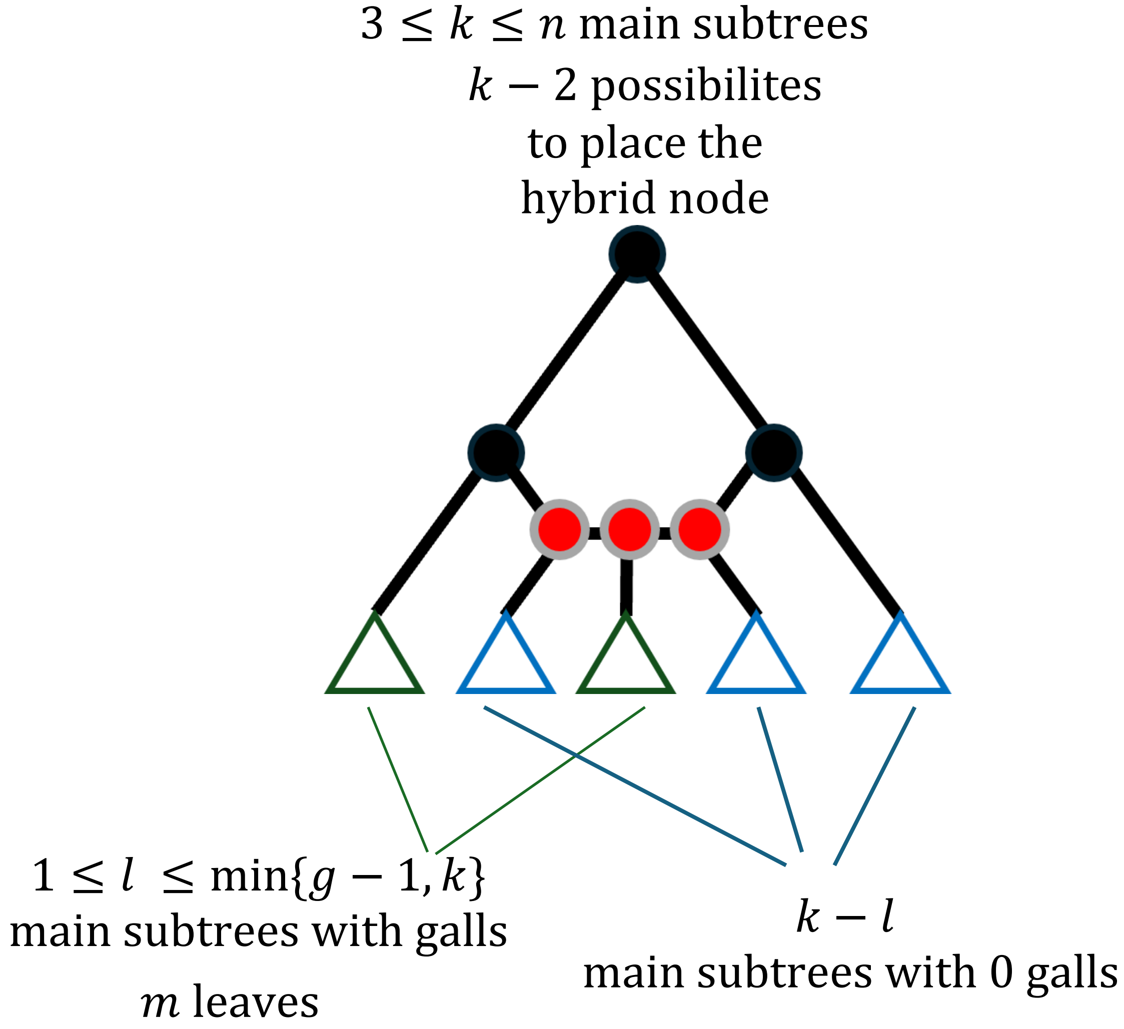




With a root gall



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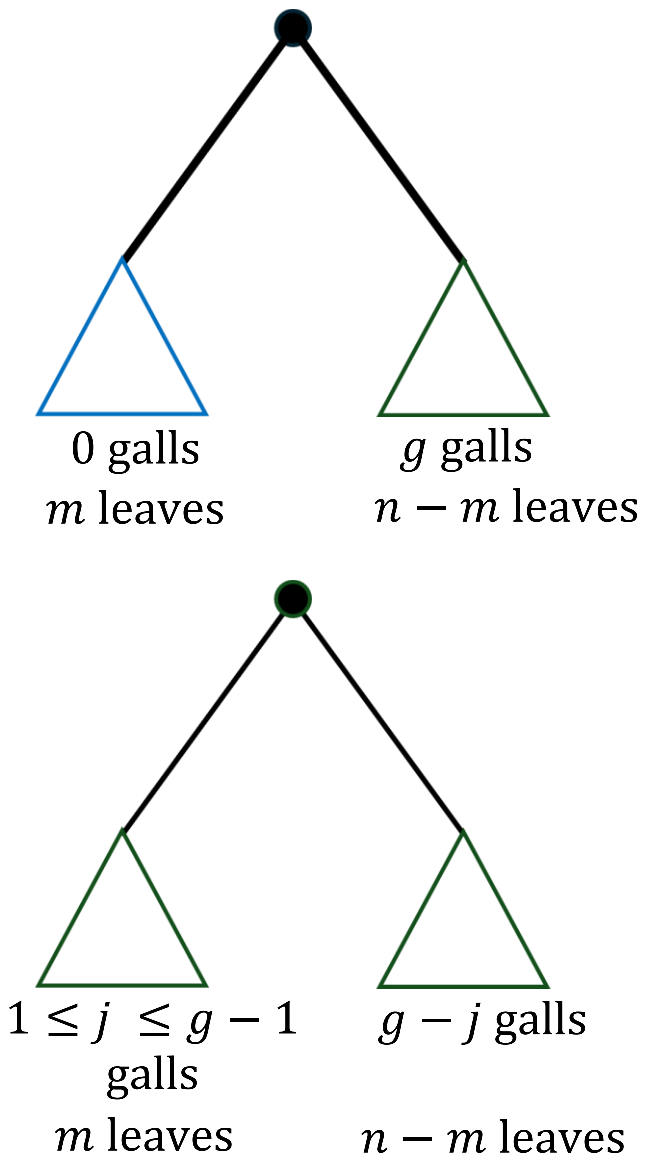

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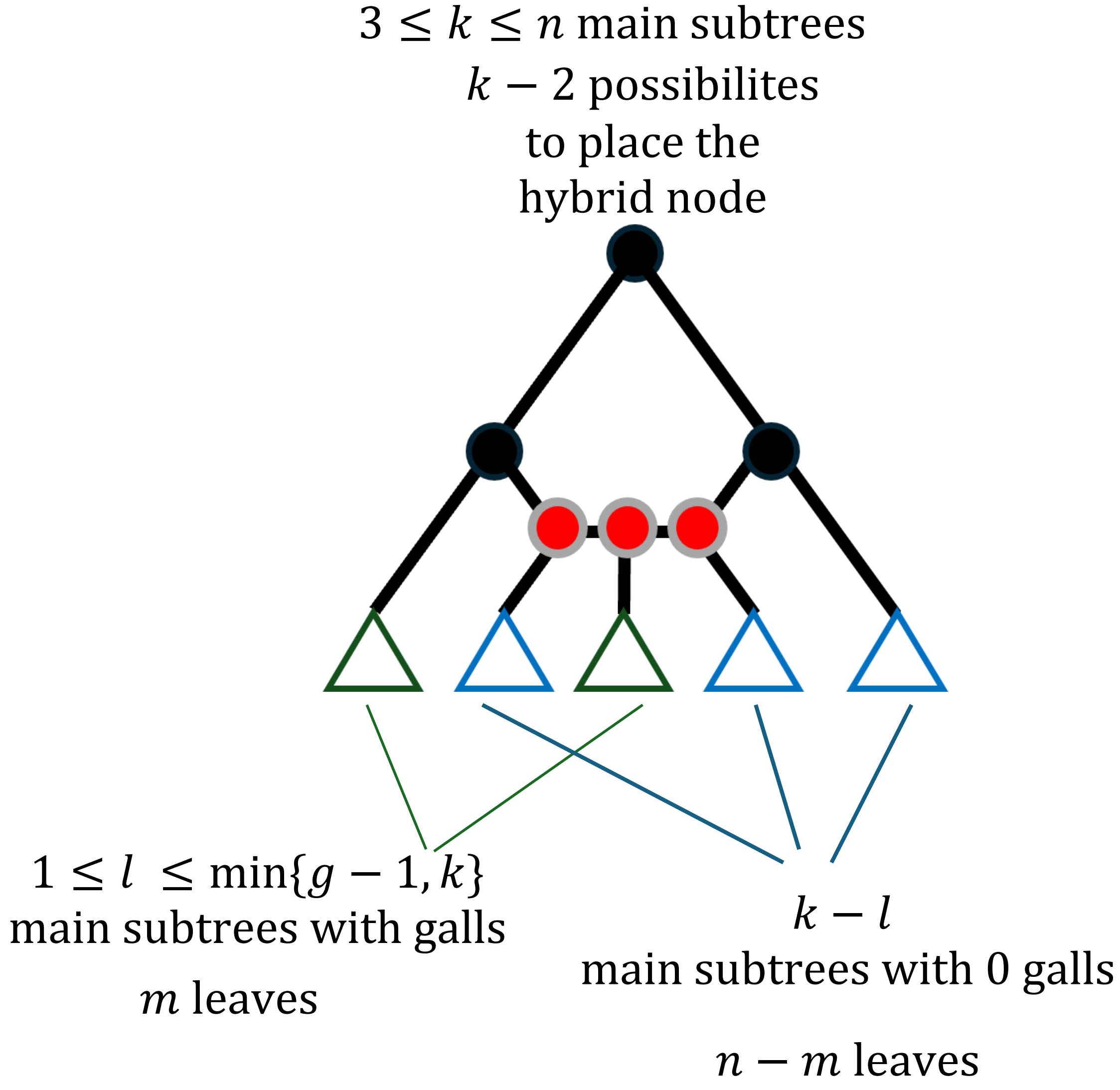

Asymptotic Analysis

Asymptotic Analysis

$$\mathcal{E}_1(t) = \frac{1}{1-\mathcal{U}(t)} - \frac{1}{[1-\mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1-\mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1-\mathcal{U}(t)][1-\mathcal{U}(t^2)]}$$

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$E_{n,g}$ − The number of galled trees with g galls and n leaves

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This includes $g = 0$

$$E_{n,g} \sim \frac{2^{0-1}}{(0)!\gamma^{0-1}\sqrt{\pi}} n^{0-\frac{3}{2}} \rho^{-n} = \frac{\gamma}{2\sqrt{\pi}} n^{-\frac{3}{2}} \rho^{-n} \sim [t^n]\mathcal{U}(t)$$

Asymptotic Analysis

The subexponential portion $\frac{2^{2g-1}}{(2g)^{!}\gamma^{4g-1}\sqrt{\pi}}n^{2g-3/2} = c_g n^{2g-3/2}$

Number of galls g	Exact value of c_g	Approximate value of c_g	$n^{2g-3/2}$
0	$\frac{\gamma}{2\sqrt{\pi}}$	0.3188	$n^{-\frac{3}{2}}$
1	$\frac{1}{\gamma^3\sqrt{\pi}}$	0.3910	$n^{\frac{1}{2}}$
2	$\frac{1}{3\gamma^7\sqrt{\pi}}$	0.0799	$n^{\frac{5}{2}}$
3	$\frac{2}{45\gamma^{11}\sqrt{\pi}}$	0.0065	$n^{\frac{9}{2}}$
4	$\frac{1}{315\gamma^{15}\sqrt{\pi}}$	2.8638×10^{-4}	$n^{\frac{13}{2}}$
5	$\frac{2}{14175\gamma^{19}\sqrt{\pi}}$	7.8062×10^{-6}	$n^{\frac{17}{2}}$

Summary

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We show an explicit formula for the asymptotic enumeration of unlabeled binary non-plane galled trees with exactly g galls. This includes unlabeled binary non-plane trees with no galls.

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However, the subexponential growth, with the increase in the number of galls by 1, is greater by a factor of $\frac{4n^2}{\gamma^4(2g+1)(2g+2)}$.

Thank You

Acknowledgements

Noah Rosenberg

Michael Fuchs National Chengchi University

Bernhard Gittenberger Technische Universität Wien

L to R: Egor Lappo, Chloe Shiff, Xiran Liu, Noah Rosenberg, Kaleda Denton, Maike Morrison, Lily Agranat-Tamir
not pictured: Daniel Cotter, Juan Esteban Rodriguez Rodriguez, Kennedy Agwamba, Zarif Ahsan, Emily Dickey, Bradley Moon, Michael Doboli, Anna Lyubarskaja, Daniel Bauman

Stanford University

Asymptotics of unlabeled galled trees with a fixed number of galls

Lily Agranat-Tamir, Michael Fuchs, Bernhard Gittenberger and Noah Rosenberg
AofA 2024

Lecture Outline

Introduction – phylogenetic trees

Definition – galled trees

Previous work – the enumeration of galled trees and galled trees with one gall

Novel work – the enumeration of galled trees with a fixed number of galls

Summary

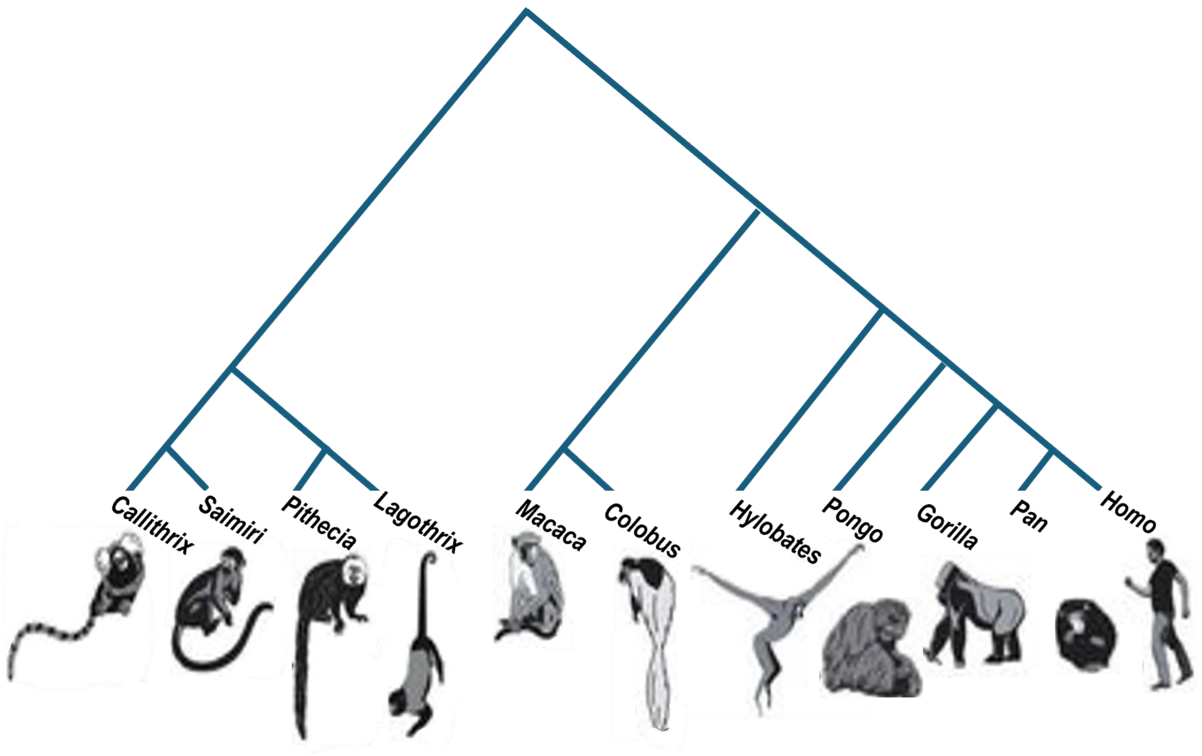
Phylogenetic Trees

Phylogenetic Trees

Species Trees

Phylogenetic Trees

Species Trees



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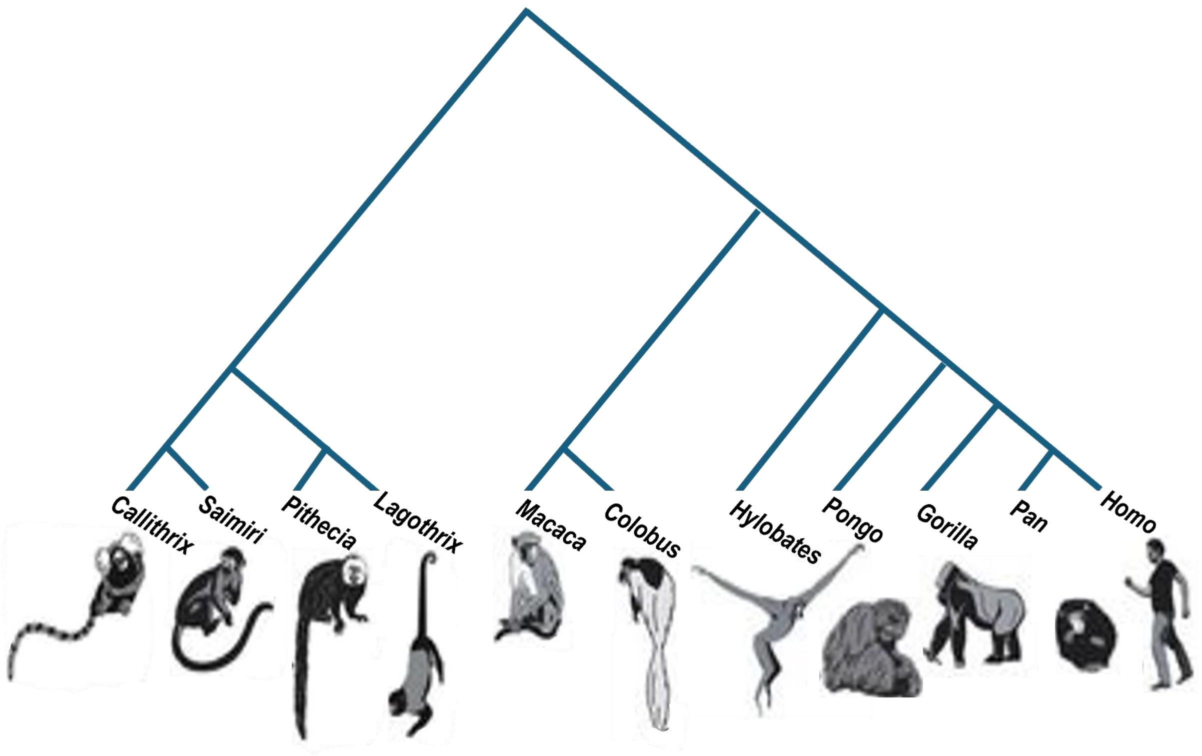

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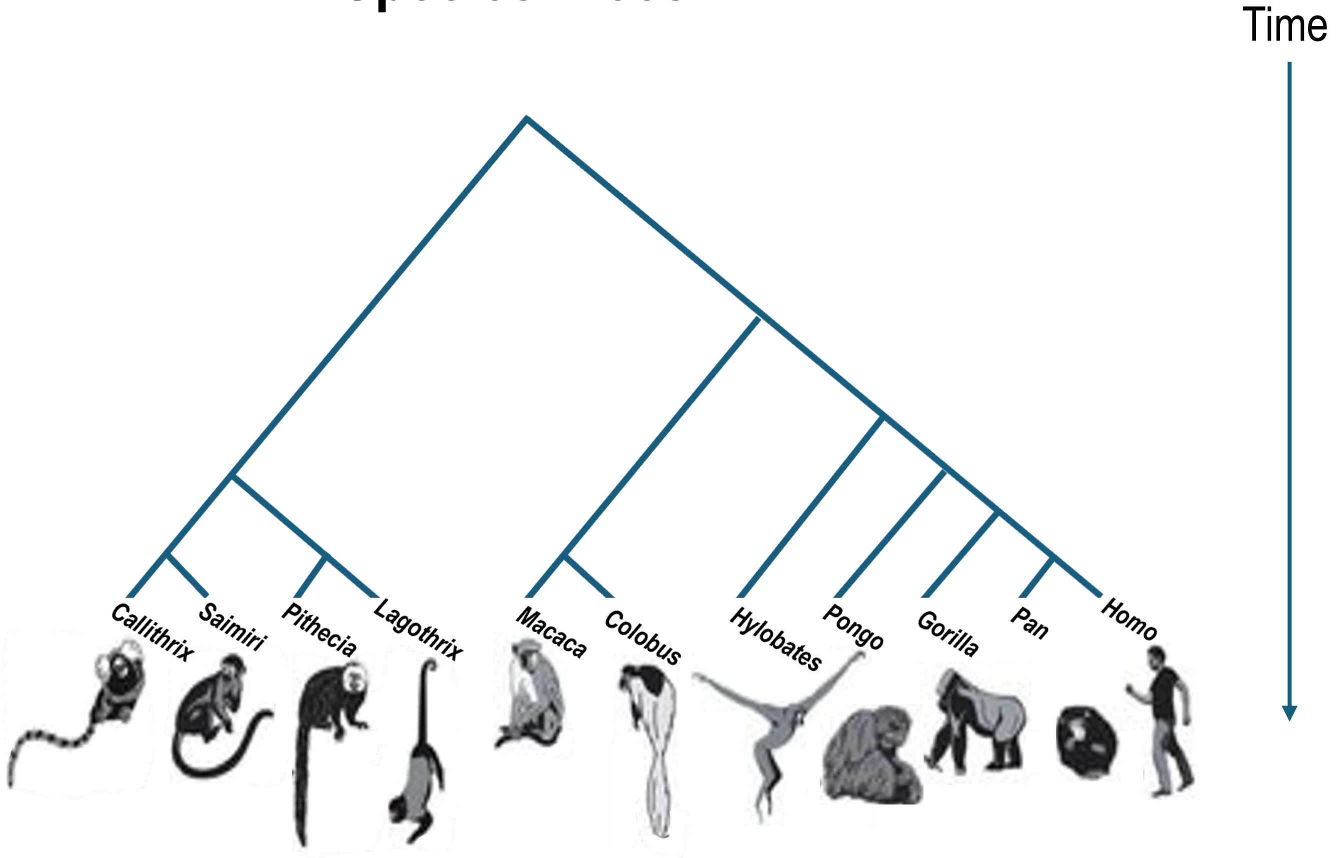

