Asymptotics of unlabeled galled trees with a fixed number of galls

Lily Agranat-Tamir, Michael Fuchs, Bernhard Gittenberger and Noah Rosenberg
AofA 2024
Introduction – phylogenetic trees

Definition – galled trees

Previous work – the enumeration of galled trees and galled trees with one gall

Novel work – the enumeration of galled trees with a fixed number of galls

Summary
Species Trees
Phylogenetic Trees

Species Trees

[Diagram showing phylogenetic relationships among various species, including Callicebus, Salimni, Pithecus, Lagothrix, Macaca, Colobus, Hylobates, Pongo, Gorilla, Pan, and Homo.]
Phylogenetic Trees

Species Trees

Adapted from http://anthropologyiselemental.ua.edu/osteology.html
Phylogenetic Trees

Species Trees

Adapted from http://anthropologyiselemental.ua.edu/osteology.html
Phylogenetic Trees

Species Trees

Common ancestor

Catta
Saimiri
Pithecus
Lagotrich
Macaca
Colobus
Hylobates
Pongo
Gorilla
Pan
Homo

Time

Adapted from http://anthropologyiselemental.ua.edu/osteology.html
Phylogenetic Trees

Species Trees

Common ancestor

Callithrix
Saimiri
Pithecus
Lagotrix
Macaca
Colobus
Hylobates
Pongo
Gorilla
Pan
Homo

Time

Adapted from http://anthropologyiselemental.ua.edu/osteology.html
Phylogenetic Trees

Species Trees

DNA based Trees

Common ancestor

Adapted from http://anthropologyselemental.ua.edu/osteology.html
Phylogenetic Trees

Species Trees

DNA based Trees

Adapted from http://anthropologyiselemental.ua.edu/osteology.html
Phylogenetic Trees

Species Trees

Common ancestor

DNA based Trees

Early globin gene duplication

Adapted from http://anthropologyiselemental.ua.edu/osteology.html
Phylogenetic Trees

Species Trees

DNA based Trees

Common ancestor

Early globin gene

duplication

duplication

Gorilla-α
Pan-α
Homo-α
Homo-β
Pan-β
Gorilla-β

Adapted from http://anthropologyiselemental.ua.edu/osteology.html
Phylogenetic Trees

Species Trees

Common ancestor

Time

DNA based Trees

Early globin gene
duplication

Gorilla-α
Pan-α
Homo-α
Homo-β
Pan-β
Gorilla-β

P. pernix
P. sufataricus
P. aerophilum

Adapted from http://anthropologyiselemental.ua.edu/osteology.html
Phylogenetic Trees

Archaic Humans

- Neanderthals
- Denisovans
- Anatomically Modern Human
Phylogenetic Trees

Early globin gene

α-chain gene

Gorilla-α  Pan-α  Homo-α  Homo-β  Pan-β  Gorilla-β

β-chain gene
Phylogenetic Trees

Early globin gene

α-chain gene

Pan-α

Homo-α

Gorilla-α

Pan-β

Homo-β

Gorilla-β

β-chain gene

Recombination
Phylogenetic Trees

Early globin gene

α-chains

Gorilla-α
Pan-α
Homo-α

β-chains

Homo-β
Pan-β
Gorilla-β

Recombination
Phylogenetic Trees

- Ph. pernix
- Ph. sufataricus
- Ph. aerophilum
Phylogenetic Trees

Horizontal Gene Transfer

P. pernix
P. sufataricus
P. aerophilum
Phylogenetic Trees

Horizontal Gene Transfer

Transformation
Conjugation
Transduction

Burmeister, 2015, Evol Med Public Health
Phylogenetic Trees
Introduction – phylogenetic trees

**Definition** – galled trees

Previous work – the enumeration of galled trees and galled trees with one gall

**Novel work** – the enumeration of galled trees with a fixed number of galls

Summary
Galled Trees
Galled Trees

A rooted binary tree
Galled Trees

A rooted binary tree  A galled tree

Bienvenu et al 2022, Kong et al 2022
Galled Trees

A rooted binary tree

A galled tree

Bienvenu et al 2022, Kong et al 2022
Galled Trees

A rooted binary tree

A galled tree

Reticulation/
Hybrid node

Bienvenu et al 2022, Kong et al 2022
Galled Trees

A rooted binary tree

A galled tree

Bienvenu et al 2022, Kong et al 2022
Galled Trees

A rooted binary tree

A galled tree

Not a galled tree

Bienvenu et al 2022, Kong et al 2022
Galled Trees

A rooted binary tree

A galled tree

Not a galled tree

Reticulation/
Hybrid node

Hybridizing
nodes

Bienvenu et al 2022, Kong et al 2022
Galled Trees

A rooted binary tree

A galled tree

Not a galled tree

Not a galled tree

Reticulation/
Hybrid node

Hybridizing
nodes

Bienvenu et al 2022, Kong et al 2022
Galled Trees

A rooted binary tree

A galled tree

Not a galled tree

Not a galled tree

Reticulation/ Hybrid node

Hybridizing nodes

Bienvenu et al 2022, Kong et al 2022
A rooted binary tree

A galled tree

Not a galled tree

Not a galled tree

The smallest galled tree

Reticulation/
Hybrid node

Hybridizing
nodes

Bienvenu et al 2022, Kong et al 2022
Galled Trees
Definitions
Galled Trees

Definitions

A rooted galled tree is a rooted binary phylogenetic network in which three properties hold:
Galled Trees

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**First**, each reticulation node $a_r$ has a unique ancestor node $r$ such that exactly two non-overlapping paths of edges exist from $r$ to $a_r$. 

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Ignoring the direction of edges, the two paths connecting $r$ and $a_r$ form a cycle $C_r$, known as a gall.
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Second, the set of nodes in the gall $C_r$, associated with reticulation node $a_r$, and the set of nodes in the gall $C_s$, associated with reticulation node $a_s$, are disjoint.
Definitions

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**Third (time consistency, normality)**, the ancestor node $r$ must be separated from $a_r$ by at least two edges.
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Third (time consistency, normality), the ancestor node $r$ must be separated from $a_r$ by at least two edges.
Lecture Outline

Introduction – phylogenetic trees

Definition – galled trees

Previous work – the enumeration of galled trees and galled trees with one gall

Novel work – the enumeration of galled trees with a fixed number of galls

Summary
Rooted Unlabeled Binary Non-plane Trees
Rooted Unlabeled Binary Non-plane Trees
Rooted Unlabeled Binary Non-plane Trees

\[ U(t) = t + \frac{1}{2} U^2(t) + \frac{1}{2} U(t^2) \]

Otter 1948, Comtet, 1974
Rooted Unlabeled Binary Non-plane Trees

\[ U(t) = t + \frac{1}{2} U^2(t) + \frac{1}{2} U(t^2) \]

Otter 1948, Comtet, 1974

\[ U(t) \approx 1 - \gamma \sqrt{1 - \frac{t}{\rho}} \]

\[ \rho \approx 0.4027 \]

\[ \gamma \approx 1.1301 \]

Landau, 1977; Flajolet and Sedgewick, 2009
Rooted Unlabeled Binary Non-plane Trees

\[ \mathcal{U}(t) = t + \frac{1}{2} \mathcal{U}^2(t) + \frac{1}{2} \mathcal{U}(t^2) \]

\[ \mathcal{U}(t) \sim 1 - \gamma \sqrt{1 - \frac{t}{\rho}} \]

\[ \rho \approx 0.4027 \]
\[ \gamma \approx 1.1301 \]

\[ [t^n] \mathcal{U}(t) \sim \frac{\gamma}{2\sqrt{\pi}} n^{-\frac{3}{2}} \rho^{-n} \]