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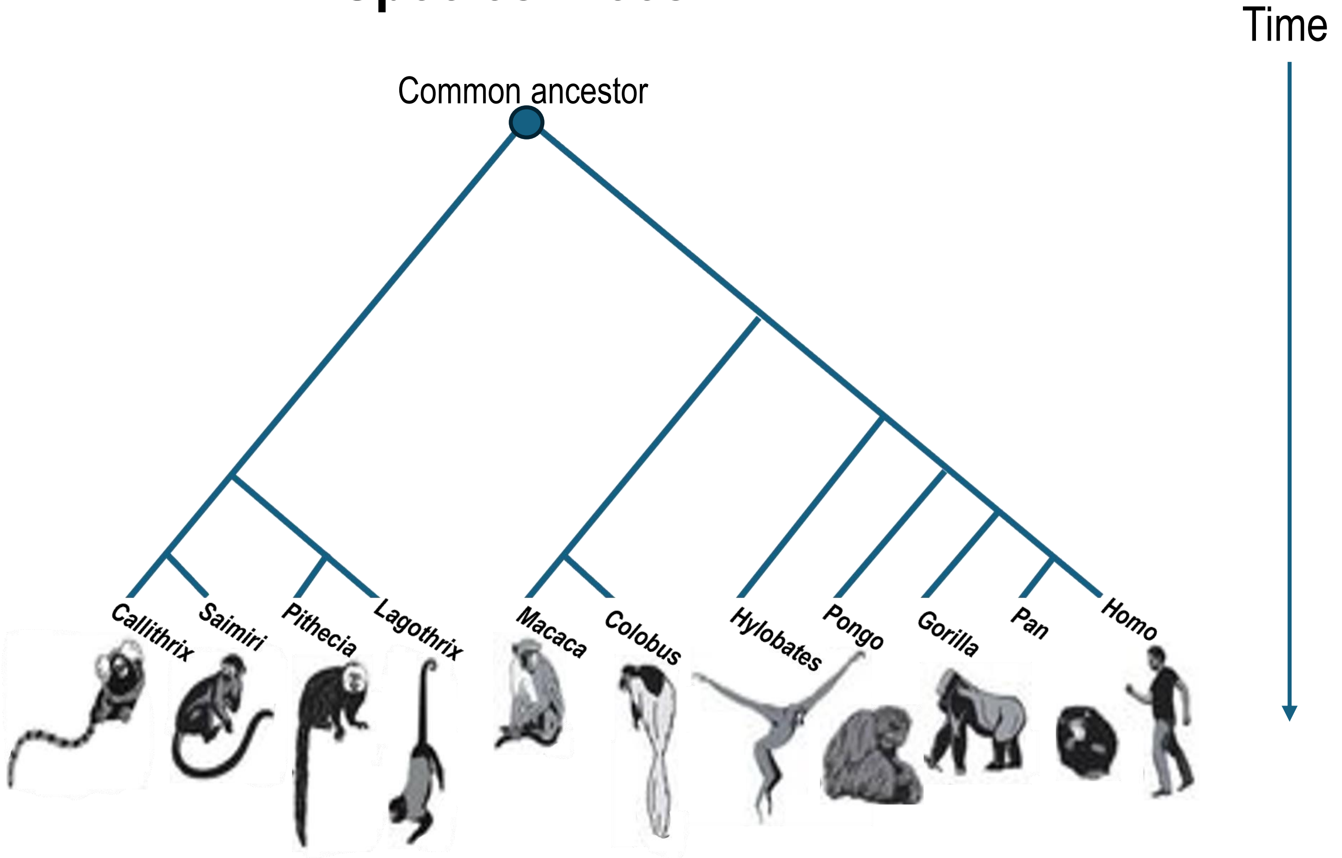

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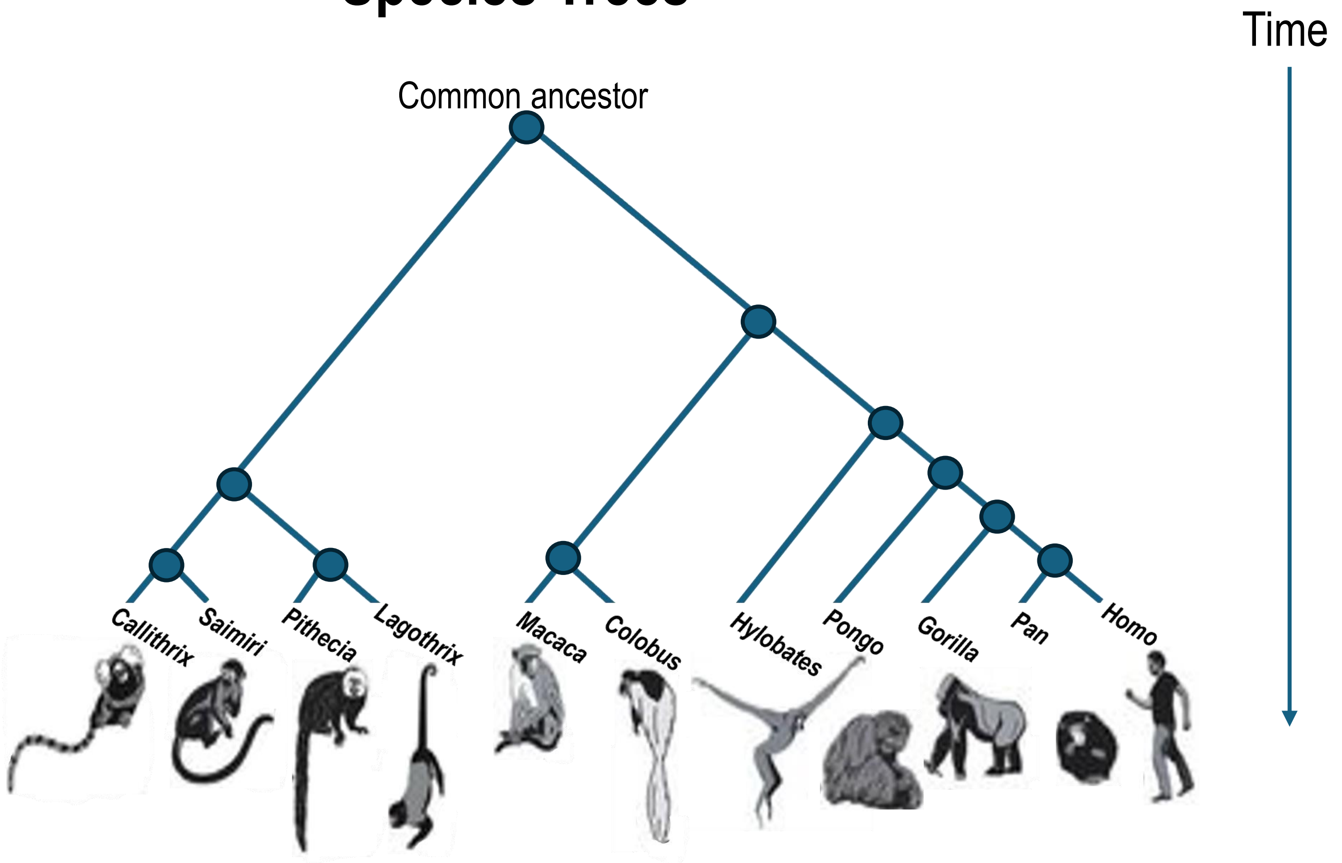

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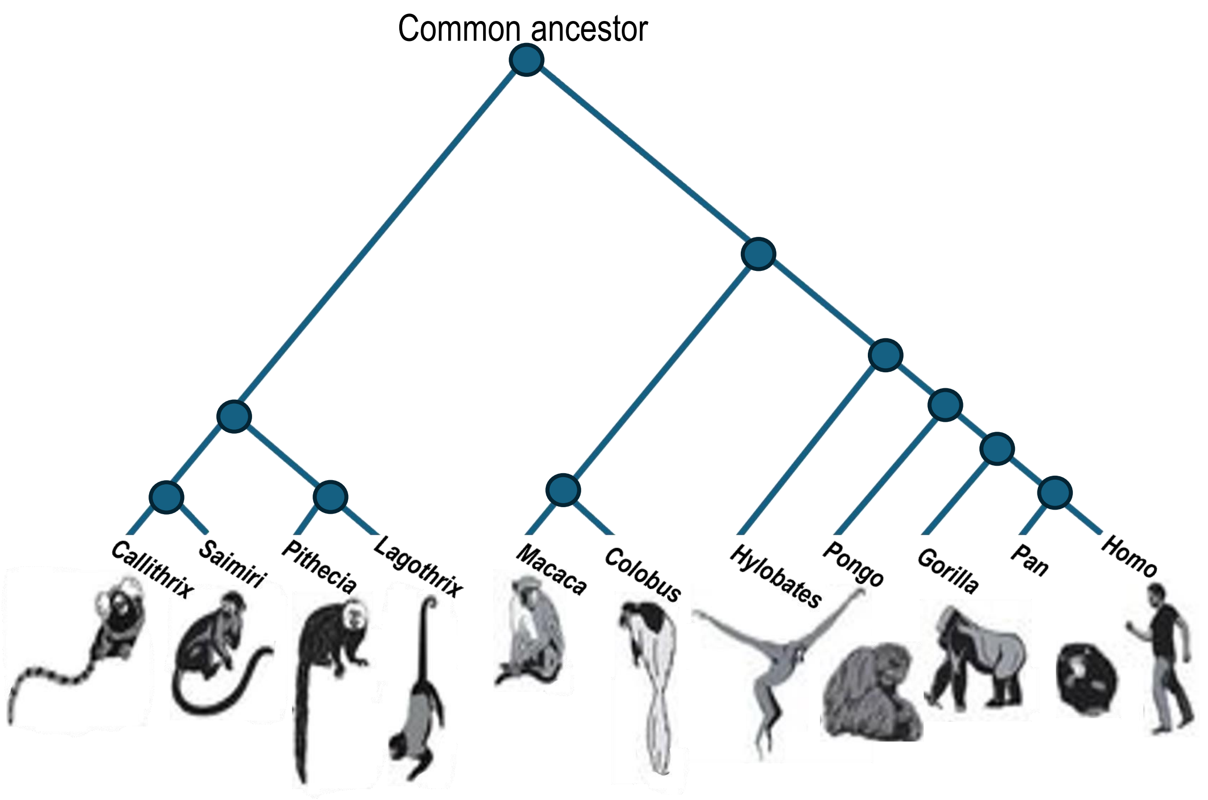

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Time

DNA based Trees

Phylogenetic Trees

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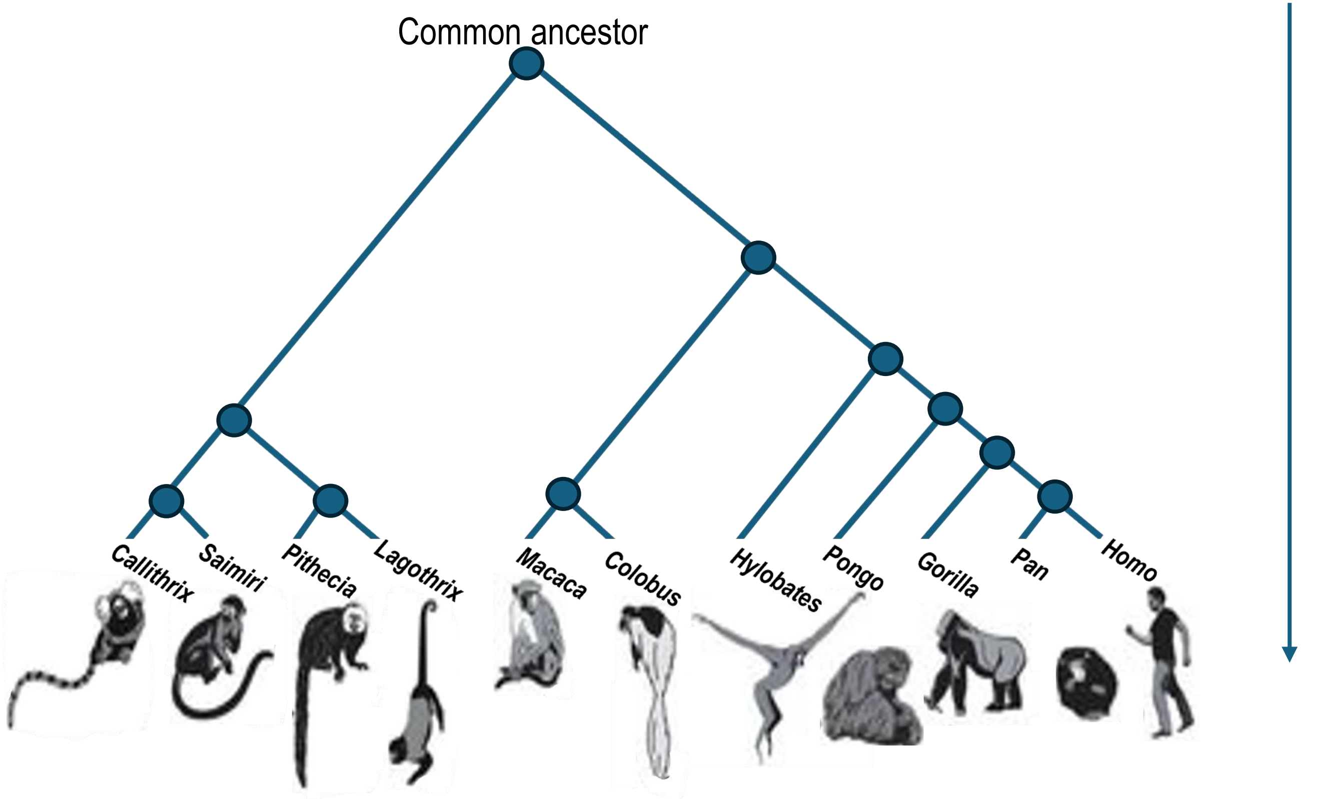

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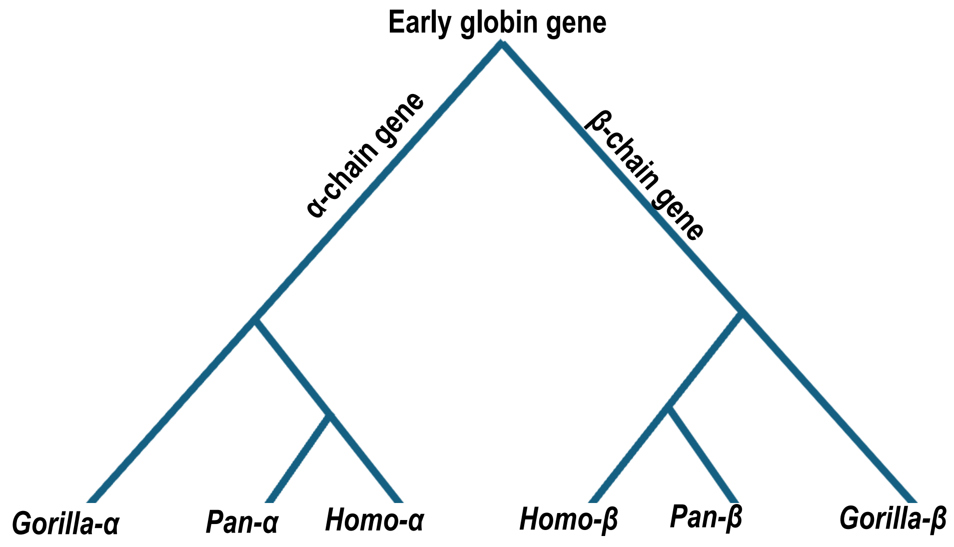

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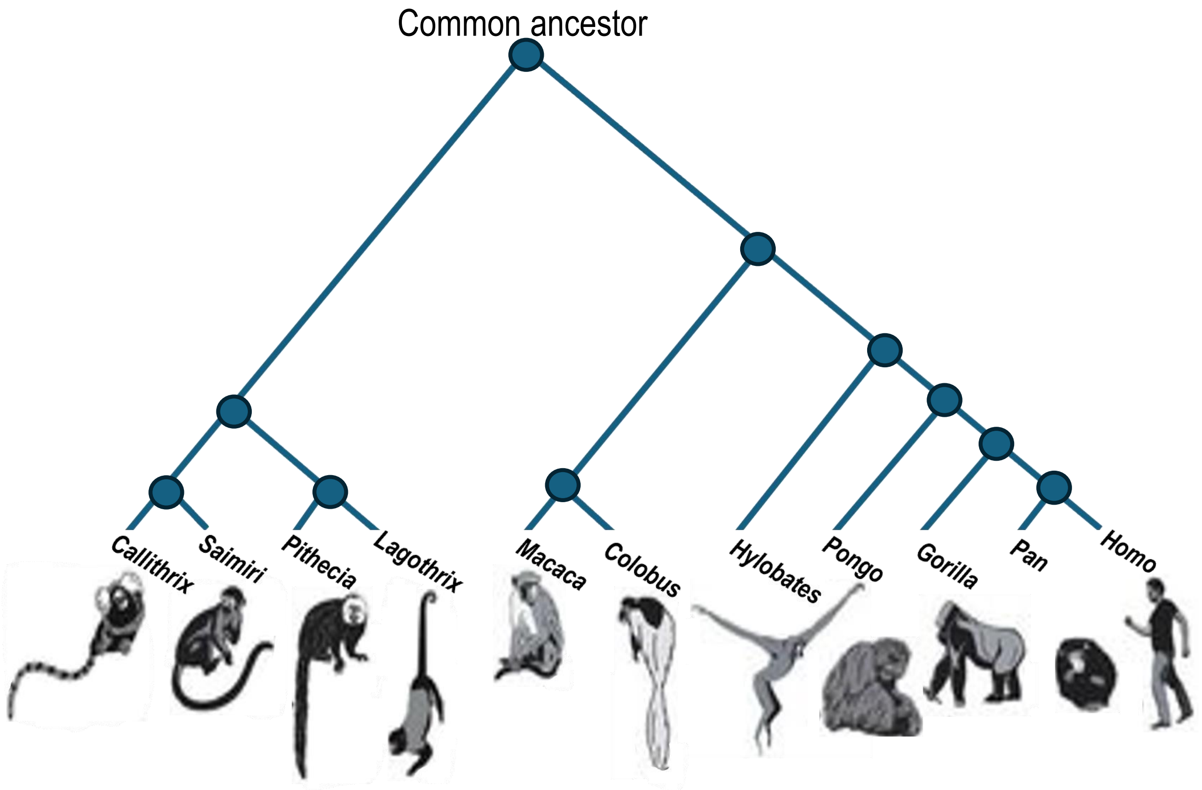

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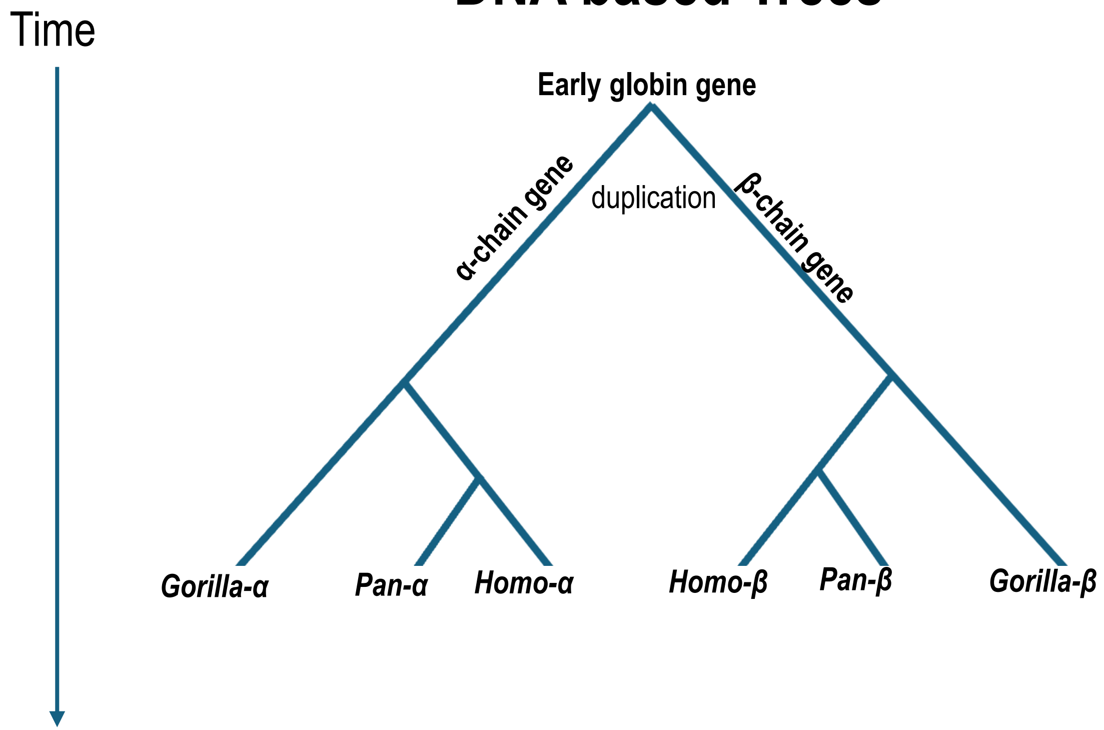

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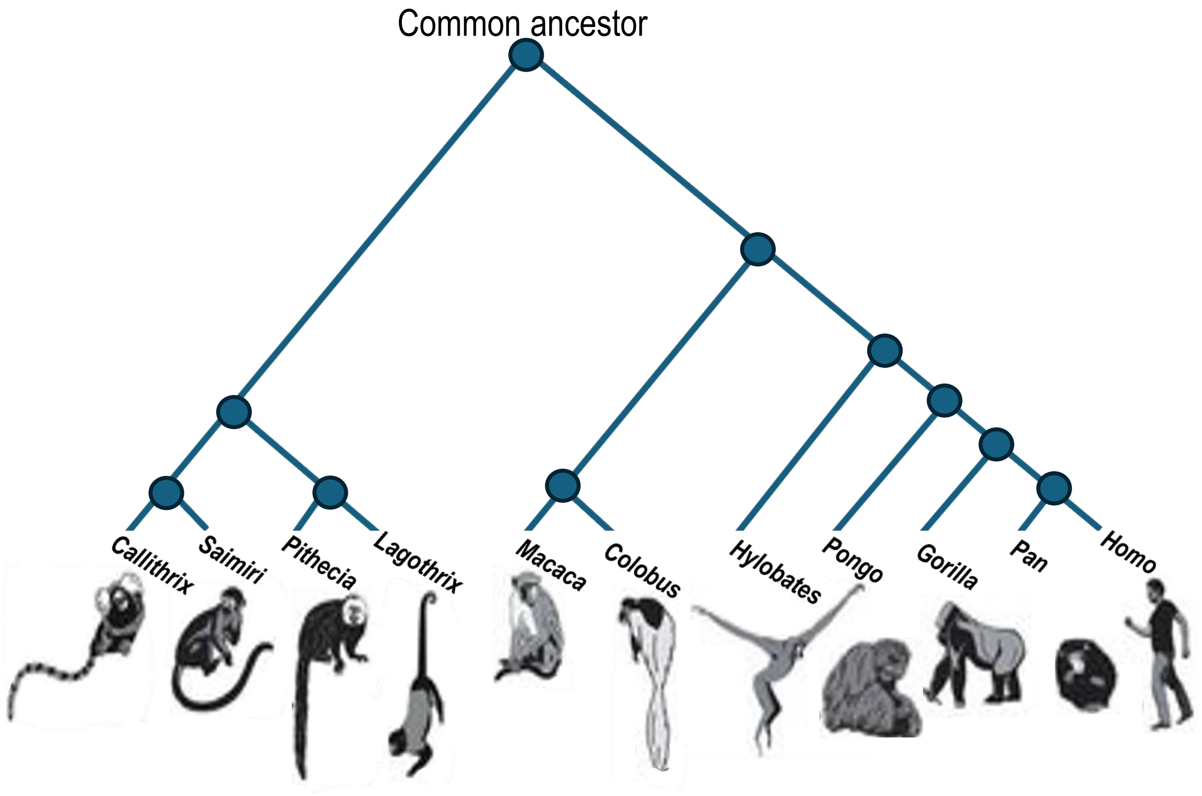

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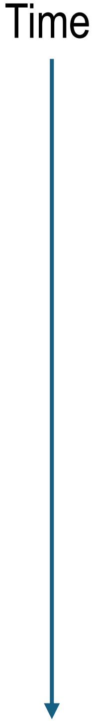

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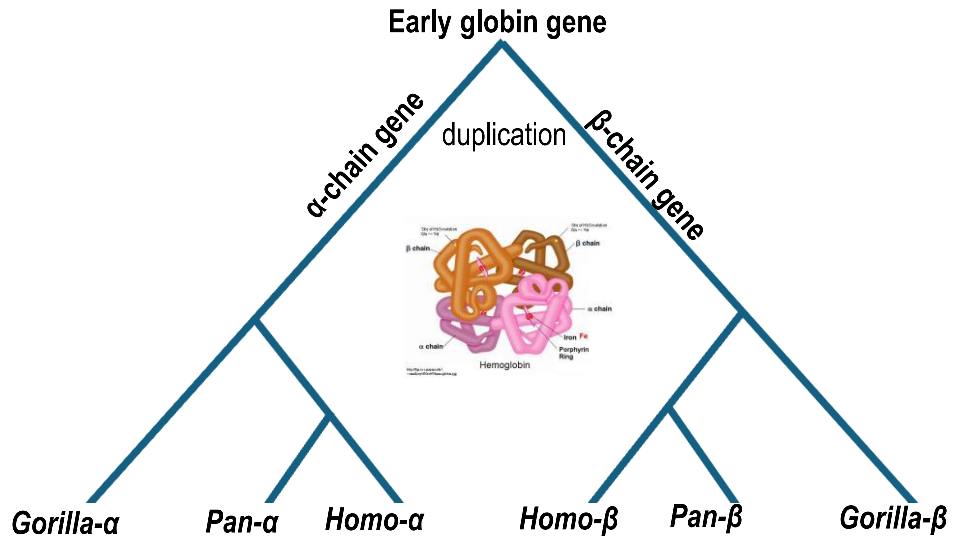

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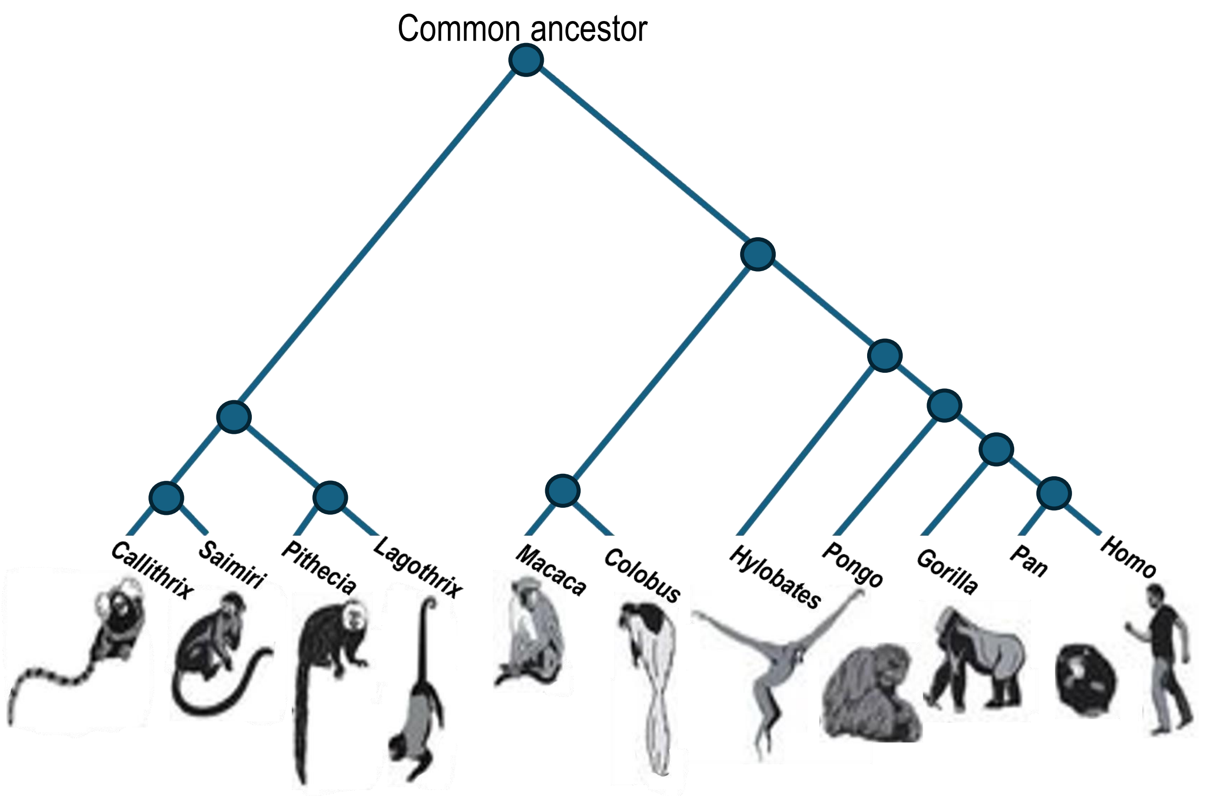

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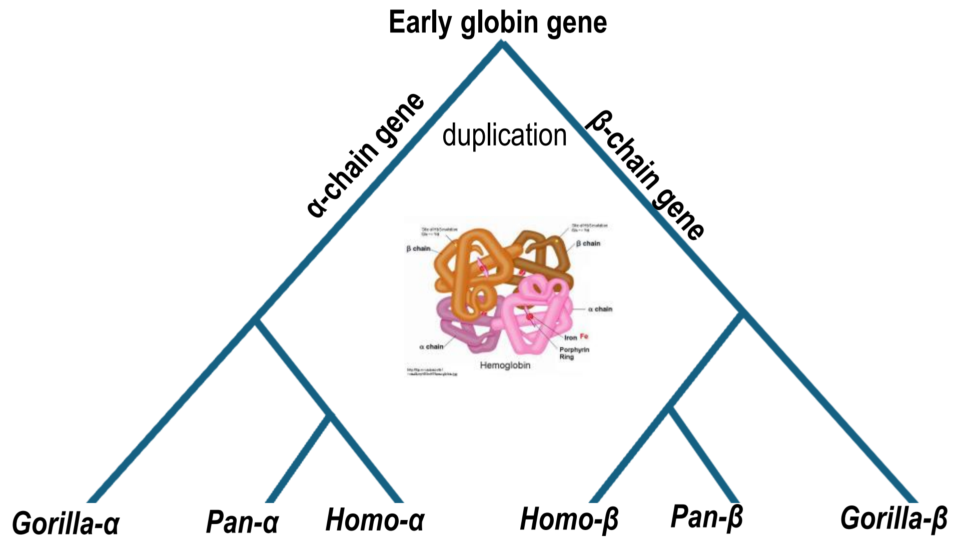

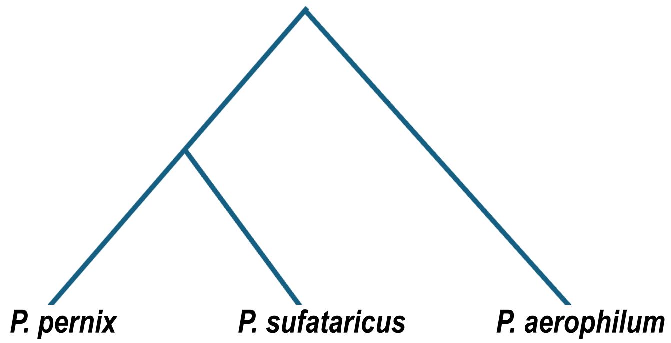

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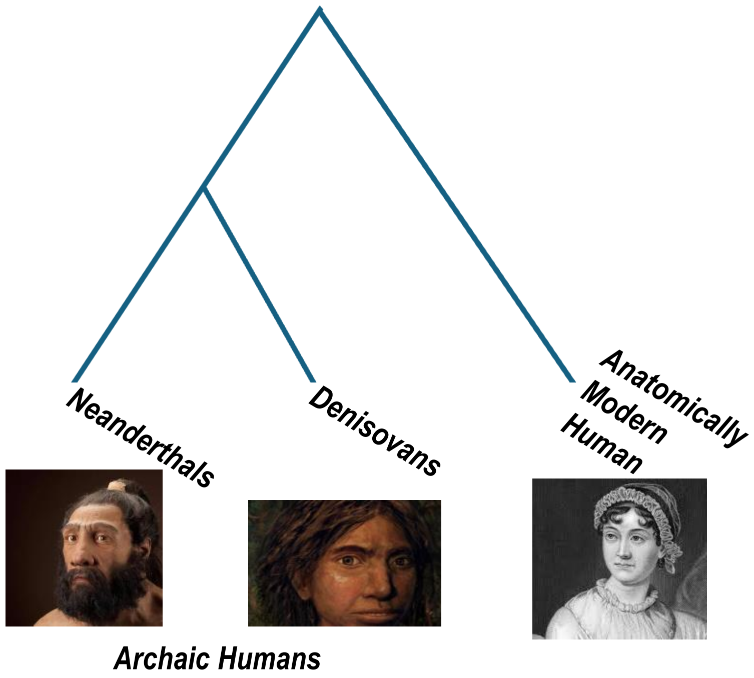

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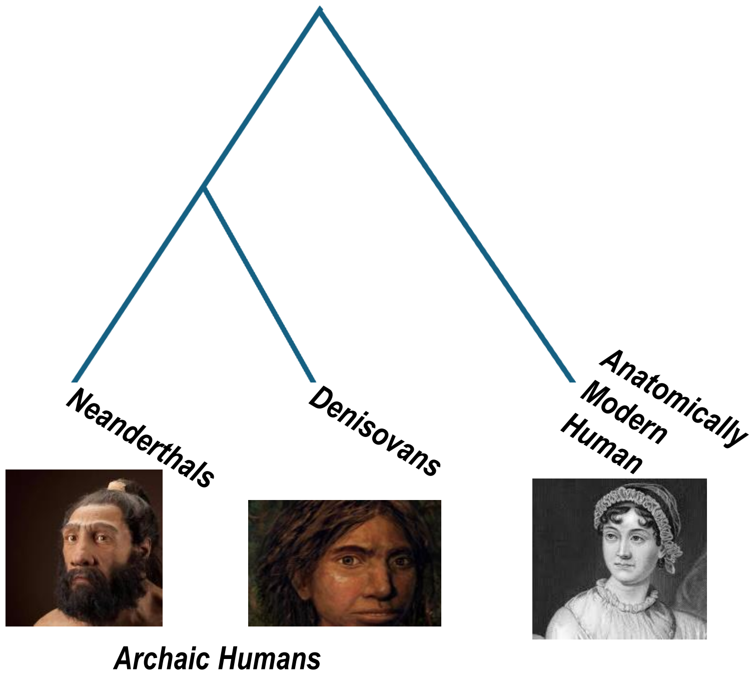

https://humanorigins.si.edu/evidence/human-fossils/species/homo-neanderthalensis

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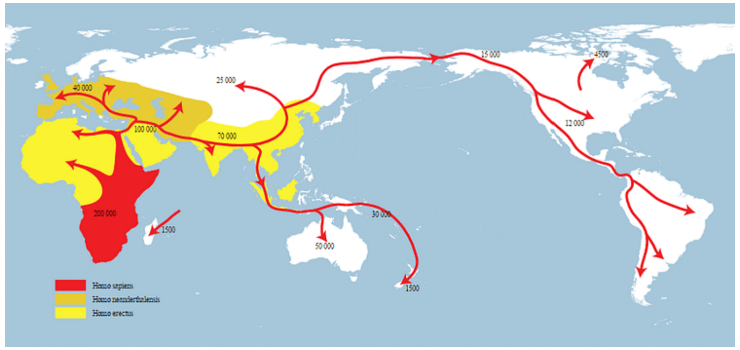

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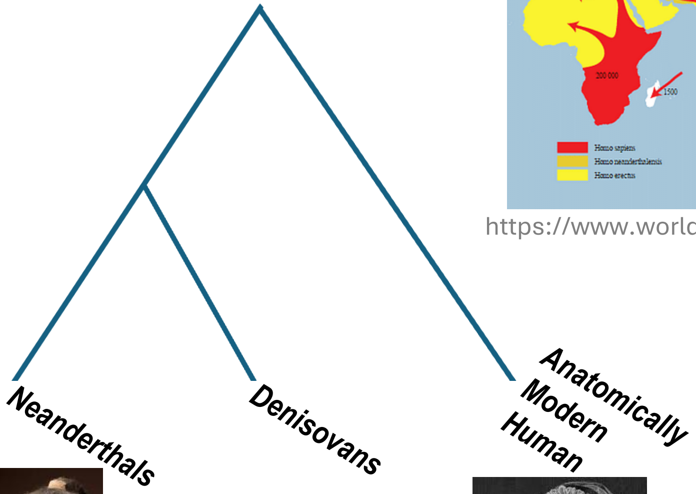

Archaic Humans

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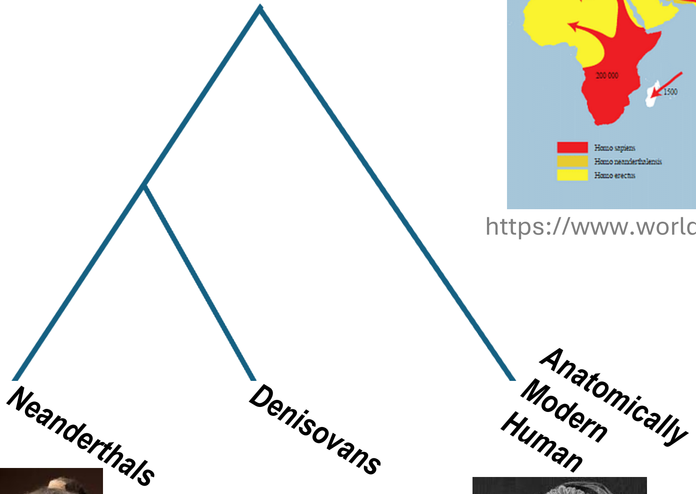

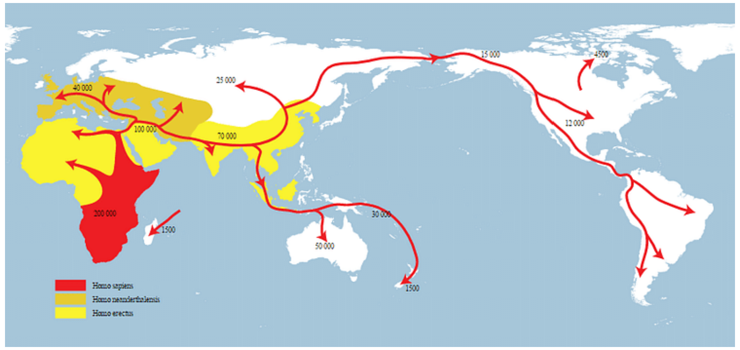

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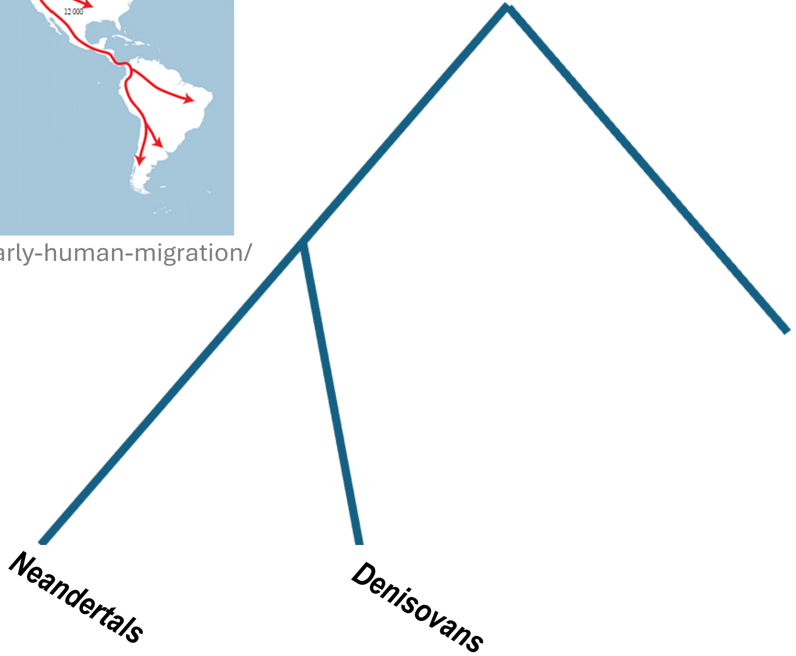
Archaic Humans

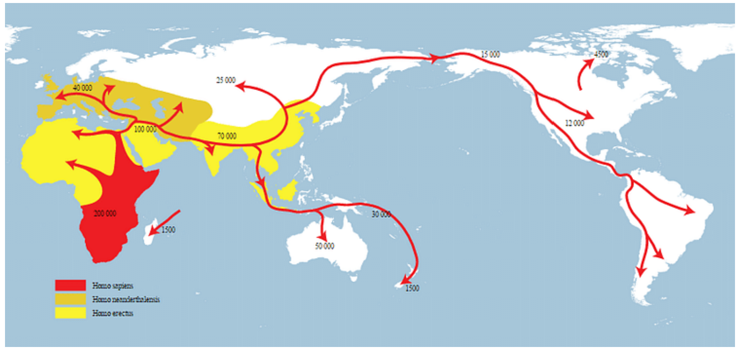


https://humanorigins.si.edu/evidence/human-fossils/species/homo-neanderthalensis

https://www.sciencefocus.com/science/denisovans

https://www.britannica.com/biography/Jane-Austen

Phylogenetic Trees



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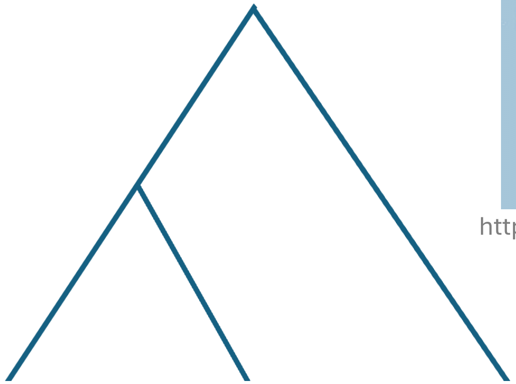

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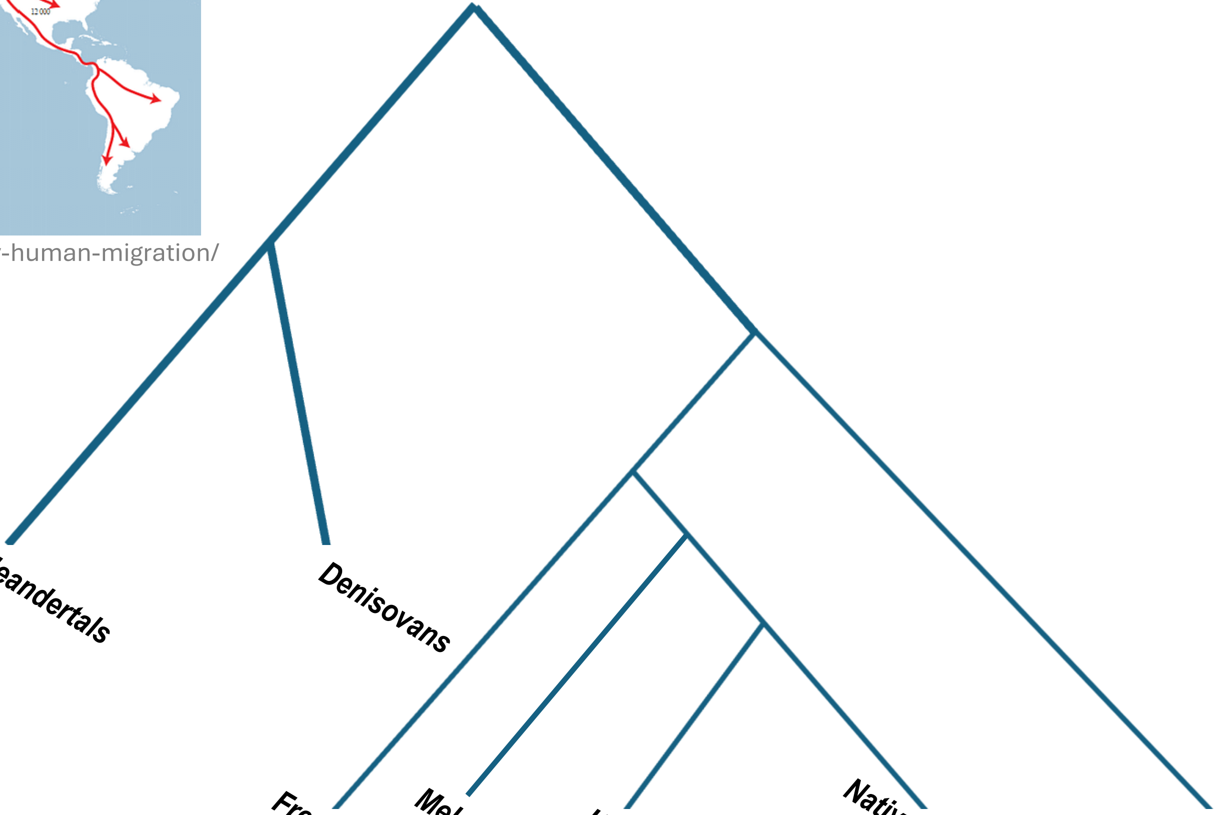