Otter 1948, Comtet, 1974

Landau, 1977; Flajolet and Sedgewick, 2009
Rooted Unlabeled Binary Non-plane Galled Trees

Agranat-Tamir et al, 2024
Rooted Unlabeled Binary Non-plane Galled Trees

Agranat-Tamir et al., 2024
\[ \mathcal{A}(t) = 1 + t + \frac{1}{2} \mathcal{A}^2(t) + \frac{1}{2} \mathcal{A}(t^2) - \frac{1}{1 - \mathcal{A}(t)} + \frac{\mathcal{A}(t)}{2[1 - \mathcal{A}(t)]^2} + \frac{\mathcal{A}(t)}{2[1 - \mathcal{A}(t^2)]} \]
\[
\mathcal{A}(t) = 1 + t + \frac{1}{2} \mathcal{A}^2(t) + \frac{1}{2} \mathcal{A}(t^2) - \frac{1}{1 - \mathcal{A}(t)} + \frac{\mathcal{A}(t)}{2[1 - \mathcal{A}(t)]^2} + \frac{\mathcal{A}(t)}{2[1 - \mathcal{A}(t^2)]}
\]

\[
[t^n] \mathcal{A}(t) \sim \frac{\delta}{2\sqrt{\pi}} n^{-\frac{3}{2}} r^{-n}
\]

\[ r \approx 0.2073; \ \delta \approx 0.2793 \]
Rooted Unlabeled Binary Non-plane Galled Trees

\[ A(t) = 1 + t + \frac{1}{2} A^2(t) + \frac{1}{2} A(t^2) - \frac{1}{1 - A(t)} + \frac{A(t)}{2[1 - A(t)]^2} + \frac{A(t)}{2[1 - A(t^2)]} \]

\[ [t^n] A(t) \approx \frac{\delta}{2\sqrt{\pi}} n^{-\frac{3}{2}} r^{-n} \]

\[ r \approx 0.2073; \quad \delta \approx 0.2793 \]

\[ \frac{1}{r} \approx 4.82; \quad \frac{1}{\rho} \approx 2.48 \]

Agranat-Tamir et al., 2024
Galled Trees with Exactly One Gall

Agranat-Tamir et al., 2024
$g$ is the number of galls; $n$ is the number of leaves
$g = 1$

$g$ is the number of galls; $n$ is the number of leaves
$g = 1$

$g$ is the number of galls; $n$ is the number of leaves
Galled Trees with Exactly One Gall

\[ n = 1 \]

No trees

\[ g = 1 \]

\( g \) is the number of galls; \( n \) is the number of leaves
Galled Trees with Exactly One Gall

\[ g = 1 \]

\[ n = 1 \]
\[ n = 2 \]

No trees

\( g \) is the number of galls; \( n \) is the number of leaves
Galled Trees with Exactly One Gall

<table>
<thead>
<tr>
<th>$n = 1$</th>
<th>$n = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No trees</td>
<td>No trees</td>
</tr>
</tbody>
</table>

$g = 1$

$g$ is the number of galls; $n$ is the number of leaves
Galled Trees with Exactly One Gall

\[ g = 1 \]

\[ n = 1 \]  \[ n = 2 \]  \[ n = 3 \]

No trees  No trees

\( g \) is the number of galls; \( n \) is the number of leaves
Galled Trees with Exactly One Gall

\( g = 1 \)

\( n = 1 \) \hspace{1cm} \( n = 2 \) \hspace{1cm} \( n = 3 \) \hspace{1cm} \( n = 4 \)

\( g \) is the number of galls; \( n \) is the number of leaves
Galled Trees with Exactly One Gall

$g = 1$

$g$ is the number of galls; $n$ is the number of leaves

$g = 1$
### Galled Trees with Exactly One Gall

<table>
<thead>
<tr>
<th>$n$</th>
<th>Trees</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>No trees</td>
</tr>
<tr>
<td>2</td>
<td>No trees</td>
</tr>
<tr>
<td>3</td>
<td><img src="image" alt="Tree with 3 leaves and 1 gall" /></td>
</tr>
<tr>
<td>4</td>
<td><img src="image" alt="Tree with 4 leaves and 1 gall" /> +14 more</td>
</tr>
<tr>
<td>5</td>
<td><img src="image" alt="Tree with 5 leaves and 1 gall" /> +47 more</td>
</tr>
<tr>
<td>6</td>
<td><img src="image" alt="Tree with 6 leaves and 1 gall" /></td>
</tr>
</tbody>
</table>

$g$ is the number of galls; $n$ is the number of leaves.

Agranat-Tamir et al, 2024
Galled Trees with Exactly One Gall

$g = 1$

$g$ is the number of galls; $n$ is the number of leaves

$\mathcal{E}_1(t) = \frac{1}{1 - \mathcal{U}(t)} - \frac{1}{[1 - \mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)][1 - \mathcal{U}(t^2)]}$

Agranat-Tamir et al., 2024
Galled Trees with Exactly One Gall

\[ \mathcal{E}_1(t) = \frac{1}{1 - \mathcal{U}(t)} - \frac{1}{[1 - \mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)][1 - \mathcal{U}(t^2)]} \]

\[ \mathcal{E}_1(t) \sim \frac{1}{2\gamma^3(1 - \frac{t}{\rho})^2} \]

\( g = 1 \)

\( g \) is the number of galls; \( n \) is the number of leaves

Agranat-Tamir et al., 2024
Galled Trees with Exactly One Gall

\[ \mathcal{E}_1(t) = \frac{1}{1 - \mathcal{U}(t)} - \frac{1}{[1 - \mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)][1 - \mathcal{U}(t^2)]} \]

\[ \mathcal{E}_1(t) \sim \frac{1}{2 \gamma^3 (1 - \frac{t}{\rho})^{\frac{3}{2}}} \]

\[ [t^n] \mathcal{E}_1(t) \sim \frac{1}{\gamma^3 \sqrt{\pi}} n^2 \rho^{-n} \]

\( g \) is the number of galls; \( n \) is the number of leaves

\( g = 1 \)

\( n = 1 \)  No trees

\( n = 2 \)  No trees

\( n = 3 \)

\( n = 4 \)

\( n = 5 \)

\( n = 6 \)  +14 more

Agranat-Tamir et al., 2024
Asymptotic enumeration of unlabeled galled trees with a fixed number of galls
Lecture Outline

Introduction – phylogenetic trees

Definition – galled trees

Previous work – the enumeration of galled trees and galled trees with one gall

**Novel work** – the enumeration of galled trees with a fixed number of galls

Summary
Recursion

No root gall

With a root gall
Recursion

No root gall

With a root gall
Recursion

No root gall

0 galls

With a root gall
No root gall

0 galls

g galls

With a root gall
Recursion

No root gall

- 0 galls
- $m$ leaves

With a root gall

- $g$ galls
- $n - m$ leaves
Recursion

No root gall

- 0 galls
- $m$ leaves

With a root gall

- $g$ galls
- $n - m$ leaves
Recursion

No root gall

- 0 galls
- *m* leaves

With a root gall

- 1 ≤ *j* ≤ *g* − 1 galls
- *g* − *j* galls

- *g* galls
- *n* − *m* leaves
Recursion

No root gall

- 0 galls
  - 0 leaves
  - $m$ leaves

- $g$ galls
  - $g$ leaves
  - $n - m$ leaves

With a root gall

- $1 \leq j \leq g - 1$
  - $g - j$ galls
  - $m$ leaves
  - $n - m$ leaves
Recursion