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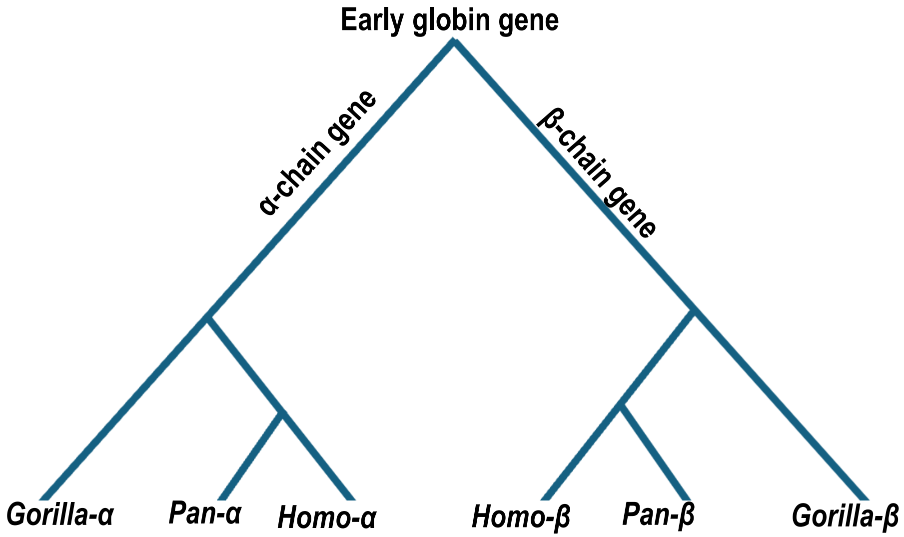

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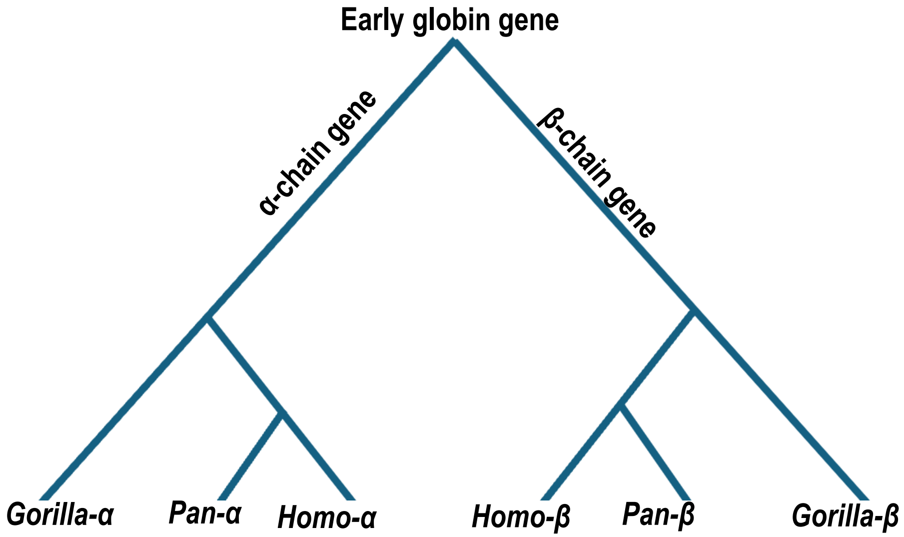
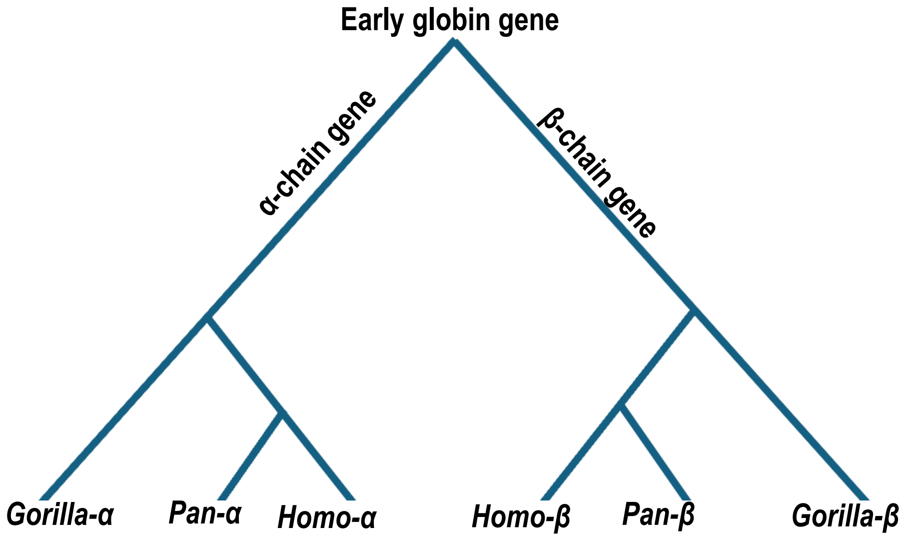
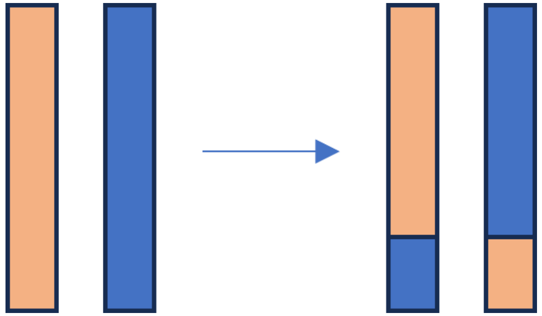


Recombination

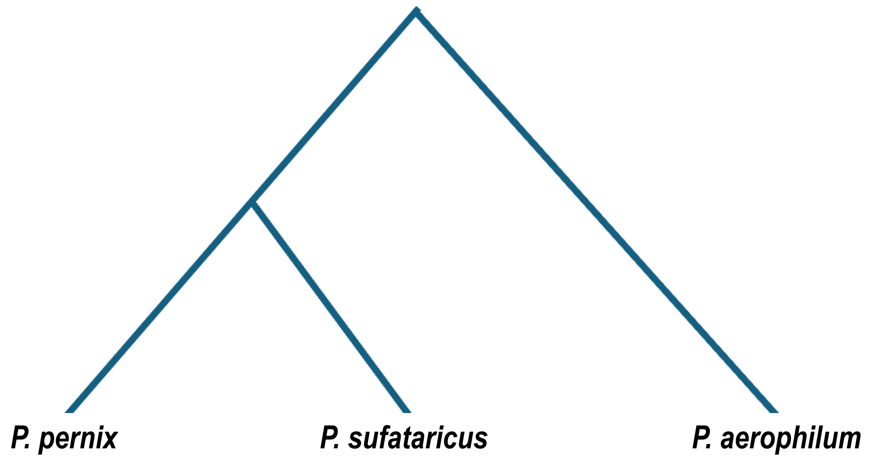
Phylogenetic Trees



Recombination

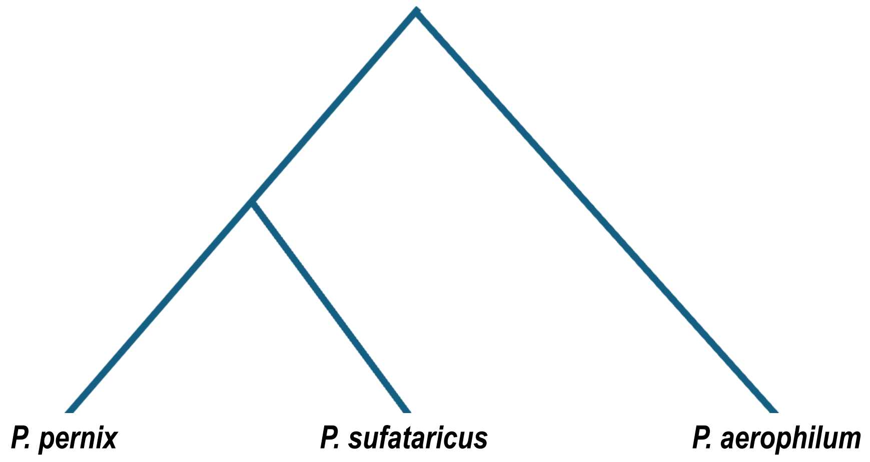


Phylogenetic Trees

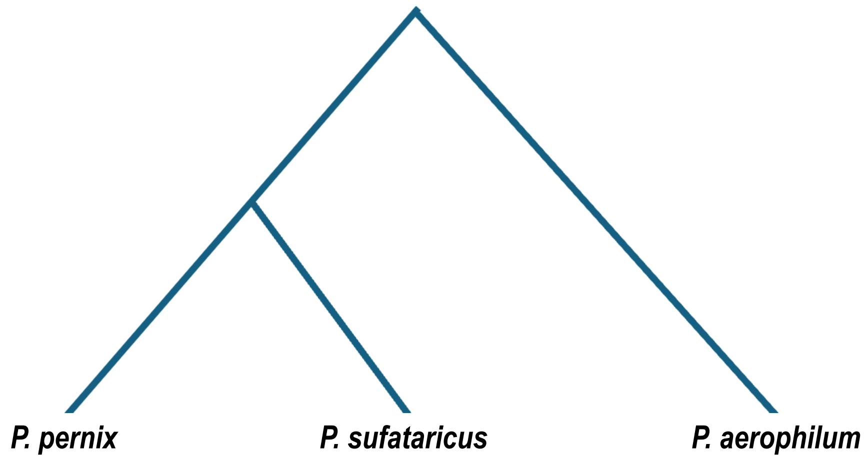


Phylogenetic Trees

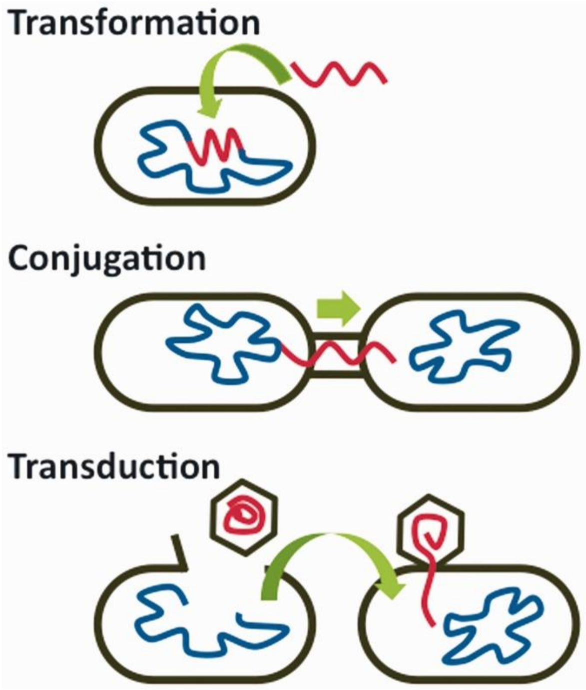
Horizontal Gene Transfer



Phylogenetic Trees



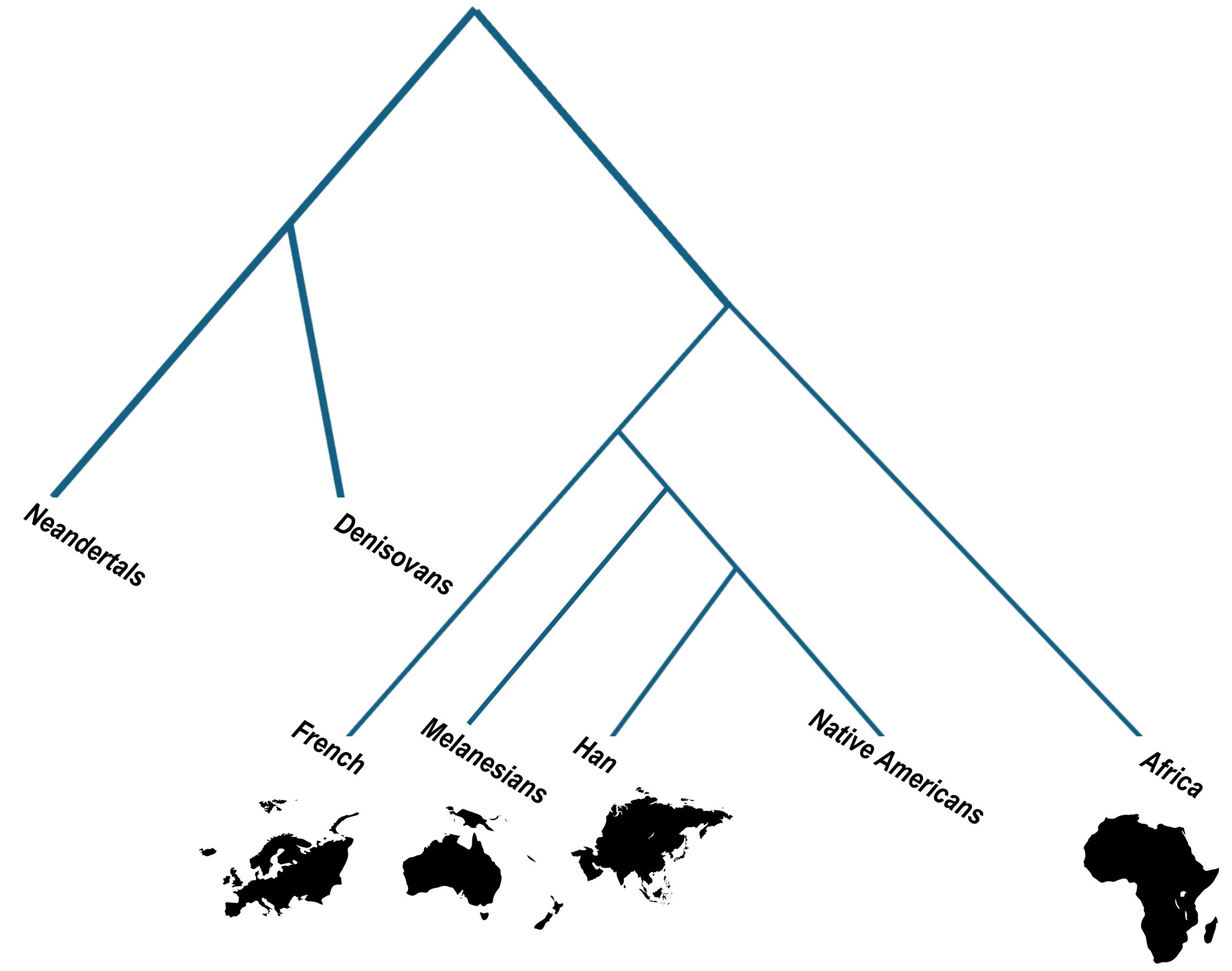
Horizontal Gene Transfer



Burmeister, 2015, *Evol Med Public Health*

Phylogenetic Trees

Phylogenetic Trees



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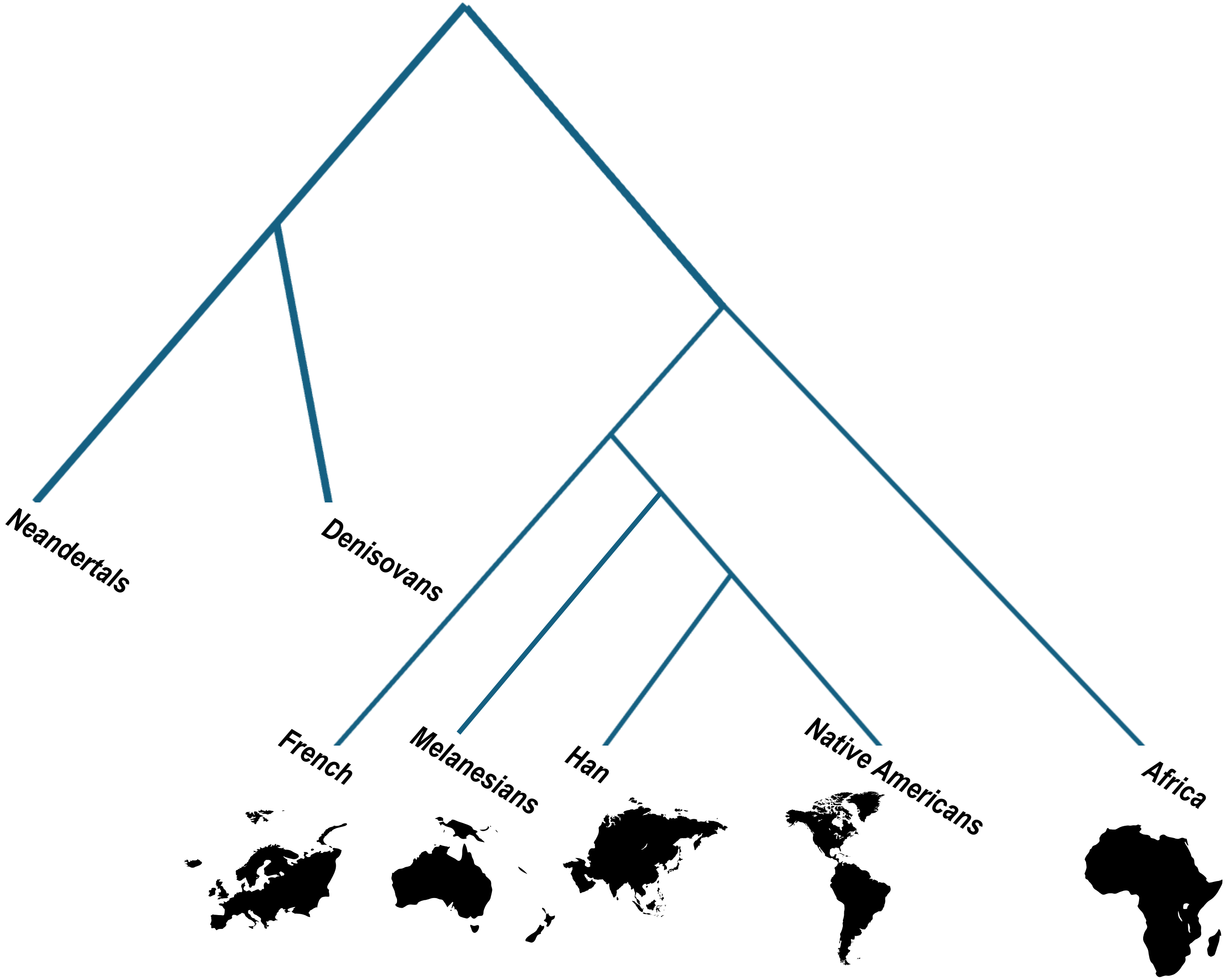

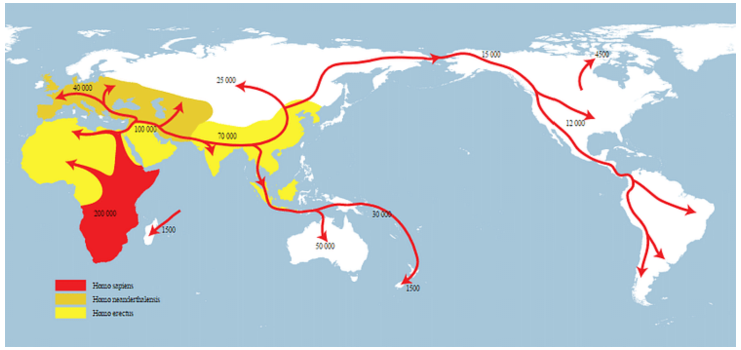

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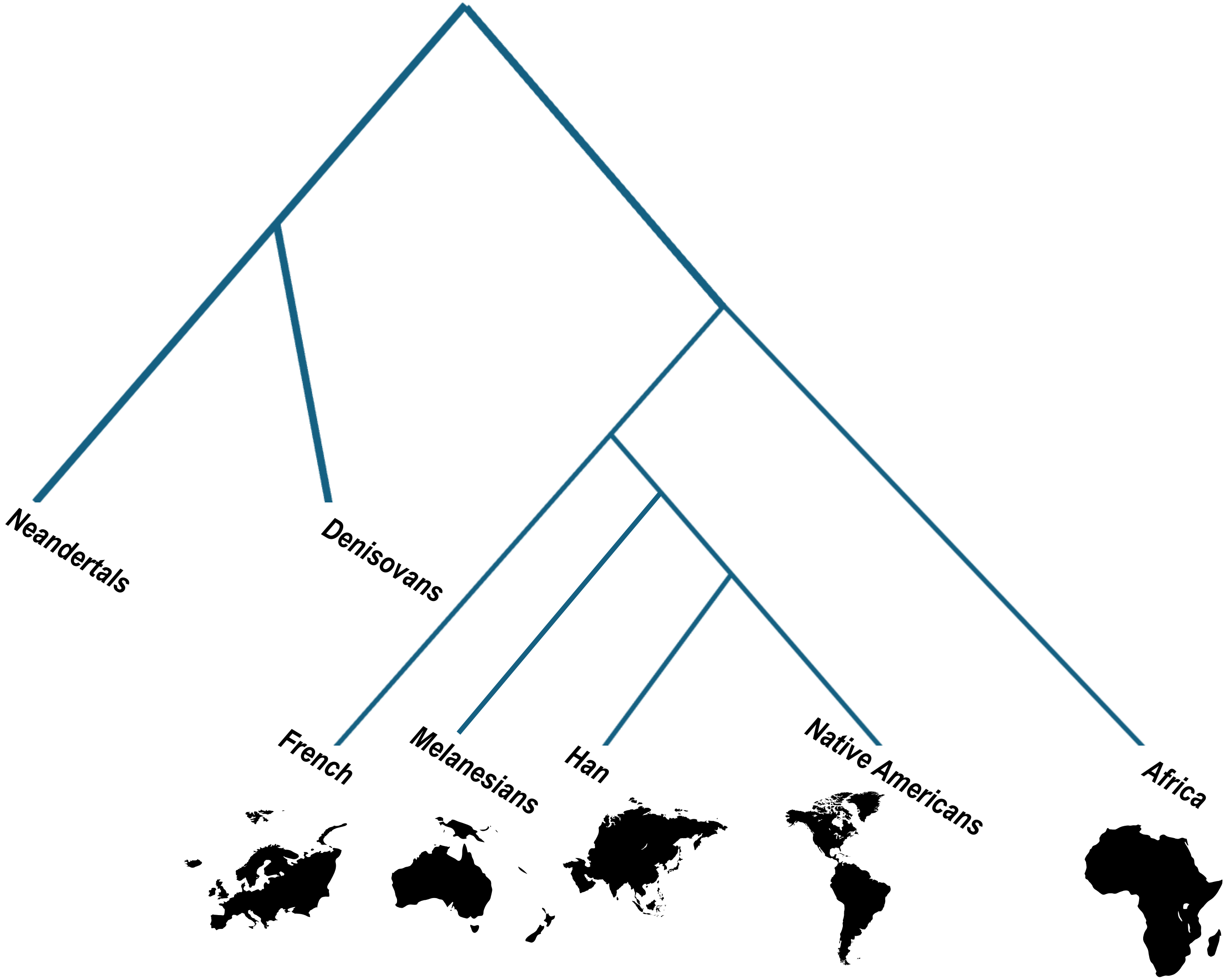

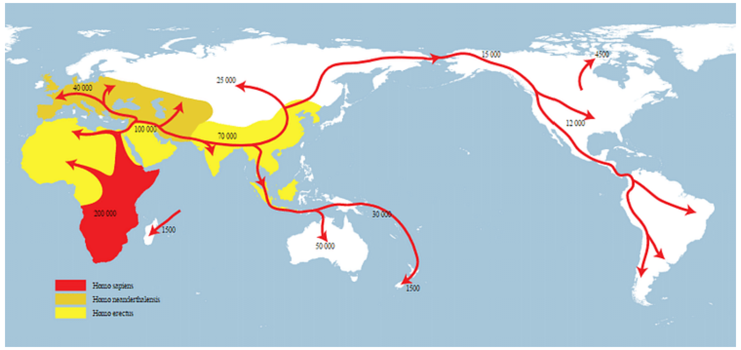

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Admixture

Lecture Outline

Introduction – phylogenetic trees

Definition – galled trees

Previous work – the enumeration of galled trees and galled trees with one gall

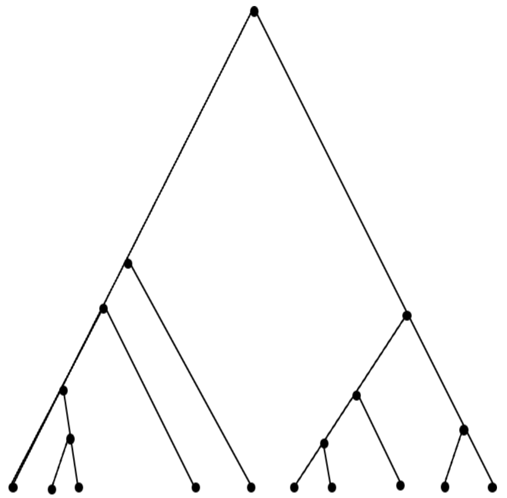
Novel work – the enumeration of galled trees with a fixed number of galls

Summary

Galled Trees

Galled Trees

A rooted binary tree



Galled Trees

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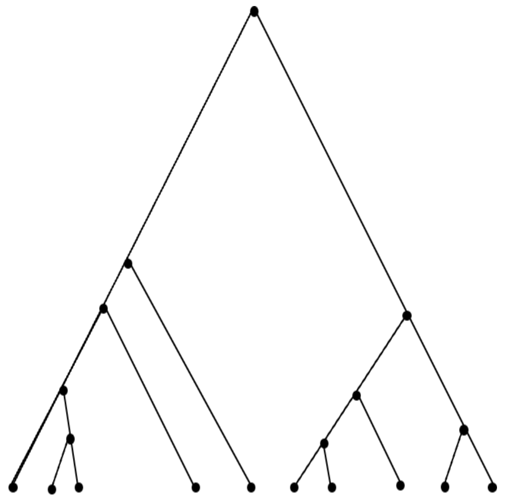
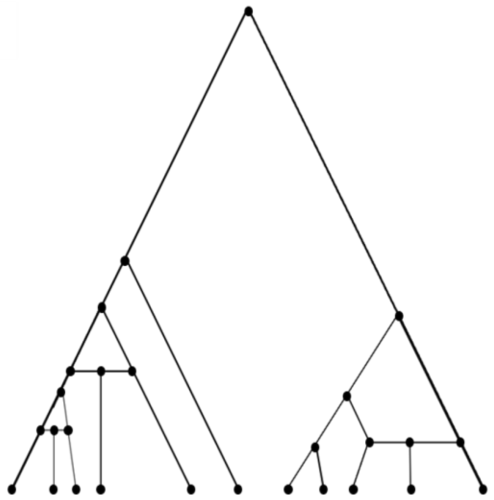


A galled tree



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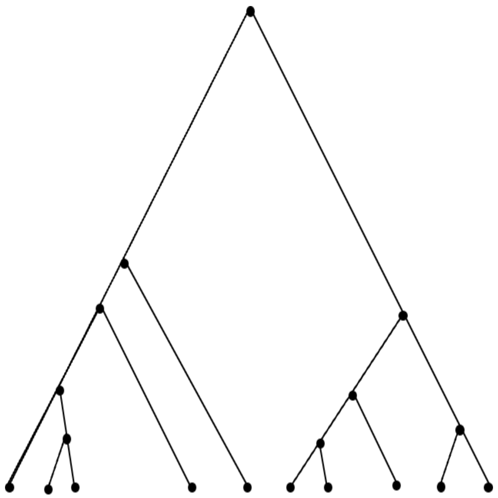
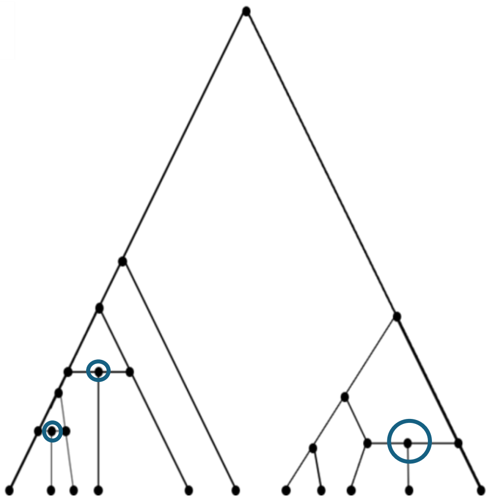
Galled Trees

A rooted binary tree



A galled tree



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Galled Trees

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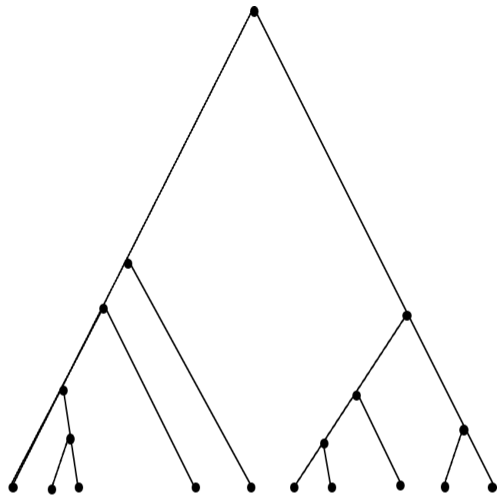
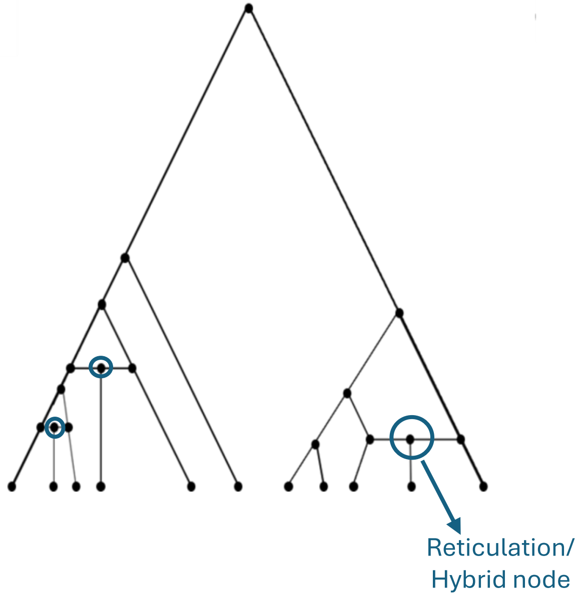


A galled tree



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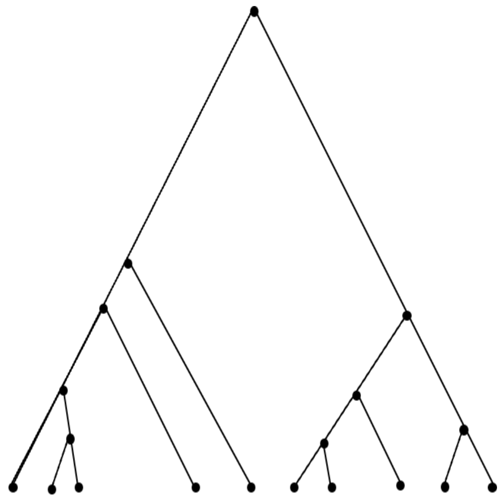

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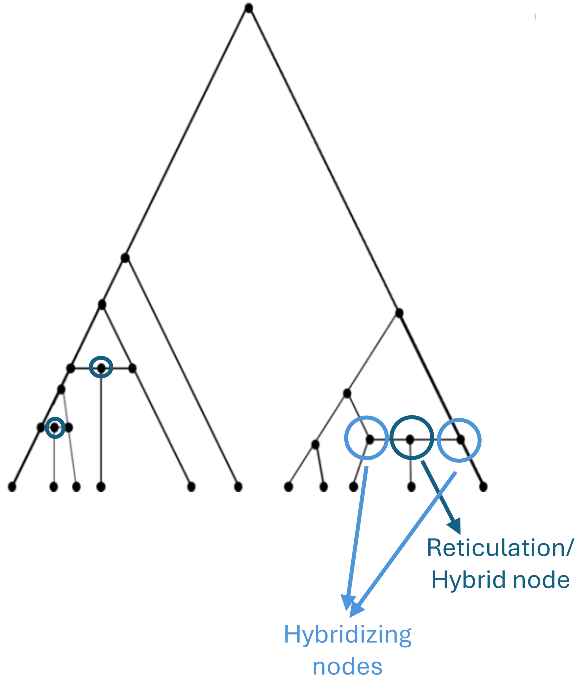

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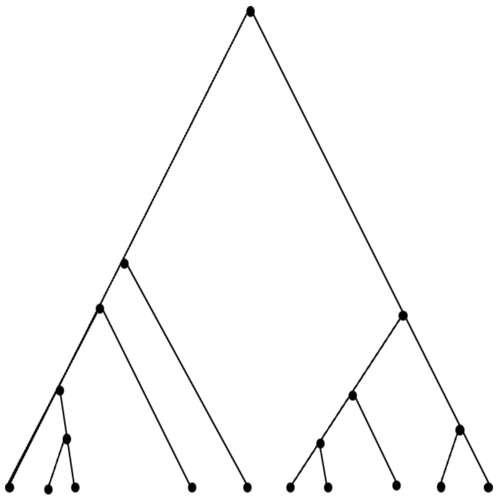

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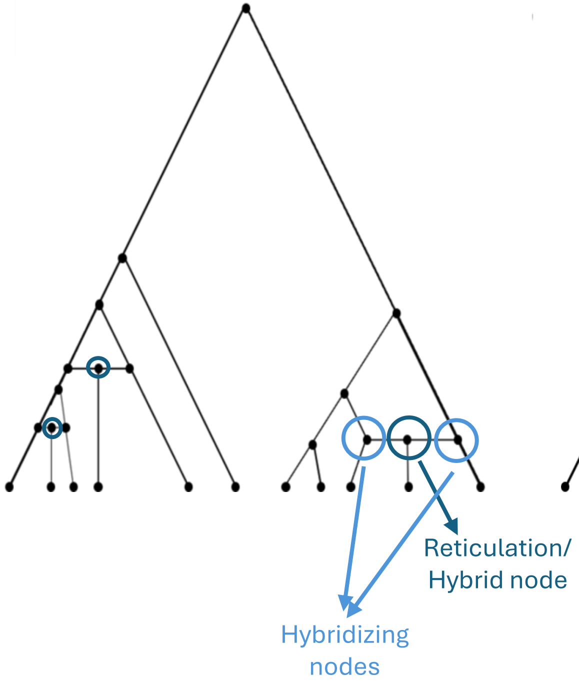
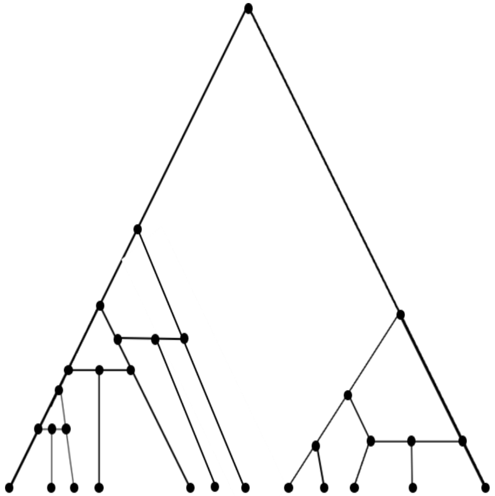


Not a galled tree



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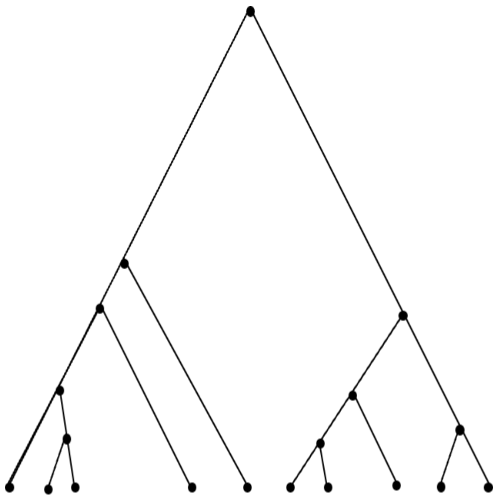

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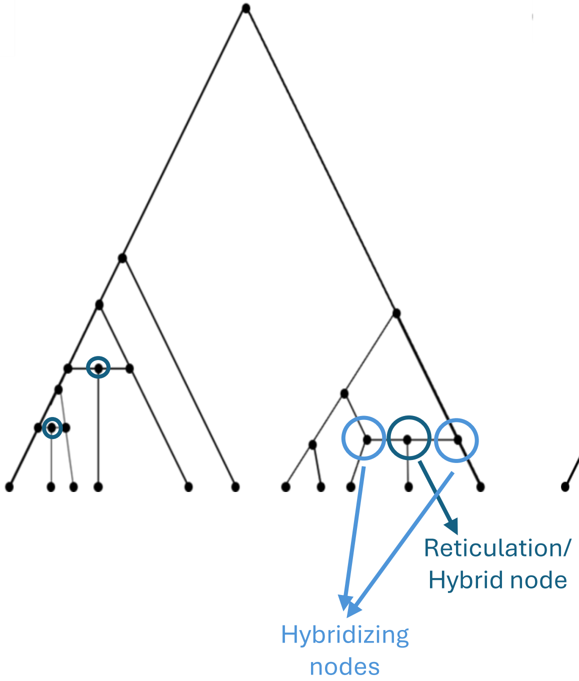

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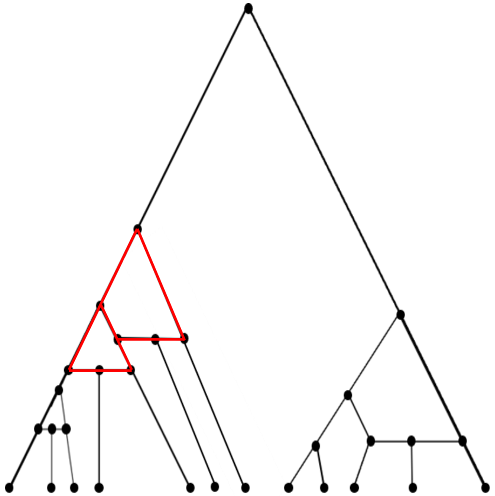

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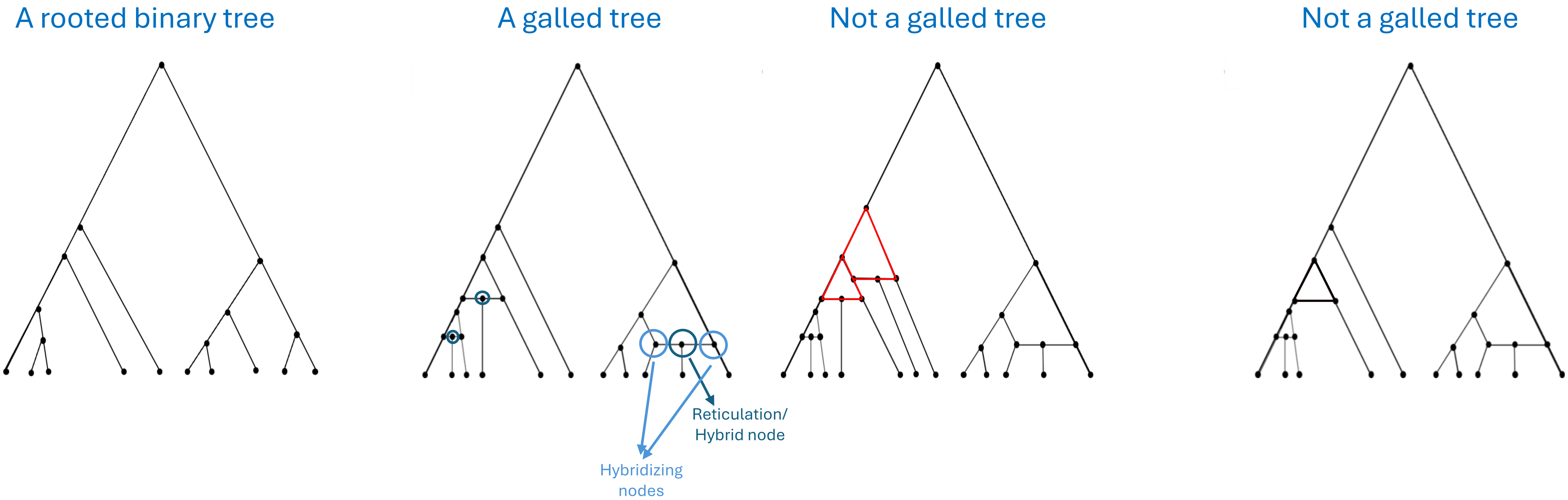

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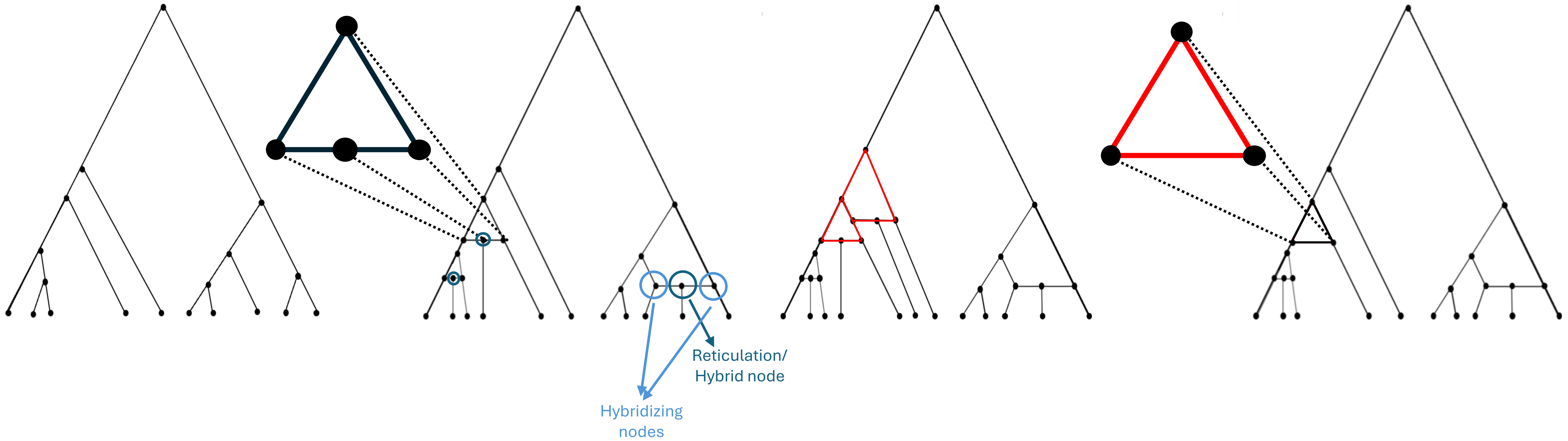
Galled Trees

A rooted binary tree | A galled tree | Not a galled tree | Not a galled tree



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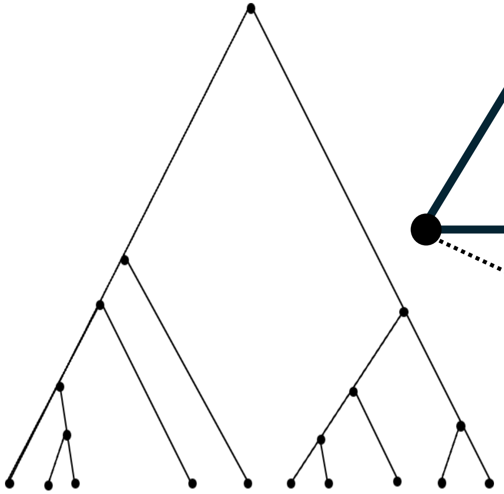