No root gall

- 0 galls
- 1 ≤ j ≤ g - 1 galls

With a root gall

- g galls
- g - j galls

m leaves

n - m leaves

n - m leaves

1 ≤ j ≤ g - 1

g - j galls
Recursion

No root gall

0 galls
m leaves

g galls
n – m leaves

1 ≤ j ≤ g – 1
galls
m leaves

With a root gall

3 ≤ k ≤ n main subtrees
Recursion

No root gall

0 galls
$m$ leaves

$g$ galls
$n - m$ leaves

$1 \leq j \leq g - 1$
galls
$m$ leaves

$g - j$ galls
$n - m$ leaves

With a root gall

$3 \leq k \leq n$ main subtrees

$k - 2$ possibilities to place the hybrid node
Recursion

No root gall

0 galls

$g$ balls

$m$ leaves

$n - m$ leaves

$1 \leq j \leq g - 1$

galls

$m$ leaves

$n - m$ leaves

With a root gall

$3 \leq k \leq n$ main subtrees

$k - 2$ possibilities to place the hybrid node

$1 \leq l \leq \min\{g - 1, k\}$

main subtrees with galls
Recursion

No root gall

- 0 galls
- \( m \) leaves
- 1 ≤ \( j \) ≤ \( g - 1 \) galls
- \( m \) leaves

With a root gall

- 3 ≤ \( k \) ≤ \( n \) main subtrees
- \( k - 2 \) possibilities to place the hybrid node

- 1 ≤ \( l \) ≤ \( \min\{g - 1, k\} \) main subtrees with galls
- \( m \) leaves
Recursion

No root gall

0 galls
m leaves

1 \leq j \leq g - 1
galls
m leaves

With a root gall

3 \leq k \leq n \text{ main subtrees}

k - 2 possibilities
to place the hybrid node

1 \leq l \leq \min\{g - 1, k\}
main subtrees with galls
m leaves

k - l
main subtrees with 0 galls

1 \leq j \leq g - 1
g - j galls

n - m leaves
Recursion

No root gall

\[ 1 \leq j \leq g - 1 \]
\[ m \text{ leaves} \]
\[ g - j \text{ galls} \]
\[ m \text{ leaves} \]
\[ n - m \text{ leaves} \]

With a root gall

\[ 3 \leq k \leq n \] main subtrees
\[ k - 2 \] possibilities to place the hybrid node

\[ 1 \leq l \leq \min\{g - 1, k\} \]
\[ m \text{ leaves} \]
\[ k - l \]
\[ n - m \text{ leaves} \]
\[ E_1(t) = \frac{1}{1 - u(t)} - \frac{1}{[1 - u(t)]^2} + \frac{u(t)}{2[1 - u(t)]^3} + \frac{u(t)}{2[1 - u(t)][1 - u(t^2)]} \]
Asymptotic Analysis

\[ E_1(t) = \frac{1}{1 - U(t)} - \frac{1}{[1 - U(t)]^2} + \frac{U(t)}{2[1 - U(t)]^3} + \frac{U(t)}{2[1 - U(t)][1 - U(t^2)]} \]

\[ E_2(t) = \frac{E_1(t)}{2[1 - U(t)]} \left[ E_1(t) + \frac{U(t) + U^2(t)}{[1 - U(t)]^3} - \frac{1}{1 - U(t)} + \frac{1}{1 - U(t^2)} \right] + \frac{E_1(t^2)}{2[1 - U(t)]} \]
\[ E_1(t) = \frac{1}{1 - \mathcal{U}(t)} - \frac{1}{[1 - \mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)][1 - \mathcal{U}(t^2)]} \]

\[ E_2(t) = \frac{E_1(t)}{2[1 - \mathcal{U}(t)]} \left[ E_1(t) + \frac{\mathcal{U}(t) + \mathcal{U}^2(t)}{[1 - \mathcal{U}(t)]^3} - \frac{1}{1 - \mathcal{U}(t)} + \frac{1}{1 - \mathcal{U}(t^2)} \right] + \frac{E_1(t^2)}{2[1 - \mathcal{U}(t)]} \]

\[ E_g(t) \sim \frac{\delta_g}{\gamma^{4g-1}(1 - t/\rho)^{2g-1/2}} \]
Asymptotic Analysis

\[ \mathcal{E}_1(t) = \frac{1}{1 - \mathcal{U}(t)} - \frac{1}{[1 - \mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)][1 - \mathcal{U}(t^2)]]} \]

\[ \mathcal{E}_2(t) = \frac{\mathcal{E}_1(t)}{2[1 - \mathcal{U}(t)]} \left[ \mathcal{E}_1(t) + \frac{\mathcal{U}(t) + \mathcal{U}^2(t)}{[1 - \mathcal{U}(t)]^3} - \frac{1}{1 - \mathcal{U}(t)} + \frac{1}{1 - \mathcal{U}(t^2)} \right] + \frac{\mathcal{E}_1(t^2)}{2[1 - \mathcal{U}(t)]} \]

\[ \mathcal{E}_g(t) \sim \frac{\delta_g}{\gamma^{4g-1}(1 - t/\rho)^{2g-1/2}} \]

\[ \delta_1 = \frac{1}{2} \quad \mathcal{E}_1(t) \sim \frac{1}{2\gamma^3(1 - \frac{t}{\rho})^3} \]
Asymptotic Analysis

\[ E_1(t) = \frac{1}{1 - \Upsilon(t)} - \frac{1}{[1 - \Upsilon(t)]^2} + \frac{\Upsilon(t)}{2[1 - \Upsilon(t)]^3} + \frac{\Upsilon(t)}{2[1 - \Upsilon(t)][1 - \Upsilon(t)^2]} \]

\[ E_2(t) = \frac{E_1(t)}{2[1 - \Upsilon(t)]} \left[ E_1(t) + \frac{\Upsilon(t) + \Upsilon^2(t)}{[1 - \Upsilon(t)]^3} - \frac{1}{1 - \Upsilon(t)} + \frac{1}{1 - \Upsilon(t^2)} \right] + \frac{E_1(t^2)}{2[1 - \Upsilon(t)]} \]

\[ E_g(t) \sim \frac{\delta_g}{\gamma^4 g^{-1} (1 - t/\rho)^2 g^{-1/2}} \]

\[ \delta_1 = \frac{1}{2} \]

\[ E_1(t) \sim \frac{1}{2\gamma^3 (1 - t/\rho)^3} \]

\[ g \geq 2 \]

\[ \delta_g = \frac{1}{2} \sum_{l=1}^{g-1} \left[ \delta_l \delta_{g-l} + (l + 1) \sum_{d \in C(g-1, l)} \prod_{j=1}^{l} \delta_d \right] \]
Asymptotic Analysis

\[ E_g(t) \sim \frac{\delta_g}{\gamma^{4g-1}(1 - t/\rho)^{2g-1/2}} \]
$E_{n,g}$ — The number of galled trees with $g$ galls and $n$ leaves