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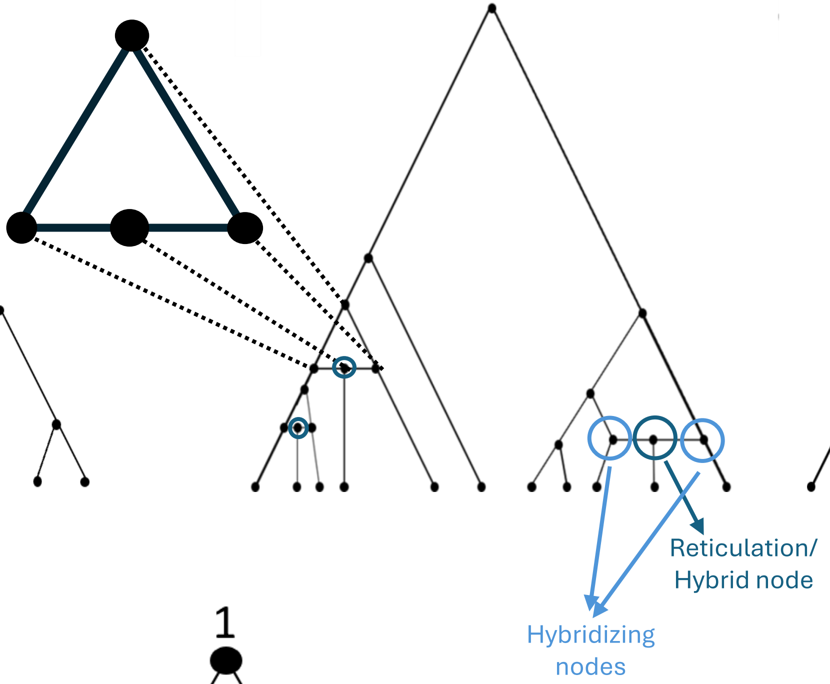

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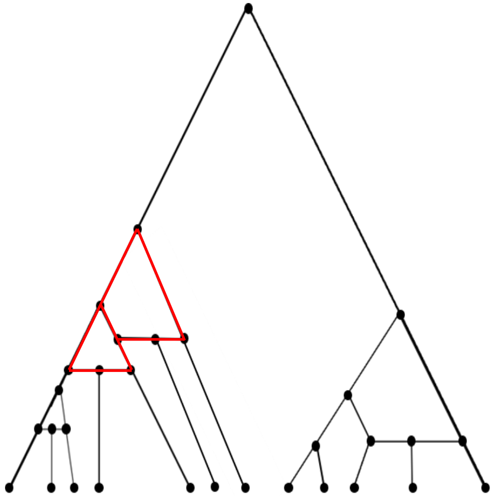

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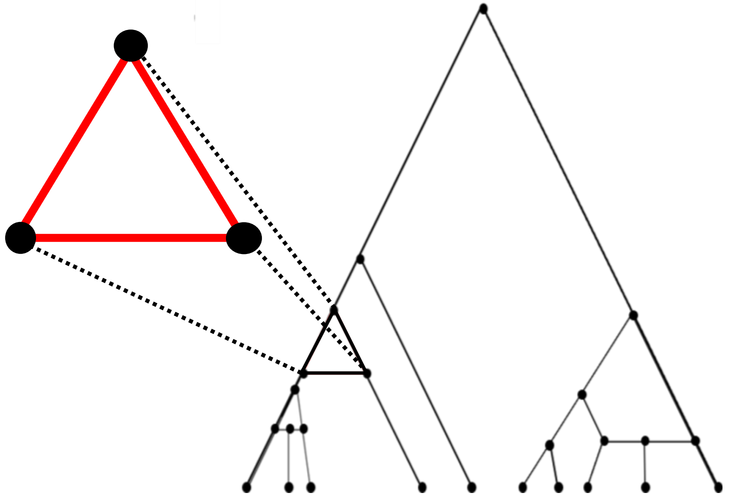
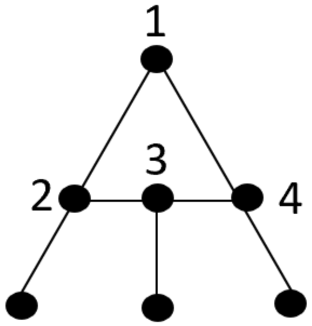


The smallest galled tree



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Galled Trees

Galled Trees

Definitions

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Ignoring the direction of edges, the two paths connecting r and a_r form a cycle C_r, known as a gall.

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Second, the set of nodes in the gall C_r, associated with reticulation node a_r, and the set of nodes in the gall C_s, associated with reticulation node a_s, are disjoint.

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https://discoverandshare.org/2021/06/24/all-about-galls/

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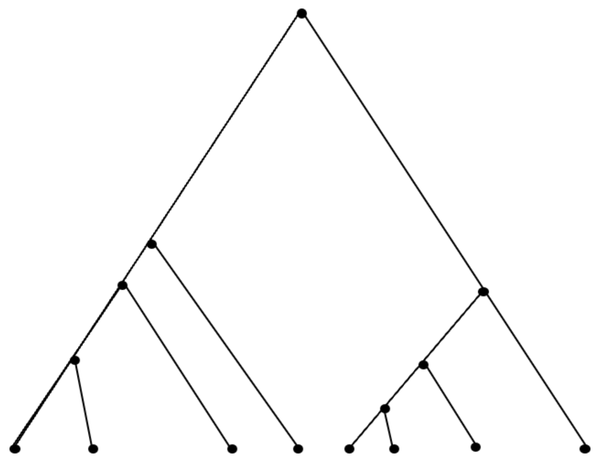
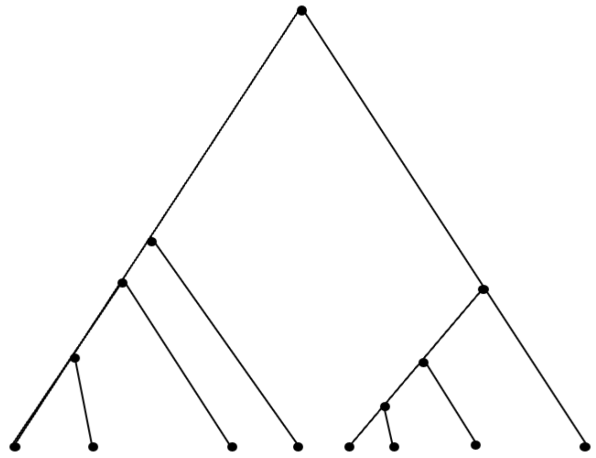
Rooted Unlabeled Binary Non-plane Trees

Rooted Unlabeled Binary Non-plane Trees



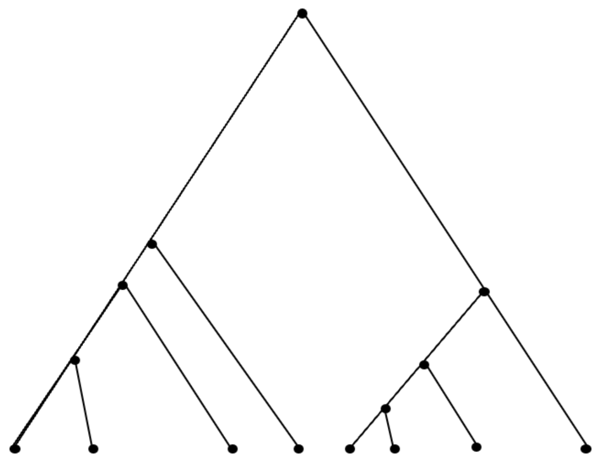
Rooted Unlabeled Binary Non-plane Trees



$$\mathcal{U}(t) = t + \frac{1}{2}\mathcal{U}^2(t) + \frac{1}{2}\mathcal{U}(t^2)$$

Otter 1948, Comtet, 1974

Rooted Unlabeled Binary Non-plane Trees



$$\mathcal{U}(t) = t + \frac{1}{2}\mathcal{U}^2(t) + \frac{1}{2}\mathcal{U}(t^2)$$

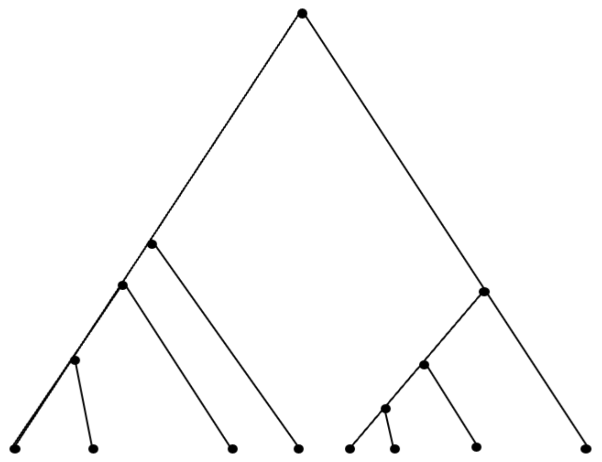
Otter 1948, Comtet, 1974

$$\mathcal{U}(t) \sim 1 - \gamma\sqrt{1 - \frac{t}{\rho}}$$

$\rho \approx 0.4027$
$\gamma \approx 1.1301$

Landau, 1977; Flajolet and Sedgewick, 2009

Rooted Unlabeled Binary Non-plane Trees



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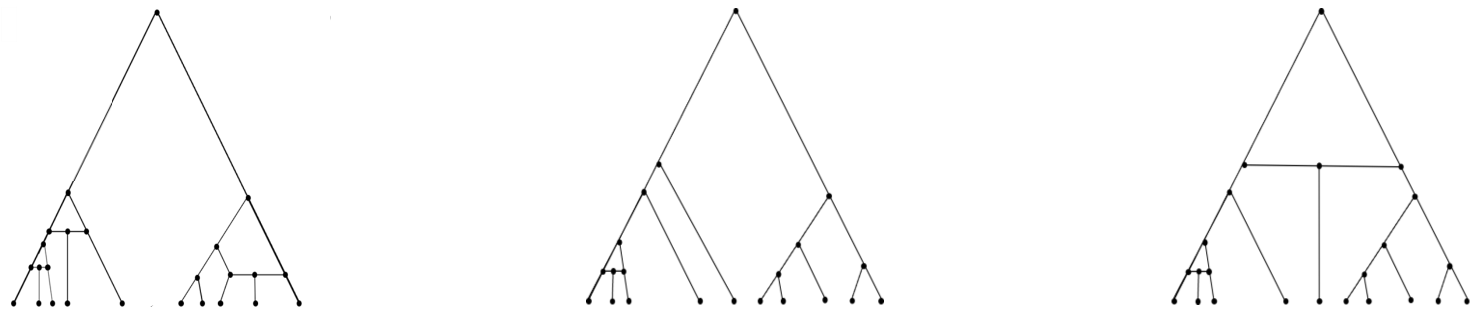
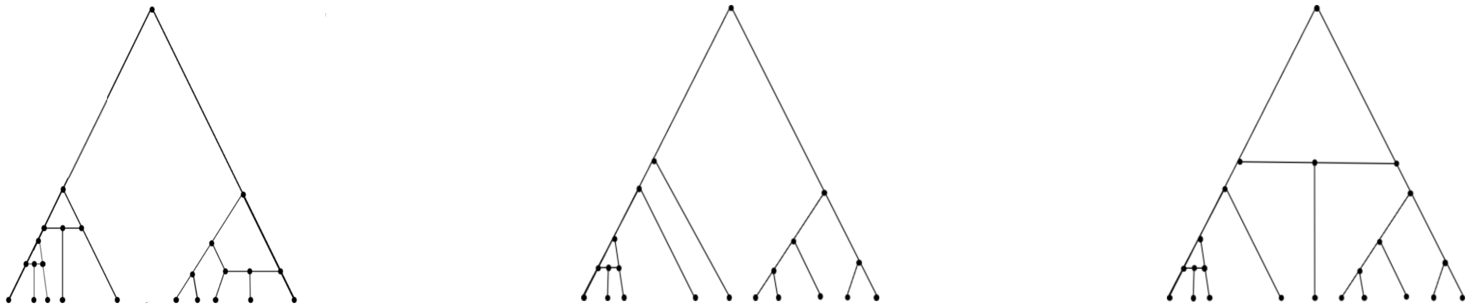
Rooted Unlabeled Binary Non-plane Galled Trees

Rooted Unlabeled Binary Non-plane Galled Trees



Agranat-Tamir *et al*, 2024

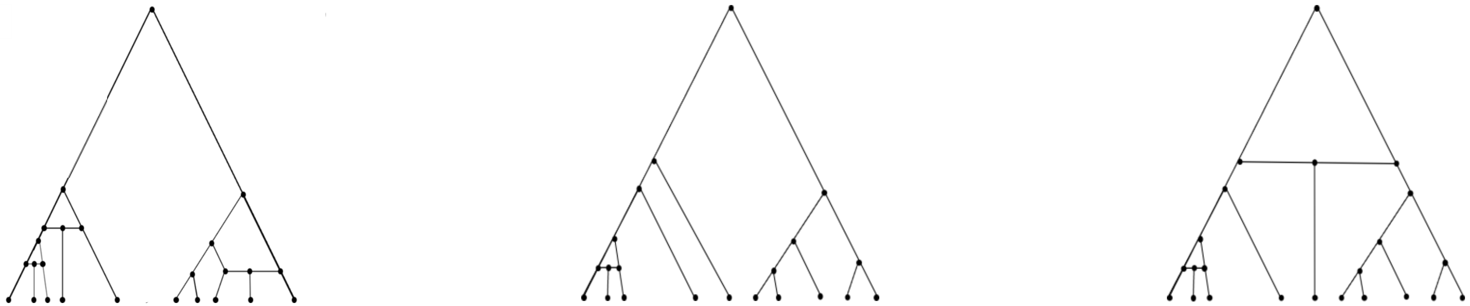
Rooted Unlabeled Binary Non-plane Galled Trees



$$\mathcal{A}(t) = 1 + t + \frac{1}{2}\mathcal{A}^2(t) + \frac{1}{2}\mathcal{A}(t^2) - \frac{1}{1-\mathcal{A}(t)} + \frac{\mathcal{A}(t)}{2[1-\mathcal{A}(t)]^2} + \frac{\mathcal{A}(t)}{2[1-\mathcal{A}(t^2)]}$$

Agranat-Tamir *et al*, 2024

Rooted Unlabeled Binary Non-plane Galled Trees



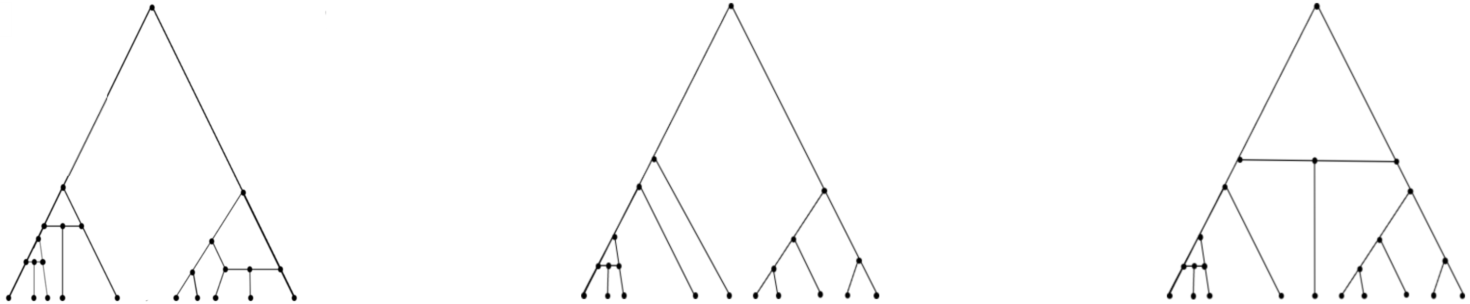
$$\mathcal{A}(t) = 1 + t + \frac{1}{2}\mathcal{A}^2(t) + \frac{1}{2}\mathcal{A}(t^2) - \frac{1}{1-\mathcal{A}(t)} + \frac{\mathcal{A}(t)}{2[1-\mathcal{A}(t)]^2} + \frac{\mathcal{A}(t)}{2[1-\mathcal{A}(t^2)]}$$

$$[t^n]\mathcal{A}(t) \sim \frac{\delta}{2\sqrt{\pi}} n^{-\frac{3}{2}} r^{-n}$$

$$r \approx 0.2073;\ \delta \approx 0.2793$$

Agranat-Tamir *et al*, 2024

Rooted Unlabeled Binary Non-plane Galled Trees



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$$\frac{1}{r} \approx 4.82;\ \frac{1}{\rho} \approx 2.48$$

Agranat-Tamir *et al*, 2024

Galled Trees with Exactly One Gall

Galled Trees with Exactly One Gall

g is the number of galls; n is the number of leaves

Galled Trees with Exactly One Gall

$g = 1$

g is the number of galls; n is the number of leaves

Agranat-Tamir *et al*, 2024

Galled Trees with Exactly One Gall

$n = 1$

$g = 1$

g is the number of galls; n is the number of leaves

Galled Trees with Exactly One Gall

$n = 1$

No trees

$g = 1$

g is the number of galls; n is the number of leaves

Galled Trees with Exactly One Gall

$n = 1$ $n = 2$

No trees

$g = 1$

g is the number of galls; n is the number of leaves

Galled Trees with Exactly One Gall

$n = 1$ | $n = 2$

No trees | No trees

$g = 1$

g is the number of galls; n is the number of leaves

Agranat-Tamir *et al*, 2024

Galled Trees with Exactly One Gall

$n = 1$

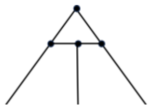
$n = 2$

$n = 3$

$g = 1$

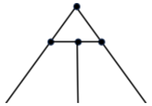
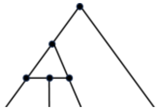
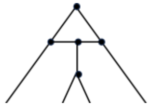
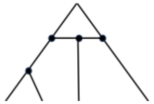
No trees

No trees



g is the number of galls; n is the number of leaves

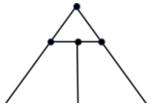
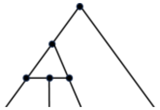
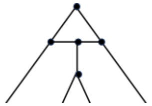
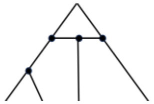
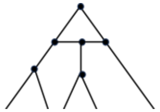
Galled Trees with Exactly One Gall

	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$g = 1$	No trees	No trees		   

g is the number of galls; n is the number of leaves

Agranat-Tamir *et al*, 2024

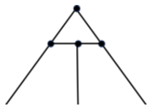
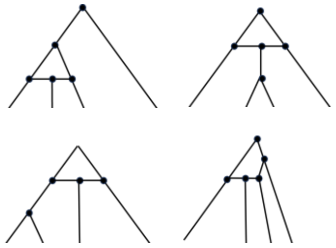
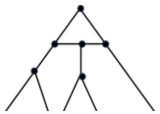
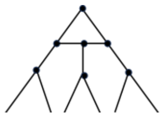
Galled Trees with Exactly One Gall

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$g = 1$	No trees	No trees		   	 +14 more

g is the number of galls; n is the number of leaves

Agranat-Tamir *et al*, 2024

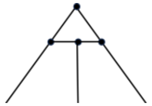
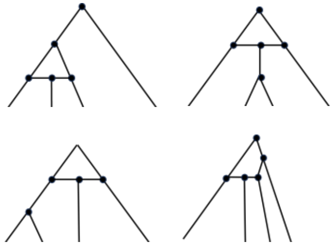
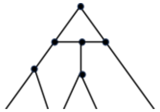
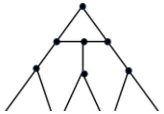
Galled Trees with Exactly One Gall

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$g = 1$	No trees	No trees			 +14 more	 +47 more

g is the number of galls; n is the number of leaves

Agranat-Tamir *et al*, 2024

Galled Trees with Exactly One Gall

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$g = 1$	No trees	No trees			 +14 more	 +47 more

g is the number of galls; n is the number of leaves

$$\mathcal{E}_1(t) = \frac{1}{1-\mathcal{U}(t)} - \frac{1}{[1-\mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1-\mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1-\mathcal{U}(t)][1-\mathcal{U}(t^2)]}$$

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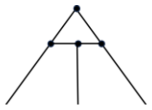
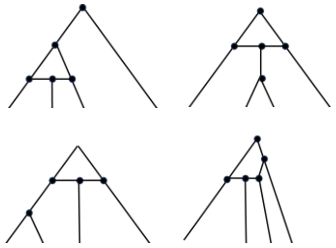
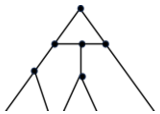
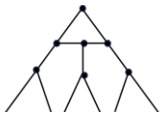
Galled Trees with Exactly One Gall

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$g = 1$	No trees	No trees			 +14 more	 +47 more

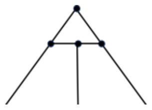
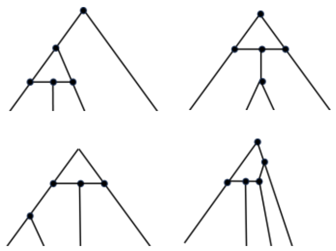
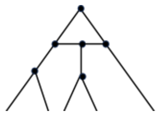
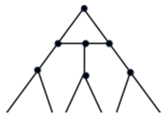
g is the number of galls; n is the number of leaves

$$\mathcal{E}_1(t) = \frac{1}{1-\mathcal{U}(t)} - \frac{1}{[1-\mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1-\mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1-\mathcal{U}(t)][1-\mathcal{U}(t^2)]}$$

$$\mathcal{E}_1(t) \sim \frac{1}{2\gamma^3 (1-\frac{t}{\rho})^{\frac{3}{2}}}$$

Agranat-Tamir *et al*, 2024

Galled Trees with Exactly One Gall

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$g = 1$	No trees	No trees			 +14 more	 +47 more

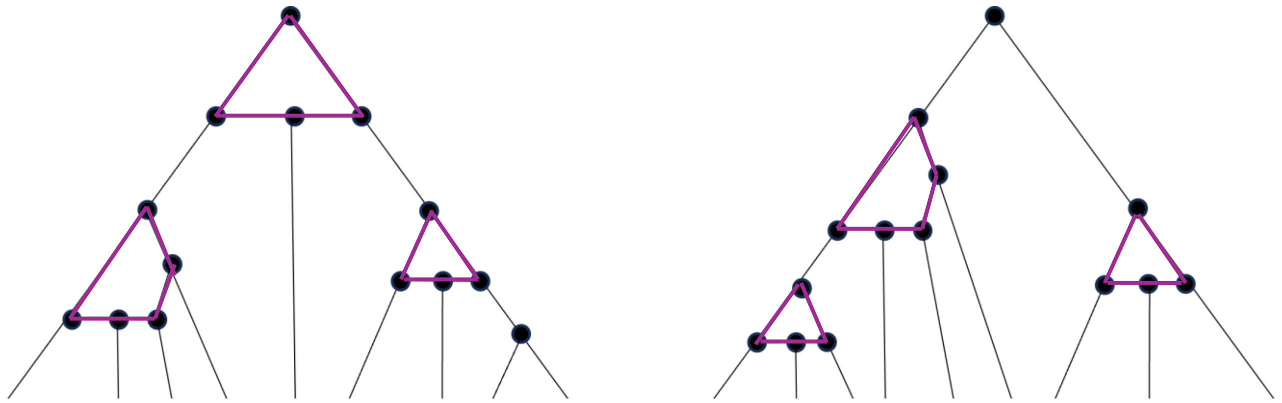
g is the number of galls; n is the number of leaves

$$\mathcal{E}_1(t) = \frac{1}{1-\mathcal{U}(t)} - \frac{1}{[1-\mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1-\mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1-\mathcal{U}(t)][1-\mathcal{U}(t^2)]}$$

$$\mathcal{E}_1(t) \sim \frac{1}{2\gamma^3\left(1-\frac{t}{\rho}\right)^{\frac{3}{2}}} \qquad [t^n]\mathcal{E}_1(t) \sim \frac{1}{\gamma^3\sqrt{\pi}} n^{\frac{1}{2}} \rho^{-n}$$