$\mathcal{E}_g(t) \sim \frac{\delta_g}{\gamma^{4g-1}(1 - t/\rho)^{2g-1/2}}$
Asymptotic Analysis

$E_{n,g} \sim \frac{\delta_g}{\gamma^{4g-1} \Gamma(2g - 1/2)} n^{2g - 3/2} \rho^{-n}$

$\mathcal{E}_g(t) \sim \frac{\delta_g}{\gamma^{4g-1} (1 - t/\rho)^{2g - 1/2}}$

$E_{n,g} \quad \text{— The number of galled trees with } g \text{ galls and } n \text{ leaves}$
$E_{n,g}$ — The number of galled trees with $g$ galls and $n$ leaves

$$E_{n,g} \sim \frac{\delta_g}{\gamma^{4g-1} \Gamma(2g - 1/2)} n^{2g - \frac{3}{2} \rho^{-1}}$$

$$E_{n,g} \sim \frac{2^{2g-1} \delta_g}{\gamma^{4g-1} (4g - 3)!! \sqrt{\pi}} n^{2g - \frac{3}{2} \rho^{-1}}$$

$$\mathcal{E}_g(t) \sim \frac{\delta_g}{\gamma^{4g-1} (1 - t/\rho)^{2g-1/2}}$$
Asymptotic Analysis

\[ E_{n,g} \sim \frac{2^{2g-1} \delta_g}{\gamma^{4g-1}(4g-3)!!\sqrt{\pi}} n^{2g-3} \rho^{-n} \]
Asymptotic Analysis

\[ 2^{2g-1} \delta_g = C_{2g-1} \]

\[ C_m \text{ is the } m^{th} \text{ Catalan number} \]

\[ \delta_g = \frac{1}{2} \sum_{l=1}^{g-1} \left[ \delta_l \delta_{g-l} + (l+1) \sum_{d \in \mathbb{G}(g-l, l)} \prod_{j=1}^{l} \delta_d \right] \]

\[ E_{n,g} \sim \frac{2^{2g-1} \delta_g}{\gamma^{4g-1}(4g-3)!! \sqrt{\pi}} n^{2g-\frac{3}{2} \rho - n} \]
Asymptotic Analysis

\[ 2^{2g-1}\delta_g = C_{2g-1} \]

\[ C_m \text{ is the } m^{th} \text{ Catalan number} \]

\[ C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42, \ldots \]

\[ \delta_g = \frac{1}{2} \sum_{l=1}^{g-1} \delta_l \delta_{g-l} + \sum_{d \in \mathbb{C}_{g-1,l}} \prod_{j=1}^{l} \delta_j \]

\[ E_{n,g} \sim \frac{2^{2g-1}\delta_g}{\gamma^{4g-1}(4g-3)!!} \sqrt{\pi} n^{2g-3} \rho^{-n} \]
Asymptotic Analysis

\[ 2^{2g-1} \delta_g = C_{2g-1} \]

\[ \delta_g = \frac{1}{2} \sum_{l=1}^{g-1} \left[ \delta_l \delta_{g-l} + (l+1) \sum_{d \in \{g-l\}} \prod_{j=1}^{l} \delta_d \right] \]

\[ C_m \text{ is the } m^{th} \text{ Catalan number} \]

\[ C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42, \ldots \]

\[ C_m = \frac{2^m (2m-1)!!}{(m+1)!} \]

\[ E_{n,g} \sim \frac{2^{2g-1} \delta_g}{\gamma^{4g-1}(4g-3)!!\sqrt{\pi}} n^{2g-\frac{3}{2}} \rho^{-n} \]
Asymptotic Analysis

\[ 2^{2g-1} \delta_g = C_{2g-1} \]

\[ \delta_g = \frac{1}{2} \sum_{l=1}^{g-1} \delta_l \delta_{g-l} + (l+1) \sum_{d \in \{g-l, l\}} \prod_{j=1}^{l} \delta_d \]

\[ C_m \text{ is the } m^{th} \text{ Catalan number} \]

\[ C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42, ... \]

\[ C_m = \frac{2^m (2m - 1)!!}{(m + 1)!} \]

\[ E_{n,g} \sim \frac{2^{2g-1}}{(2g)! \gamma^{4g-1} \sqrt{\pi}} n^{2g-3} \frac{3}{2} \rho^{-n} \]

\[ E_{n,g} \sim \frac{2^{2g-1} \delta_g}{\gamma^{4g-1} (4g - 3)!! \sqrt{\pi}} n^{2g-3} \frac{3}{2} \rho^{-n} \]
Asymptotic Analysis
Asymptotic Analysis

\[ E_{n,g} \sim \frac{2^{2g-1}}{(2g)!} \gamma^{4g-1} \sqrt{\pi} n^{2g-3/2} \rho^{-n} \]
Asymptotic Analysis

\[ E_{n,g} \sim \frac{2^{2g-1}}{(2g)!} \gamma^{4g-1} \sqrt{\frac{1}{\pi}} n^{2g-3} 2^{\rho-n} \]

This includes \( g = 0 \)
\[ E_{n,g} \sim \frac{2^{2g-1}}{(2g)! \gamma^{4g-1} \sqrt{\pi}} n^{2g-3} \rho^{-n} \]

This includes \( g = 0 \)

\[ E_{n,0} \sim \frac{2^{0-1}}{(0)! \gamma^{0-1} \sqrt{\pi}} n^{0-3} \rho^{-n} = \frac{\gamma}{2 \sqrt{\pi}} n^{-3} \rho^{-n} \sim [t^n] \mathcal{U}(t) \]
Asymptotic Analysis

The subexponential portion 
\[ \frac{2^{2g-1}}{(2g)!^{1/4} \sqrt[2]{\pi}} n^{2g-3/2} = c_g n^{2g-3/2} \]

<table>
<thead>
<tr>
<th>Number of galls ( g )</th>
<th>Exact value of ( c_g )</th>
<th>Approximate value of ( c_g )</th>
<th>( n^{2g-3/2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \frac{\gamma}{2\sqrt{\pi}} )</td>
<td>0.3188</td>
<td>( n^{3/2} )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{\gamma^3 \sqrt{\pi}} )</td>
<td>0.3910</td>
<td>( \frac{1}{n^2} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{3\gamma^7 \sqrt{\pi}} )</td>
<td>0.0799</td>
<td>( \frac{5}{n^2} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{2}{45\gamma^{11} \sqrt{\pi}} )</td>
<td>0.0065</td>
<td>( \frac{9}{n^2} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{1}{315\gamma^{15} \sqrt{\pi}} )</td>
<td>2.8638 \times 10^{-4}</td>
<td>( \frac{13}{n^2} )</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{2}{14175\gamma^{19} \sqrt{\pi}} )</td>
<td>7.8062 \times 10^{-6}</td>
<td>( \frac{17}{n^2} )</td>
</tr>
</tbody>
</table>
We show an explicit formula for the asymptotic enumeration of unlabeled binary non-plane galled trees with exactly $g$ galls. This includes unlabeled binary non-plane trees with no galls.
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From this, the exponential growth of galled trees with any fixed number of galls is the same as that of unlabeled trees with no galls (different from that of all galled trees).
We show an explicit formula for the asymptotic enumeration of unlabeled binary non-plane galled trees with exactly $g$ galls. This includes unlabeled binary non-plane trees with no galls.

From this, the exponential growth of galled trees with any fixed number of galls is the same as that of unlabeled trees with no galls (different from that of all galled trees).

However, the subexponential growth, with the increase in the number of galls by 1, is greater by a factor of $\frac{4n^2}{\gamma^4(2g+1)(2g+2)}$. 
Thank You
Acknowledgements

Noah Rosenberg

Michael Fuchs  National Chengchi University

Bernhard Gittenberger  Technische Universität Wien

L to R: Egor Lappo, Chloe Shiff, Xiran Liu, Noah Rosenberg, Kaleda Denton, Maike Morrison, Lily Agranat-Tamir

not pictured: Daniel Cotter, Juan Esteban Rodriguez Rodriguez, Kennedy Agwamba, Zarif Ahsan, Emily Dickey, Bradley Moon, Michael Doboli, Anna Lyubarskaja, Daniel Bauman
Asymptotics of unlabeled galled trees with a fixed number of galls

Lily Agranat-Tamir, Michael Fuchs, Bernhard Gittenberger and Noah Rosenberg
AofA 2024
Introduction – phylogenetic trees

Definition – galled trees

Previous work – the enumeration of galled trees and galled trees with one gall

Novel work – the enumeration of galled trees with a fixed number of galls

Summary
Phylogenetic Trees

Species Trees
Phylogenetic Trees

Species Trees
Phylogenetic Trees

Species Trees

Adapted from http://anthropologyiselemental.ua.edu/osteology.html
Phylogenetic Trees

Species Trees

Adapted from http://anthropologyiselemental.ua.edu/osteology.html
Phylogenetic Trees

Species Trees

Common ancestor

Time

Adapted from http://anthropology.seelemental.ua.edu/osteology.html
Phylogenetic Trees

Species Trees

DNA based Trees

Adapted from http://anthropologyis elemental.ua.edu/osteology.html
Phylogenetic Trees

Species Trees

DNA based Trees

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Phylogenetic Trees

Species Trees

Common ancestor

Callithrix
Saimiri
Pithecus
Lagothrix
Macaca
Colobus
Hyllobates
Pongo
Gorilla
Pan
Homo

DNA based Trees

Time

Early globin gene
duplication

Gorilla-α
Pan-α
Homo-α
Homo-β
Pan-β
Gorilla-β

Adapted from http://anthropologyiselemental.ua.edu/osteology.html
Phylogenetic Trees

Species Trees

DNA based Trees

Common ancestor

Time

Early globin gene
duplication

Adapted from http://anthropologyiselemental.ua.edu/osteology.html
Phylogenetic Trees

Species Trees

Common ancestor

DNA based Trees

Early globin gene

duplication

Gorilla-α

Pan-α

Homo-α

Homo-β

Pan-β

Gorilla-β

P. pernix

P. sufataricus

P. aerophilum

Adapted from http://anthropologyiselemental.ua.edu/osteology.html
Phylogenetic Trees

Archaic Humans

Neanderthals

Denisovans

Anatomically Modern Human
Phylogenetic Trees

https://humanorigins.si.edu/evidence/human-fossils/species/homo-neanderthalensis
https://www.sciencefocus.com/science/denisovans
https://www.britannica.com/biography/Jane-Austen
Phylogenetic Trees

https://www.worldhistory.org/article/1070/early-human-migration/

https://humanorigins.si.edu/evidence/human-fossils/species/homo-neanderthalensis
https://www.sciencefocus.com/science/denisovans
https://www.britannica.com/biography/Jane-Austen
Phylogenetic Trees

Early globin gene

α-chain gene

Gorilla-α

Pan-α

Homo-α

Homo-β

Pan-β

Gorilla-β
Phylogenetic Trees

Early globin gene

- α-chain gene
- β-chain gene

Gorilla-α
Pan-α
Homo-α
Homo-β
Pan-β
Gorilla-β

Recombination
Phylogenetic Trees

Early globin gene

Gorilla-α  Pan-α  Homo-α  Homo-β  Pan-β  Gorilla-β

Recombination
Phylogenetic Trees

- *P. aerophilum*
- *P. sufataricus*
- *P. pernix*
Phylogenetic Trees

Horizontal Gene Transfer

P. pernix  P. sufataricus  P. aerophilum
Phylogenetic Trees

P. aerophilum

P. sufaricus

P. pernix

Horizontal Gene Transfer

Transformation

Conjugation

Transduction

Burmeister, 2015, Evol Med Public Health
Phylogenetic Trees

https://www.worldhistory.org/article/1070/early-human-migration/
Phylogenetic Trees

Admixture

https://www.worldhistory.org/article/1070/early-human-migration/
Introduction – phylogenetic trees

**Definition** – galled trees

Previous work – the enumeration of galled trees and galled trees with one gall

Novel work – the enumeration of galled trees with a fixed number of galls

Summary
Galled Trees

A rooted binary tree
Galled Trees

A rooted binary tree

A galled tree

Bienvenu et al 2022, Kong et al 2022
Galled Trees

A rooted binary tree

A galled tree

Bienvenu et al 2022, Kong et al 2022
Galled Trees

A rooted binary tree

A galled tree

Reticulation/
Hybrid node

Bienvenu et al 2022, Kong et al 2022
Galled Trees

A rooted binary tree

A galled tree

Bienvenu et al 2022, Kong et al 2022
Galled Trees

A rooted binary tree

A galled tree

Not a galled tree

Reticulation/
Hybrid node

Hybridizing
nodes

Bienvenu et al 2022, Kong et al 2022
Galled Trees

A rooted binary tree

A galled tree

Not a galled tree

Bienvenu et al 2022, Kong et al 2022
Galled Trees

A rooted binary tree

A galled tree

Not a galled tree

Not a galled tree

Bienvenu et al 2022, Kong et al 2022
Galled Trees

A rooted binary tree  A galled tree  Not a galled tree  Not a galled tree

Bienvenu et al 2022, Kong et al 2022
A rooted binary tree

A galled tree

Not a galled tree

Not a galled tree

The smallest galled tree

Reticulation/Hybrid node

Hybridizing nodes

Bienvenu et al 2022, Kong et al 2022
Galled Trees
Definitions
A rooted galled tree is a rooted binary phylogenetic network in which three properties hold:
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First, each reticulation node $a_r$ has a unique ancestor node $r$ such that exactly two non-overlapping paths of edges exist from $r$ to $a_r$. 
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Ignoring the direction of edges, the two paths connecting $r$ and $a_r$ form a cycle $C_r$, known as a gall.
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Second, the set of nodes in the gall $C_r$, associated with reticulation node $a_r$, and the set of nodes in the gall $C_s$, associated with reticulation node $a_s$, are disjoint.
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Third (time consistency, normality), the ancestor node \( r \) must be separated from \( a_r \) by at least two edges.
Galled Trees

**Definitions**

*A rooted galled tree* is a rooted binary phylogenetic network in which three properties hold:

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https://discoverandshare.org/2021/06/24/all-about-galls/
Lecture Outline

Introduction – phylogenetic trees

Definition – galled trees

**Previous work** – the enumeration of galled trees and galled trees with one gall

**Novel work** – the enumeration of galled trees with a fixed number of galls

Summary
Rooted Unlabeled Binary Non-plane Trees
Rooted Unlabeled Binary Non-plane Trees
Rooted Unlabeled Binary Non-plane Trees

\[ u(t) = t + \frac{1}{2} u^2(t) + \frac{1}{2} u(t^2) \]