Goal

Asymptotic enumeration of unlabeled galled trees
with a fixed number of galls



Lecture Outline

Introduction – phylogenetic trees

Definition – galled trees

Previous work – the enumeration of galled trees and galled trees with one gall

Novel work – the enumeration of galled trees with a fixed number of galls

Summary

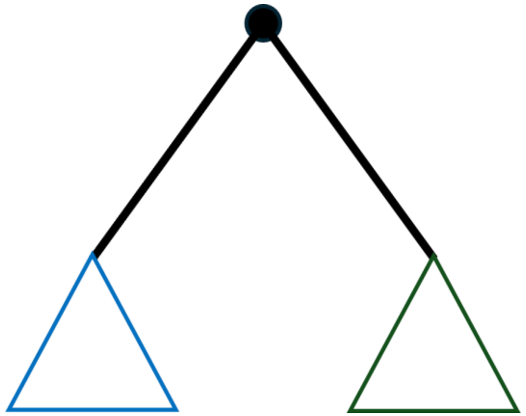
Recursion

No root gall

With a root gall

Recursion

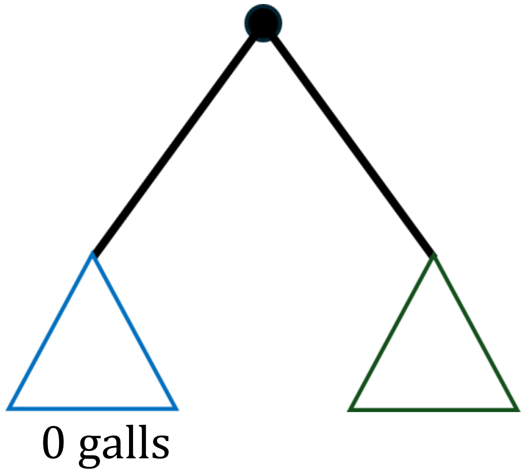
No root gall



With a root gall

Recursion

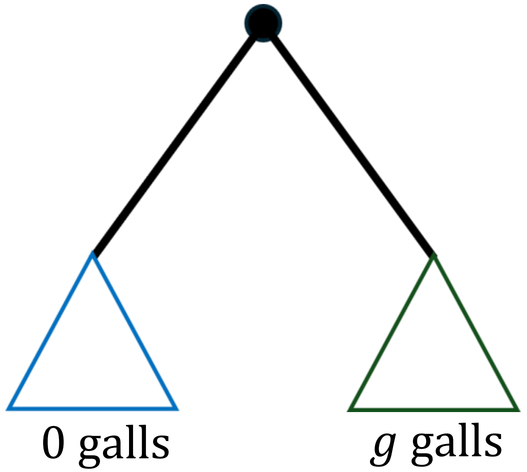
No root gall



With a root gall

Recursion

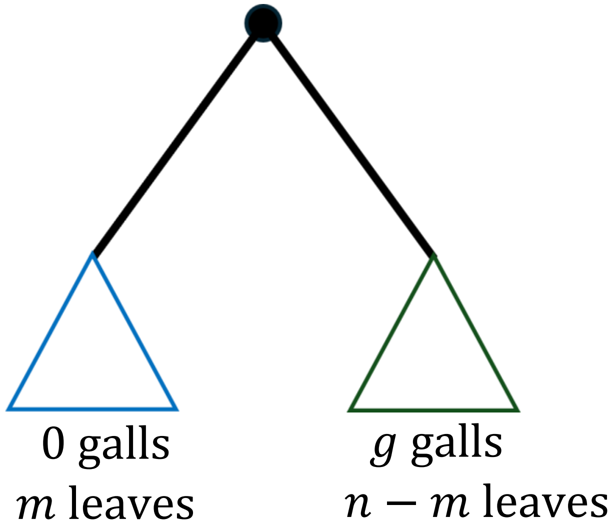
No root gall



With a root gall

Recursion

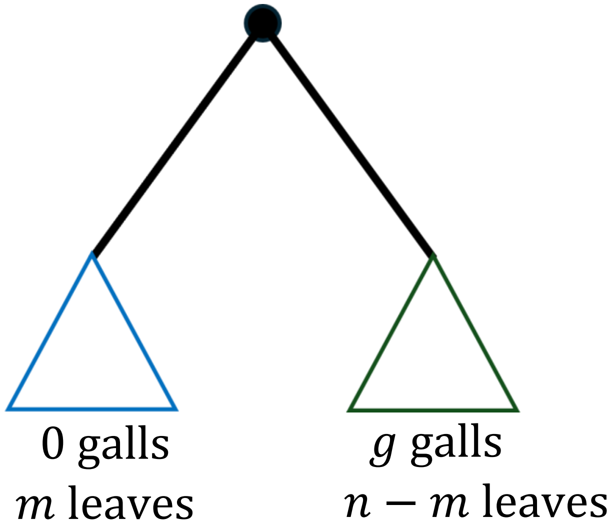
No root gall

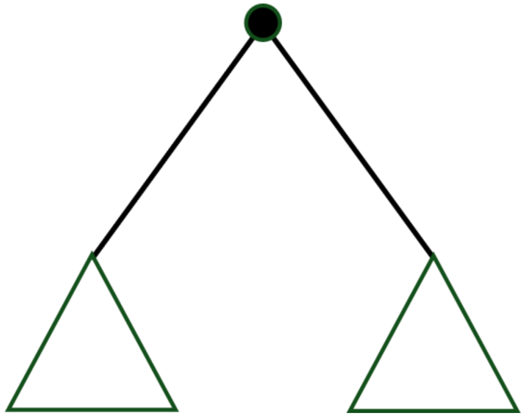


With a root gall

Recursion

No root gall

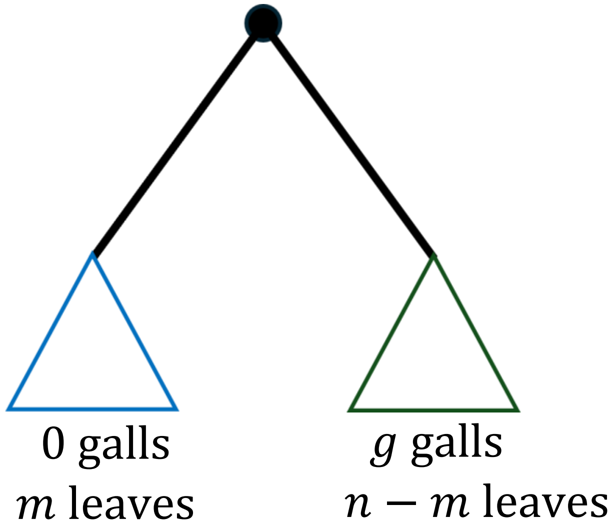


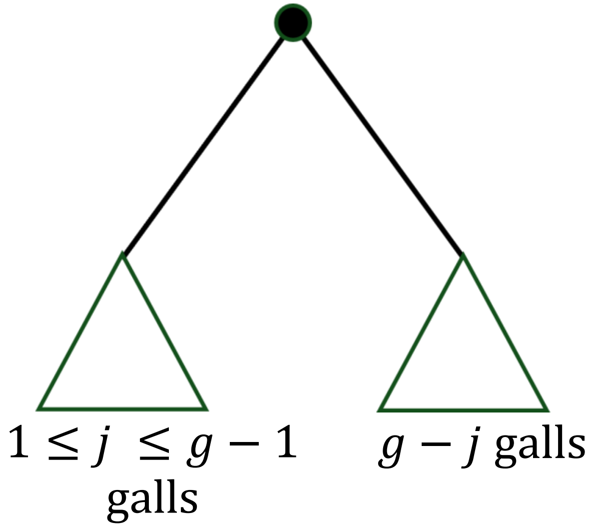


With a root gall

Recursion

No root gall

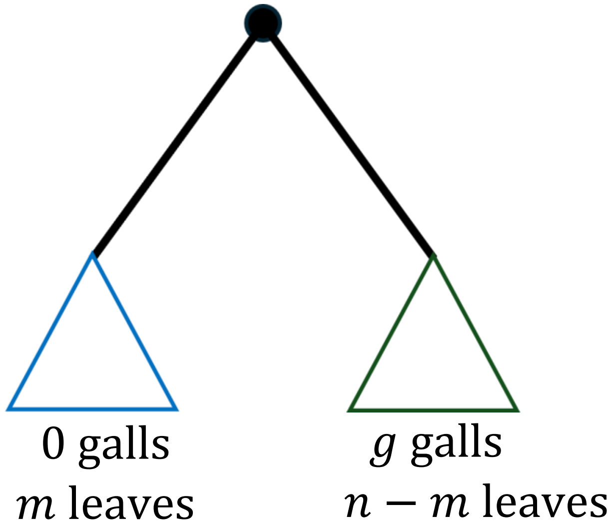


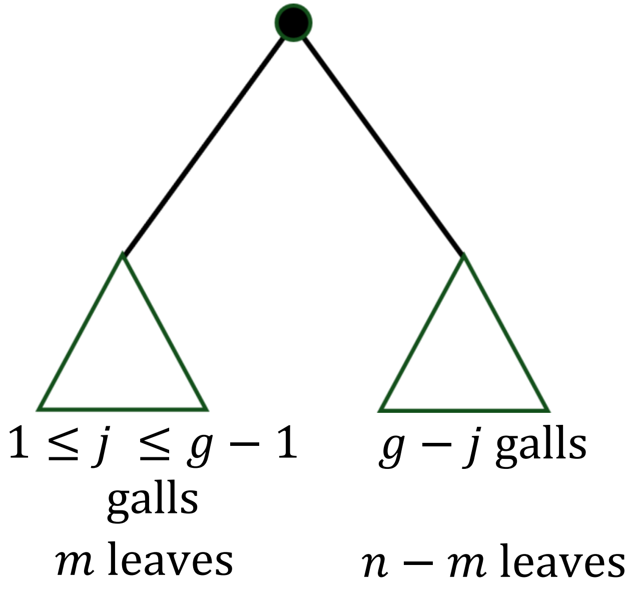


With a root gall

Recursion

No root gall

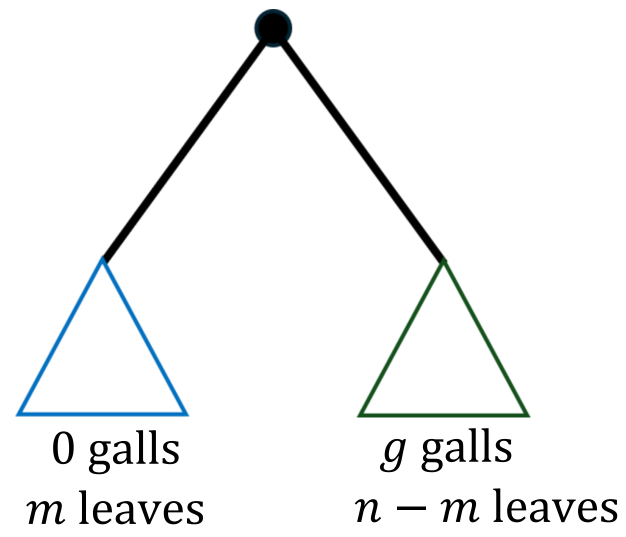


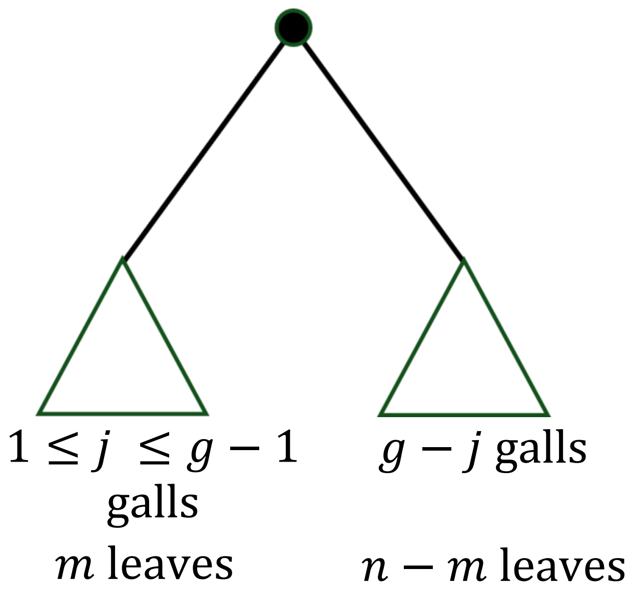


With a root gall

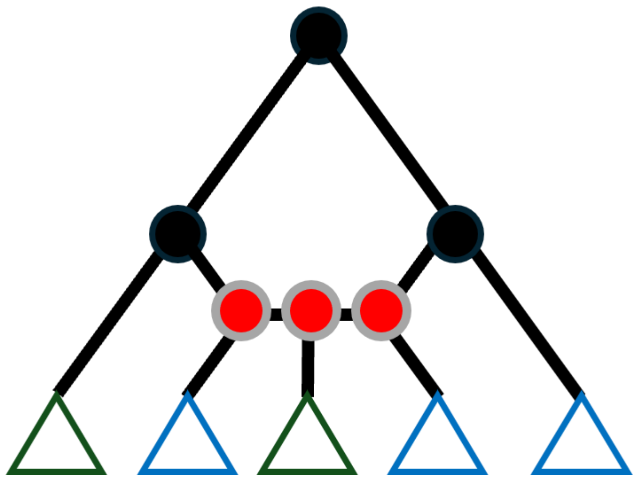
Recursion

No root gall



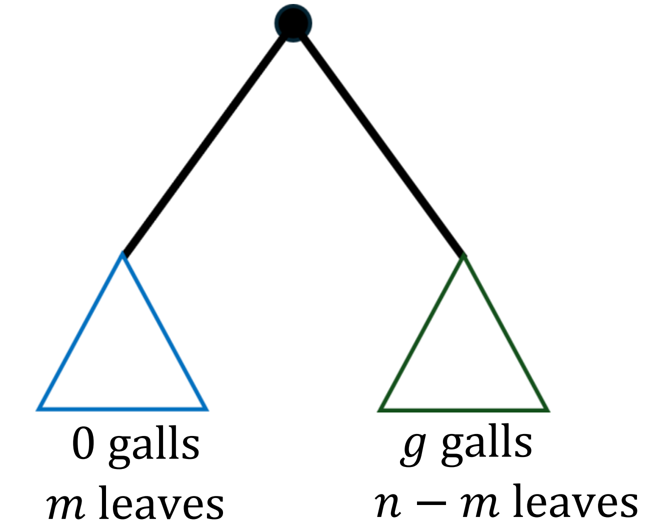


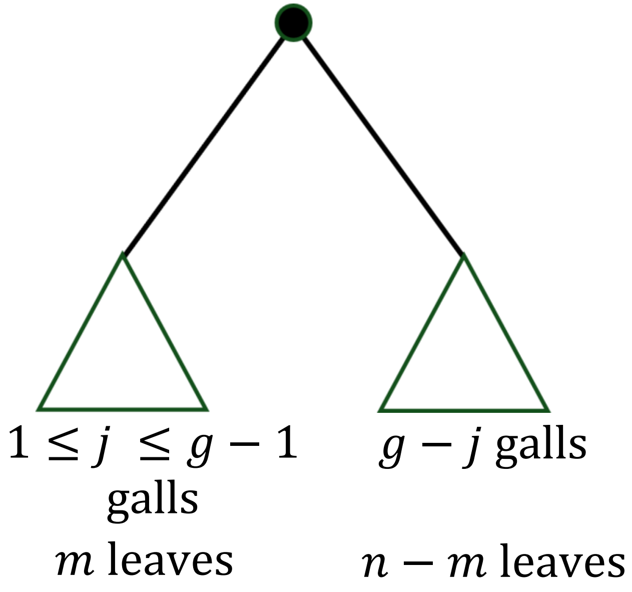
With a root gall



Recursion

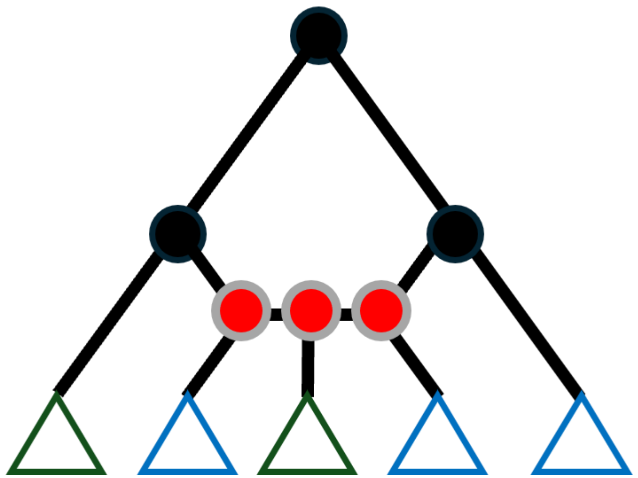
No root gall





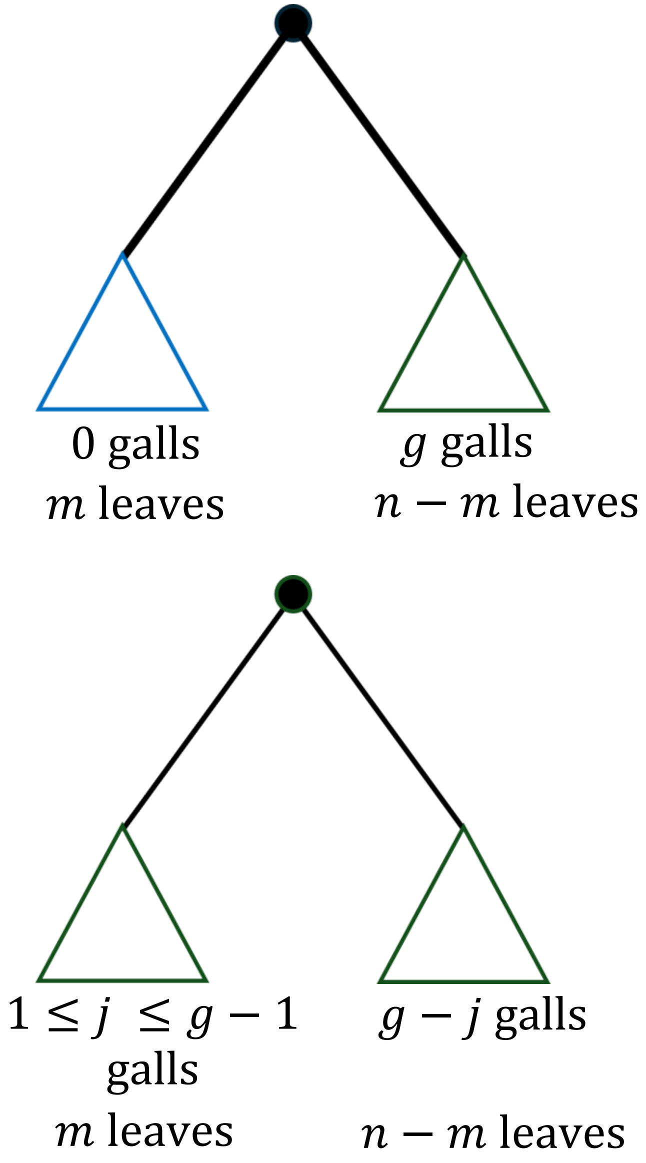
With a root gall

$3 \leq k \leq n$ main subtrees



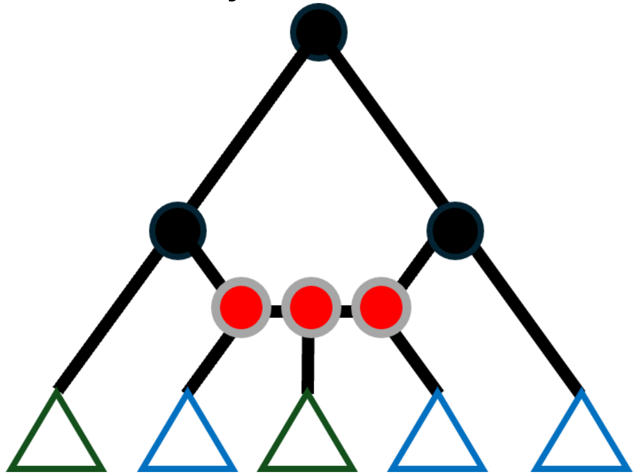
Recursion

No root gall



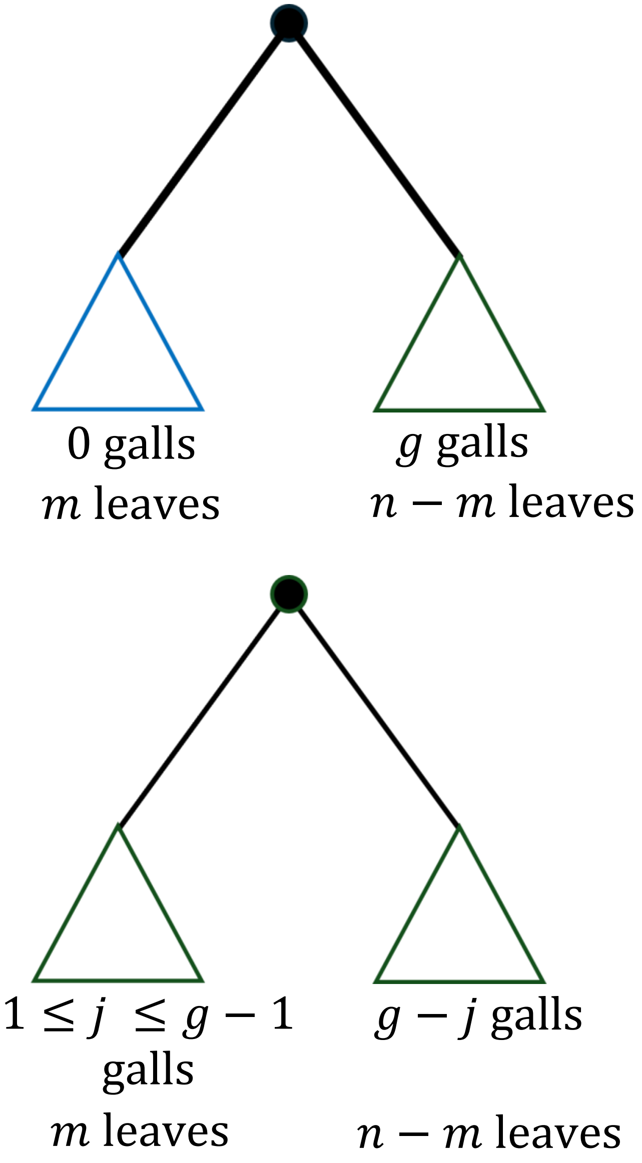
With a root gall

$3 \leq k \leq n$ main subtrees
$k - 2$ possibilites
to place the
hybrid node

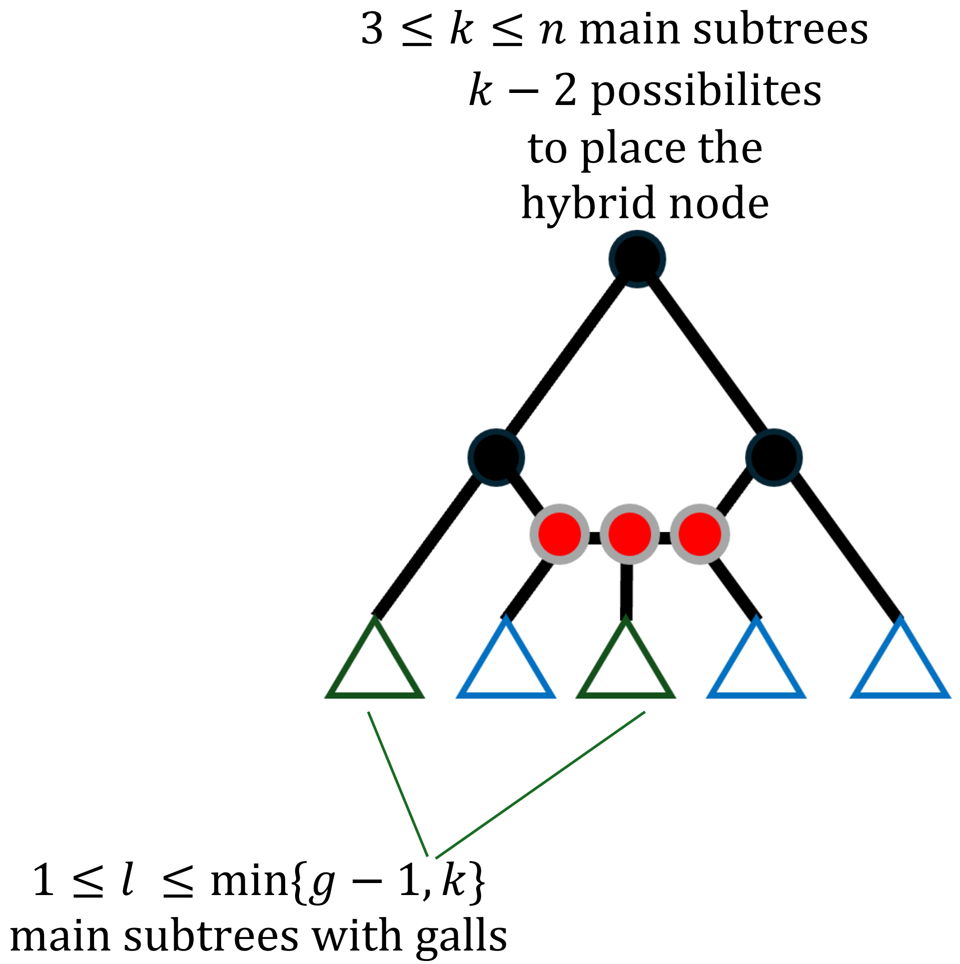


Recursion

No root gall

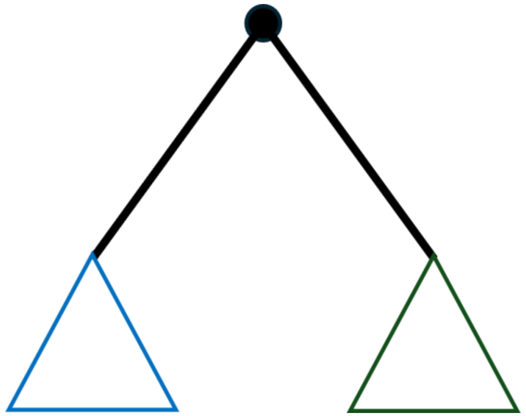


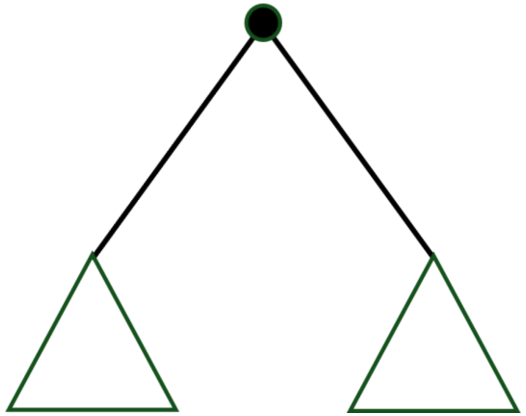
With a root gall



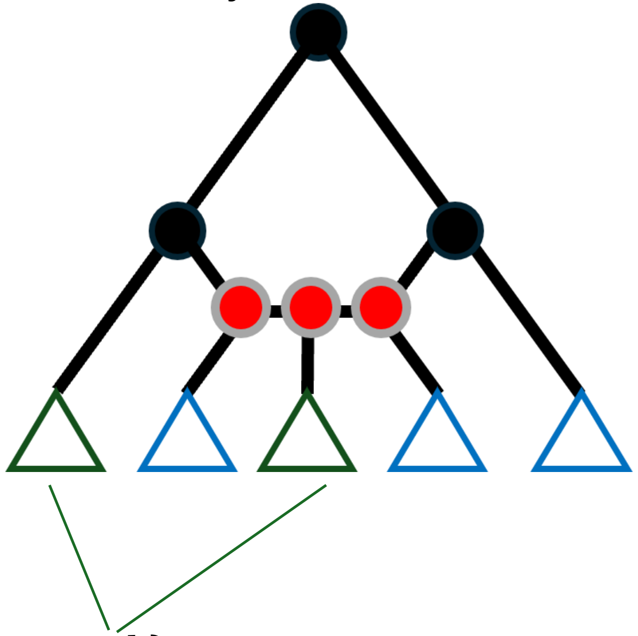
Recursion

No root gall



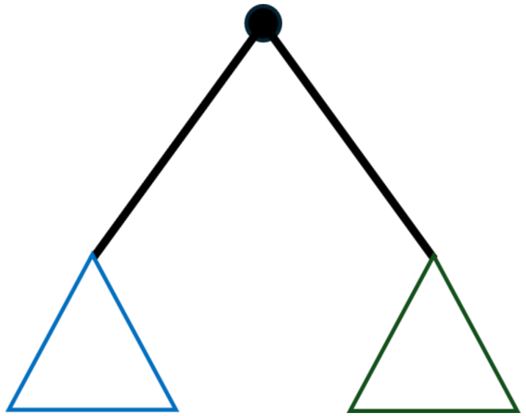


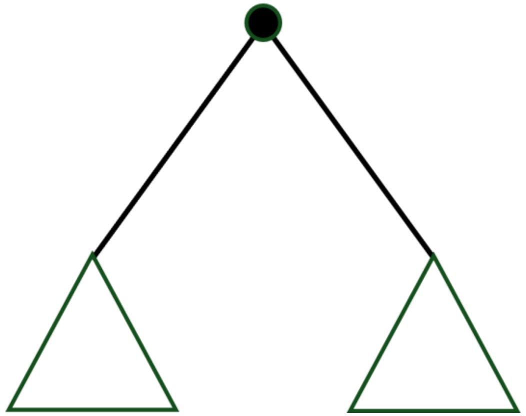
With a root gall



Recursion

No root gall

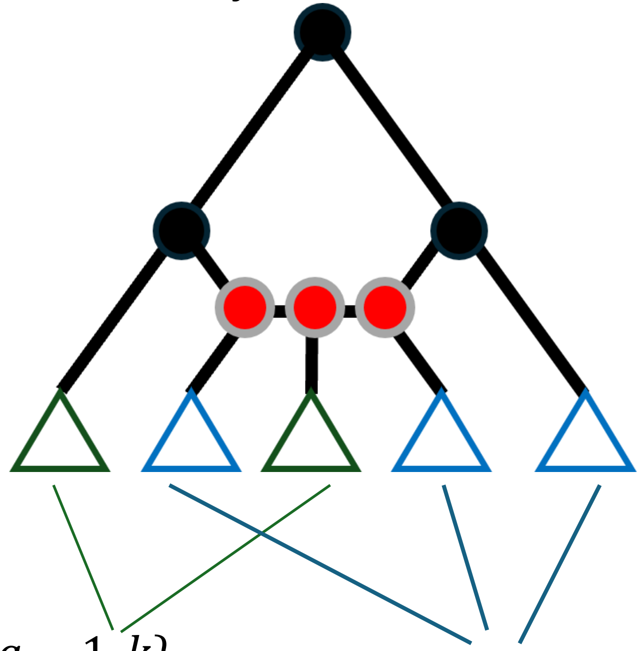




With a root gall

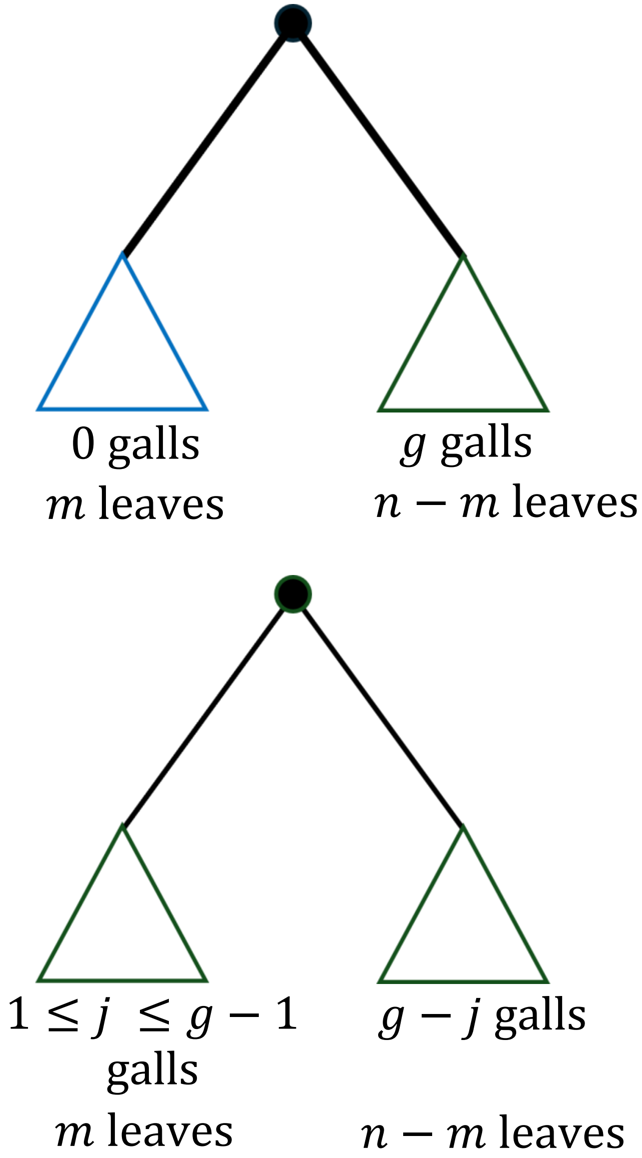
$3 \leq k \leq n$ main subtrees

$k - 2$ possibilites to place the hybrid node

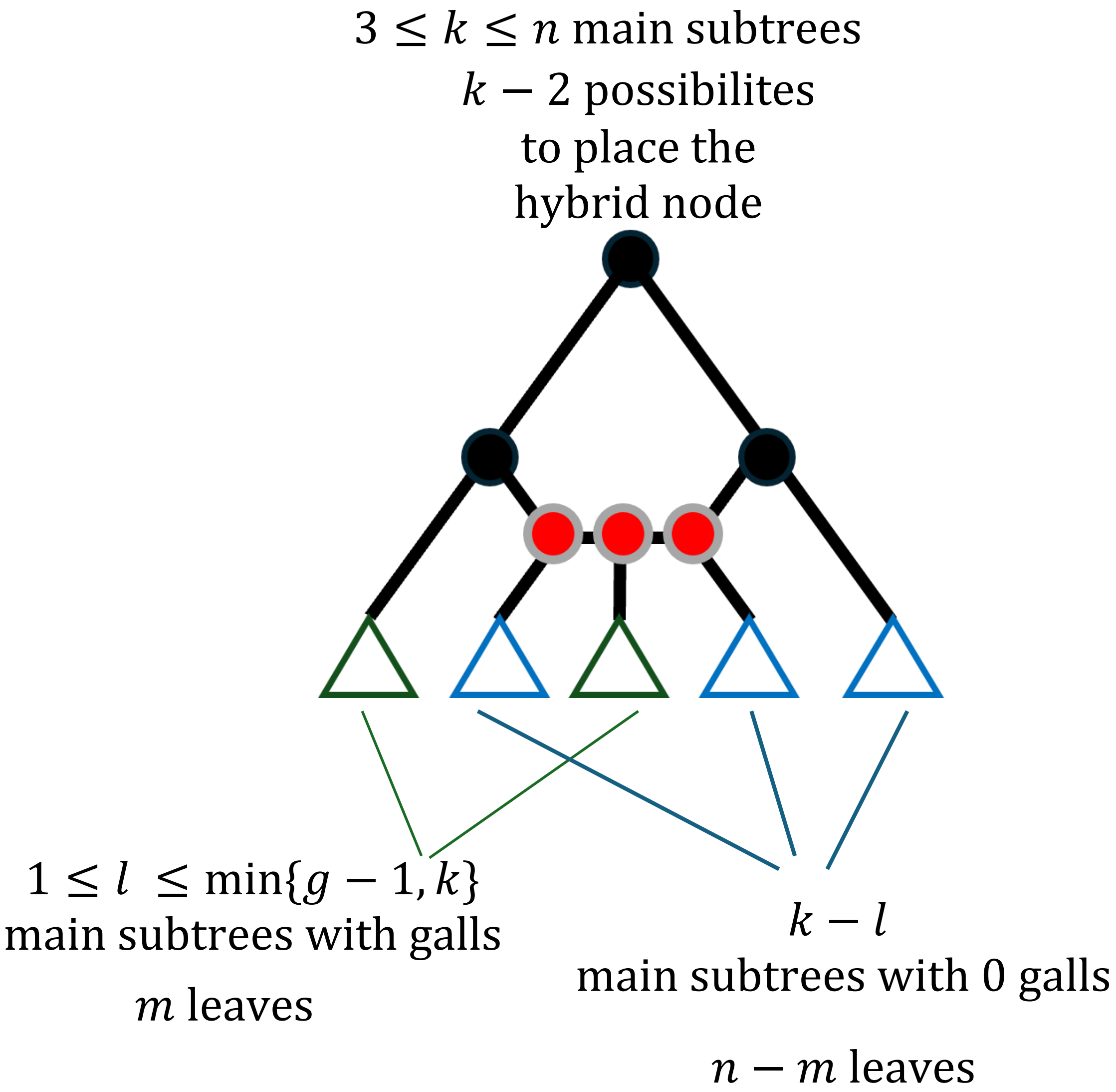


Recursion

No root gall



With a root gall



Asymptotic Analysis

Asymptotic Analysis

$$\mathcal{E}_1(t) = \frac{1}{1-\mathcal{U}(t)} - \frac{1}{[1-\mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1-\mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1-\mathcal{U}(t)][1-\mathcal{U}(t^2)]}$$

Asymptotic Analysis

$$\mathcal{E}_1(t) = \frac{1}{1-\mathcal{U}(t)} - \frac{1}{[1-\mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1-\mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1-\mathcal{U}(t)][1-\mathcal{U}(t^2)]}$$

$$\mathcal{E}_2(t) = \frac{\mathcal{E}_1(t)}{2[1-\mathcal{U}(t)]}\left[\mathcal{E}_1(t) + \frac{\mathcal{U}(t)+\mathcal{U}^2(t)}{[1-\mathcal{U}(t)]^3} - \frac{1}{1-\mathcal{U}(t)} + \frac{1}{1-\mathcal{U}(t^2)}\right] + \frac{\mathcal{E}_1(t^2)}{2[1-\mathcal{U}(t)]}$$

Asymptotic Analysis

$$\mathcal{E}_1(t) = \frac{1}{1-\mathcal{U}(t)} - \frac{1}{[1-\mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1-\mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1-\mathcal{U}(t)][1-\mathcal{U}(t^2)]}$$

$$\mathcal{E}_2(t) = \frac{\mathcal{E}_1(t)}{2[1-\mathcal{U}(t)]}\left[\mathcal{E}_1(t) + \frac{\mathcal{U}(t)+\mathcal{U}^2(t)}{[1-\mathcal{U}(t)]^3} - \frac{1}{1-\mathcal{U}(t)} + \frac{1}{1-\mathcal{U}(t^2)}\right] + \frac{\mathcal{E}_1(t^2)}{2[1-\mathcal{U}(t)]}$$

$$\mathcal{E}_g(t) \sim \frac{\delta_g}{\gamma^{4g-1}(1-t/\rho)^{2g-1/2}}$$

Asymptotic Analysis

$$\mathcal{E}_1(t) = \frac{1}{1-\mathcal{U}(t)} - \frac{1}{[1-\mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1-\mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1-\mathcal{U}(t)][1-\mathcal{U}(t^2)]}$$

$$\mathcal{E}_2(t) = \frac{\mathcal{E}_1(t)}{2[1-\mathcal{U}(t)]}\left[\mathcal{E}_1(t) + \frac{\mathcal{U}(t)+\mathcal{U}^2(t)}{[1-\mathcal{U}(t)]^3} - \frac{1}{1-\mathcal{U}(t)} + \frac{1}{1-\mathcal{U}(t^2)}\right] + \frac{\mathcal{E}_1(t^2)}{2[1-\mathcal{U}(t)]}$$

$$\mathcal{E}_g(t) \sim \frac{\delta_g}{\gamma^{4g-1}(1-t/\rho)^{2g-1/2}}$$

$$\delta_1 = \frac{1}{2}$$

$$\mathcal{E}_1(t) \sim \frac{1}{2\gamma^3(1-\frac{t}{\rho})^{\frac{3}{2}}}$$

Asymptotic Analysis

$$\mathcal{E}_1(t) = \frac{1}{1-\mathcal{U}(t)} - \frac{1}{[1-\mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1-\mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1-\mathcal{U}(t)][1-\mathcal{U}(t^2)]}$$

$$\mathcal{E}_2(t) = \frac{\mathcal{E}_1(t)}{2[1-\mathcal{U}(t)]}\left[\mathcal{E}_1(t) + \frac{\mathcal{U}(t)+\mathcal{U}^2(t)}{[1-\mathcal{U}(t)]^3} - \frac{1}{1-\mathcal{U}(t)} + \frac{1}{1-\mathcal{U}(t^2)}\right] + \frac{\mathcal{E}_1(t^2)}{2[1-\mathcal{U}(t)]}$$

$$\mathcal{E}_g(t) \sim \frac{\delta_g}{\gamma^{4g-1}(1-t/\rho)^{2g-1/2}}$$

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$g \geq 2$

$$\delta_g = \frac{1}{2}\sum_{l=1}^{g-1}\left[\delta_l\delta_{g-l} + (l+1)\sum_{d \epsilon C(g-1,l)}\prod_{j=1}^{l}\delta_{d_j}\right]$$

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$$E_{n,g} \sim \frac{2^{0-1}}{(0)!\,\gamma^{0-1}\sqrt{\pi}} n^{0-\frac{3}{2}} \rho^{-n} = \frac{\gamma}{2\sqrt{\pi}} n^{-\frac{3}{2}} \rho^{-n} \sim [t^n]\mathcal{U}(t)$$

Asymptotic Analysis

The subexponential portion $\frac{2^{2g-1}}{(2g)!\gamma^{4g-1}\sqrt{\pi}} n^{2g-3/2} = c_g n^{2g-3/2}$

Number of galls g	**Exact value of c_g**	**Approximate value of c_g**	$n^{2g-3/2}$
0	$\frac{\gamma}{2\sqrt{\pi}}$	0.3188	$n^{-\frac{3}{2}}$
1	$\frac{1}{\gamma^3\sqrt{\pi}}$	0.3910	$n^{\frac{1}{2}}$
2	$\frac{1}{3\gamma^7\sqrt{\pi}}$	0.0799	$n^{\frac{5}{2}}$
3	$\frac{2}{45\gamma^{11}\sqrt{\pi}}$	0.0065	$n^{\frac{9}{2}}$
4	$\frac{1}{315\gamma^{15}\sqrt{\pi}}$	2.8638×10^{-4}	$n^{\frac{13}{2}}$
5	$\frac{2}{14175\gamma^{19}\sqrt{\pi}}$	7.8062×10^{-6}	$n^{\frac{17}{2}}$

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From this, the exponential growth of galled trees with any fixed number of galls is the same as that of unlabeled trees with no galls (different from that of all galled trees).

However, the subexponential growth, with the increase in the number of galls by 1, is greater by a factor of $\frac{4n^2}{\gamma^4(2g+1)(2g+2)}$.

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