Otter 1948, Comtet, 1974
Rooted Unlabeled Binary Non-plane Trees

\[ u(t) = t + \frac{1}{2} u^2(t) + \frac{1}{2} u(t^2) \]

Landau, 1977; Flajolet and Sedgewick, 2009

\[ u(t) \sim 1 - \gamma \sqrt{1 - \frac{t}{\rho}} \]

\[ \rho \approx 0.4027 \]
\[ \gamma \approx 1.1301 \]

Otter 1948, Comtet, 1974

Landau, 1977; Flajolet and Sedgewick, 2009
Rooted Unlabeled Binary Non-plane Trees

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\[ u(t) \sim 1 - \gamma \sqrt{1 - \frac{t}{\rho}} \]

\[ \rho \approx 0.4027 \]
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\[ [t^n] u(t) \sim \frac{\gamma}{2\sqrt{\pi}} n^{-\frac{3}{2}} \rho^{-n} \]

Landau, 1977; Flajolet and Sedgewick, 2009
Rooted Unlabeled Binary Non-plane Galled Trees
Rooted Unlabeled Binary Non-plane Galled Trees

Agranat-Tamir et al., 2024
\[ \mathcal{A}(t) = 1 + t + \frac{1}{2} \mathcal{A}^2(t) + \frac{1}{2} \mathcal{A}(t^2) - \frac{1}{1 - \mathcal{A}(t)} + \frac{\mathcal{A}(t)}{2[1 - \mathcal{A}(t)]^2} + \frac{\mathcal{A}(t)}{2[1 - \mathcal{A}(t^2)]} \]
Rooted Unlabeled Binary Non-plane Galled Trees

\[ A(t) = 1 + t + \frac{1}{2} A^2(t) + \frac{1}{2} A(t^2) - \frac{1}{1 - A(t)} + \frac{A(t)}{2[1 - A(t)]^2} + \frac{A(t)}{2[1 - A(t^2)]} \]

\[ [t^n] A(t) \sim \frac{\delta}{2\sqrt{\pi}} n^{-3} r^{-n} \]

\[ r \approx 0.2073; \quad \delta \approx 0.2793 \]
Rooted Unlabeled Binary Non-plane Galled Trees

\[
\mathcal{A}(t) = 1 + t + \frac{1}{2} \mathcal{A}(t)^2 + \frac{1}{2} \mathcal{A}(t^2) - \frac{1}{1 - \mathcal{A}(t)} + \frac{\mathcal{A}(t)}{2[1 - \mathcal{A}(t)]^2} + \frac{\mathcal{A}(t)}{2[1 - \mathcal{A}(t^2)]}
\]

\[
[t^n] \mathcal{A}(t) \sim \frac{\delta}{2\sqrt{\pi}} n^{-\frac{3}{2}} r^{-n}
\]

\[
r \approx 0.2073; \quad \delta \approx 0.2793
\]

\[
\frac{1}{r} \approx 4.82; \quad \frac{1}{\rho} \approx 2.48
\]
Galled Trees with Exactly One Gall

Agranat-Tamir et al, 2024
$g$ is the number of galls; $n$ is the number of leaves
Galled Trees with Exactly One Gall

\[ g = 1 \]

\( g \) is the number of galls; \( n \) is the number of leaves
$g = 1$

$g$ is the number of galls; $n$ is the number of leaves
Galled Trees with Exactly One Gall

\[ n = 1 \]

No trees

\[ g = 1 \]

\( g \) is the number of galls; \( n \) is the number of leaves
Galled Trees with Exactly One Gall

<table>
<thead>
<tr>
<th>$n = 1$</th>
<th>$n = 2$</th>
</tr>
</thead>
</table>

| $g = 1$ |

No trees

$g$ is the number of galls; $n$ is the number of leaves
Galled Trees with Exactly One Gall

<table>
<thead>
<tr>
<th>$g$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

$g$ is the number of galls; $n$ is the number of leaves.
Galled Trees with Exactly One Gall

\( g = 1 \)
\( n = 1 \)
\( n = 2 \)
\( n = 3 \)

No trees

\( g \) is the number of galls; \( n \) is the number of leaves
Galled Trees with Exactly One Gall

$g = 1$

$g$ is the number of galls; $n$ is the number of leaves
Galled Trees with Exactly One Gall

\( g = 1 \)

\( n = 1 \)  \hspace{1cm} \( n = 2 \)  \hspace{1cm} \( n = 3 \)  \hspace{1cm} \( n = 4 \)  \hspace{1cm} \( n = 5 \)

- No trees
- No trees

\( g \) is the number of galls; \( n \) is the number of leaves

+14 more

Agranat-Tamir et al., 2024
Galled Trees with Exactly One Gall

\[ g = 1 \]

\[ n = 1 \] No trees
\[ n = 2 \] No trees
\[ n = 3 \]
\[ n = 4 \] +14 more
\[ n = 5 \] +47 more
\[ n = 6 \]

\( g \) is the number of galls; \( n \) is the number of leaves

Agranat-Tamir et al, 2024
Galled Trees with Exactly One Gall

\[ g = 1 \]

\[ n = 1 \quad n = 2 \quad n = 3 \quad n = 4 \quad n = 5 \quad n = 6 \]

\[ g \text{ is the number of galls; } n \text{ is the number of leaves} \]

\[ \mathcal{E}_1(t) = \frac{1}{1 - \mathcal{U}(t)} - \frac{1}{[1 - \mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)][1 - \mathcal{U}(t^2)]} \]

Agranat-Tamir et al., 2024
Galled Trees with Exactly One Gall

\[ \mathcal{E}_1(t) = \frac{1}{1 - \mathcal{U}(t)} - \frac{1}{[1 - \mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)][1 - \mathcal{U}(t^2)]} \]

\[ \mathcal{E}_1(t) \sim \frac{1}{2y^3(1 - t^3 / \rho^2)} \]

\( g = 1 \)  
No trees  
No trees

\( g \) is the number of galls; \( n \) is the number of leaves

\( n = 1 \)  
\( n = 2 \)  
\( n = 3 \)  
\( n = 4 \)  
\( n = 5 \)  
\( n = 6 \)


Agranat-Tamir et al., 2024
No trees

\( g = 1 \)

\( n = 1 \) \hspace{1cm} n = 2 \hspace{1cm} n = 3 \hspace{1cm} n = 4 \hspace{1cm} n = 5 \hspace{1cm} n = 6 \\

No trees \hspace{1cm} No trees \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} +14 more

\( g \) is the number of galls; \( n \) is the number of leaves

\[ E_1(t) = \frac{1}{1 - \mathcal{U}(t)} - \frac{1}{[1 - \mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)][1 - \mathcal{U}(t^2)]} \]

\[ E_1(t) \sim \frac{1}{2\gamma^3(1 - \frac{t}{\rho})^\frac{3}{2}} \]

\[ [t^n]E_1(t) \sim \frac{1}{\gamma^3\sqrt{\pi}} n^\frac{3}{2} \rho^{-n} \]
Asymptotic enumeration of unlabeled galled trees with a fixed number of galls
Introduction – phylogenetic trees

Definition – galled trees

Previous work – the enumeration of galled trees and galled trees with one gall

**Novel work** – the enumeration of galled trees with a fixed number of galls

Summary
Recursion

No root gall

With a root gall
Recursion

No root gall

With a root gall
Recursion

No root gall

0 galls

With a root gall
Recursion

No root gall

0 galls

$g$ galls

With a root gall
Recursion

No root gall

0 galls

$g$ galls

$m$ leaves

$n - m$ leaves

With a root gall
Recursion

No root gall

- 0 galls
- $m$ leaves

With a root gall

- $g$ galls
- $n - m$ leaves
Recursion

No root gall

0 galls
$m$ leaves

$g$ galls
$n - m$ leaves

With a root gall

$1 \leq j \leq g - 1$
galls

$g - j$ galls
Recursion

No root gall

\[
\begin{align*}
0 \text{ galls} & \quad m \text{ leaves} \\
g \text{ galls} & \quad n - m \text{ leaves}
\end{align*}
\]

With a root gall

\[
\begin{align*}
1 \leq j \leq g - 1 \quad g - j \text{ galls} & \quad m \text{ leaves} \\
& \quad n - m \text{ leaves}
\end{align*}
\]
Recursion

No root gall

\[
0 \text{ galls} \\
m \text{ leaves}
\]

\[
g \text{ galls} \\
(n - m) \text{ leaves}
\]

1 \leq j \leq g - 1

g - j \text{ galls} \\
m \text{ leaves} \\
n - m \text{ leaves}

With a root gall

\[
\text{No root gall}
\]

\[
\text{With a root gall}
\]
Recursion

No root gall

- 0 galls
  - $m$ leaves
- $g$ galls
  - $n - m$ leaves

With a root gall

- $3 \leq k \leq n$ main subtrees

- $1 \leq j \leq g - 1$
  - galls
  - $m$ leaves
- $g - j$ galls
  - $n - m$ leaves
Recursion

No root gall

- 0 galls
  - m leaves
- g galls
  - n - m leaves

With a root gall

- 3 ≤ k ≤ n main subtrees
- k - 2 possibilities to place the hybrid node

1 ≤ j ≤ g - 1 galls
  - m leaves
- g - j galls
  - n - m leaves
Recursion

No root gall

- 0 galls, $m$ leaves
- $g$ galls, $n - m$ leaves

1 ≤ $j$ ≤ $g - 1$ galls, $m$ leaves
- $g - j$ galls, $n - m$ leaves

With a root gall

- 3 ≤ $k$ ≤ $n$ main subtrees
- $k - 2$ possibilities to place the hybrid node

1 ≤ $l$ ≤ min{$g - 1, k$} main subtrees with galls

No root gall

- 0 galls, $m$ leaves
- $g$ galls, $n - m$ leaves

1 ≤ $j$ ≤ $g - 1$ galls, $m$ leaves
- $g - j$ galls, $n - m$ leaves

With a root gall

- 3 ≤ $k$ ≤ $n$ main subtrees
- $k - 2$ possibilities to place the hybrid node

1 ≤ $l$ ≤ min{$g - 1, k$} main subtrees with galls
Recursion

No root gall

- 0 galls, $m$ leaves
- $g$ galls, $n - m$ leaves
- $1 \leq j \leq g - 1$ galls, $m$ leaves
- $g - j$ galls, $n - m$ leaves

With a root gall

- $3 \leq k \leq n$ main subtrees
- $k - 2$ possibilities to place the hybrid node

- $1 \leq l \leq \min\{g - 1, k\}$ main subtrees with galls
  - $m$ leaves
Recursion

No root gall

0 galls
\( m \) leaves

\( g \) galls
\( n - m \) leaves

1 \( \leq j \) \( \leq g - 1 \)
galls
\( m \) leaves

\( g - j \) galls
\( n - m \) leaves

With a root gall

3 \( \leq k \) \( \leq n \) main subtrees

\( k - 2 \) possibilities
to place the hybrid node

1 \( \leq l \) \( \leq \min\{g - 1, k\} \)
main subtrees with galls
\( m \) leaves

\( k - l \)
main subtrees with 0 galls
Recursion

No root gall

1 ≤ j ≤ g − 1
galls

0 galls

m leaves

With a root gall

3 ≤ k ≤ n main subtrees

k − 2 possibilities
to place the hybrid node

1 ≤ l ≤ min\{g − 1, k\}
main subtrees with galls

m leaves

k − l
main subtrees with 0 galls

n − m leaves

1 ≤ j ≤ g − 1
galls

m leaves

g − j galls

n − m leaves
Asymptotic Analysis

\[ E_1(t) = \frac{1}{1 - \mathcal{U}(t)} - \frac{1}{[1 - \mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)][1 - \mathcal{U}(t^2)]} \]
\[
\begin{align*}
\mathcal{E}_1(t) &= \frac{1}{1 - \mathcal{U}(t)} - \frac{1}{[1 - \mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)][1 - \mathcal{U}(t)^2]} \\
\mathcal{E}_2(t) &= \frac{\mathcal{E}_1(t)}{2[1 - \mathcal{U}(t)]} \left[ \mathcal{E}_1(t) + \frac{\mathcal{U}(t) + \mathcal{U}^2(t)}{[1 - \mathcal{U}(t)]^3} - \frac{1}{1 - \mathcal{U}(t)} + \frac{1}{1 - \mathcal{U}(t)^2} \right] + \frac{\mathcal{E}_1(t^2)}{2[1 - \mathcal{U}(t)]}
\end{align*}
\]
Asymptotic Analysis

\[ E_1(t) = \frac{1}{1 - \mathcal{U}(t)} - \frac{1}{[1 - \mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)][1 - \mathcal{U}(t)^2]} \]

\[ E_2(t) = \frac{E_1(t)}{2[1 - \mathcal{U}(t)]} \left[ E_1(t) + \frac{\mathcal{U}(t) + \mathcal{U}^2(t)}{[1 - \mathcal{U}(t)]^3} - \frac{1}{1 - \mathcal{U}(t)} + \frac{1}{1 - \mathcal{U}(t)^2} \right] + \frac{E_1(t^2)}{2[1 - \mathcal{U}(t)]} \]

\[ \mathcal{E}_g(t) \sim \frac{\delta_g}{\gamma^{4g-1}(1-t/\rho)^{2g-1/2}} \]
Asymptotic Analysis

\[ E_1(t) = \frac{1}{1 - U(t)} - \frac{1}{[1 - U(t)]^2} + \frac{U(t)}{2[1 - U(t)]^3} + \frac{U(t)}{2[1 - U(t)][1 - U(t)^2]} \]

\[ E_2(t) = \frac{E_1(t)}{2[1 - U(t)]} \left[ E_1(t) + \frac{U(t) + U^2(t)}{[1 - U(t)]^3} - \frac{1}{1 - U(t)} + \frac{1}{1 - U(t)^2} \right] + \frac{E_1(t^2)}{2[1 - U(t)]} \]

\[ E_g(t) \sim \frac{\delta_g}{\gamma^{4g-1}(1 - t/\rho)^{2g-1/2}} \]

\[ \delta_1 = \frac{1}{2} \quad E_1(t) \sim \frac{1}{2\gamma^3(1 - t^3/\rho^3)} \]
Asymptotic Analysis

$$\mathcal{E}_1(t) = \frac{1}{1 - \mathcal{U}(t)} - \frac{1}{[1 - \mathcal{U}(t)]^2} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)]^3} + \frac{\mathcal{U}(t)}{2[1 - \mathcal{U}(t)][1 - \mathcal{U}(t)^2]}$$

$$\mathcal{E}_2(t) = \frac{\mathcal{E}_1(t)}{2[1 - \mathcal{U}(t)]} \left[ \mathcal{E}_1(t) + \frac{\mathcal{U}(t) + \mathcal{U}(t)^2}{[1 - \mathcal{U}(t)]^3} - \frac{1}{1 - \mathcal{U}(t)} + \frac{1}{1 - \mathcal{U}(t)^2} \right] + \frac{\mathcal{E}_1(t)^2}{2[1 - \mathcal{U}(t)]}$$

$$\mathcal{E}_g(t) \sim \frac{\delta_g}{\gamma^4 g^{-1} (1 - t / \rho)^2 g^{-1/2}}$$

$$\delta_1 = \frac{1}{2} \quad \mathcal{E}_1(t) \sim \frac{1}{2 \gamma^3 (1 - \frac{t}{\rho})^\frac{3}{2}}$$

$$g \geq 2$$

$$\delta_g = \frac{1}{2} \sum_{l=1}^{g-1} \left[ \delta_l \delta_{g-l} + (l + 1) \sum_{d \in \mathcal{C}(g-1,l)} \prod_{j=1}^{l} \delta_{d_j} \right]$$
\[ \varepsilon_g(t) \sim \frac{\delta_g}{\gamma^{4g-1}(1 - t/\rho)^{2g-1/2}} \]
$E_{n,g}$ — The number of galled trees with $g$ galls and $n$ leaves

$\mathcal{E}_g(t) \sim \frac{\delta_g}{\gamma^{4g-1}(1 - t/\rho)^{2g-1/2}}$
$E_{n,g} \sim \frac{\delta_g}{\gamma^{4g-1}\Gamma(2g - 1/2)} n^{2g-3/2} \rho^{-n}$

$E_{n,g} -$ The number of galled trees with $g$ galls and $n$ leaves
$E_{n,g} \sim \delta_g \frac{\Gamma(2g - 1/2)}{\gamma^{4g-1}(2g - 1/2)} n^{2g-3/2} \rho^{-n}$

$E_{n,g} \sim \frac{2^{2g-1} \delta_g}{\gamma^{4g-1}(4g - 3)!! \sqrt{\pi}} n^{2g-3/2} \rho^{-n}$

$\mathcal{E}_g(t) \sim \frac{\delta_g}{\gamma^{4g-1}(1 - t/\rho)^{2g-1/2}}$
Asymptotic Analysis

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Asymptotic Analysis

\[ 2^{2g-1} \delta_g = C_{2g-1} \]

\[ \delta_g = \frac{1}{2} \sum_{l=1}^{g-1} \left[ \delta_l \delta_{g-l} + (l+1) \sum_{d \in \text{dec}(g-1,l)} \prod_{j=1}^{l} \delta_d \right] \]

\[ E_{n,g} \sim \frac{2^{2g-1} \delta_g}{\gamma^{4g-1}(4g-3)!! \sqrt{\pi}} n^{2g-3} \rho^{-n} \]

\( C_m \) is the \( m^{th} \) Catalan number.
Asymptotic Analysis

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\[ C_m \text{ is the } m^{th} \text{ Catalan number} \]

\[ c_0 = 1, c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 14, c_5 = 42, \ldots \]

\[ E_{n,g} \sim \frac{2^{2g-1} \delta_g}{\gamma^{4g-1} (4g-3)!! \sqrt{\pi}} n^{2g-3/2} \rho^{-n} \]
Asymptotic Analysis

\[ 2^{2g-1} \delta_g = C_{2g-1} \quad \text{\(C_m\) is the \(m^{th}\) Catalan number} \]
\[ C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42, \ldots \]

\[ \delta_g = \frac{1}{2} \sum_{l=1}^{g-1} \left[ \delta_l \delta_{g-l} + (l+1) \sum_{\delta \in \mathfrak{g}_{g-1, l}} \prod_{j=1}^{l} \delta_j \right] \]

\[ C_m = \frac{2^m (2m - 1)!!}{(m + 1)!} \]

\[ E_{n,g} \sim \frac{2^{2g-1} \delta_g}{\gamma^{4g-1}(4g-3)!!} \sqrt{\frac{n}{\pi}} n^{2g-\frac{3}{2}} \rho^{-n} \]
Asymptotic Analysis

\[ 2^{2g-1} \delta_g = C_{2g-1} \]

\[ \delta_g = \frac{1}{2} \sum_{l=1}^{g-1} \left[ \delta_l \delta_{g-l} + (l+1) \sum_{\text{dec}(g-l, l)} \prod_{j=1}^{l} \delta_j \right] \]

\[ C_m \] is the \( m \)th Catalan number

\[ C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42, ... \]

\[ C_m = \frac{2^m(2m-1)!!}{(m+1)!} \]

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This includes $g = 0$
Asymptotic Analysis

\[ E_{n,g} \sim \frac{2^{2g-1}}{(2g)! \gamma^{4g-1} \sqrt{\pi}} n^{2g-\frac{3}{2}} \rho^{-n} \]

This includes \( g = 0 \)

\[ E_{n,g} \sim \frac{2^{0-1}}{(0)! \gamma^{0-1} \sqrt{\pi}} n^{0-\frac{3}{2}} \rho^{-n} = \frac{\gamma}{2\sqrt{\pi}} n^{-\frac{3}{2}} \rho^{-n} \sim [t^n] \mathcal{U}(t) \]
Asymptotic Analysis

The subexponential portion \( \frac{2^{2g-1}}{(2g)!\sqrt{\pi}} n^{2g-3/2} = c_g n^{2g-3/2} \)

<table>
<thead>
<tr>
<th>Number of galls ( g )</th>
<th>Exact value of ( c_g )</th>
<th>Approximate value of ( c_g )</th>
<th>( n^{2g-3/2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \frac{\gamma}{2\sqrt{\pi}} )</td>
<td>0.3188</td>
<td>( n^{-\frac{3}{2}} )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{\gamma^3\sqrt{\pi}} )</td>
<td>0.3910</td>
<td>( \frac{1}{n^2} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{3\gamma^7\sqrt{\pi}} )</td>
<td>0.0799</td>
<td>( \frac{5}{n^2} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{2}{45\gamma^{11}\sqrt{\pi}} )</td>
<td>0.0065</td>
<td>( \frac{9}{n^2} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{1}{315\gamma^{15}\sqrt{\pi}} )</td>
<td>( 2.8638 \times 10^{-4} )</td>
<td>( \frac{13}{n^2} )</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{2}{14175\gamma^{19}\sqrt{\pi}} )</td>
<td>( 7.8062 \times 10^{-6} )</td>
<td>( \frac{17}{n^2} )</td>
</tr>
</tbody>
</table>
We show an explicit formula for the asymptotic enumeration of unlabeled binary non-plane galled trees with exactly $g$ galls. This includes unlabeled binary non-plane trees with no galls.
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From this, the exponential growth of galled trees with any fixed number of galls is the same as that of unlabeled trees with no galls (different from that of all galled trees).
We show an explicit formula for the asymptotic enumeration of unlabeled binary non-plane galled trees with exactly $g$ galls. This includes unlabeled binary non-plane trees with no galls.

From this, the exponential growth of galled trees with any fixed number of galls is the same as that of unlabeled trees with no galls (different from that of all galled trees).

However, the subexponential growth, with the increase in the number of galls by 1, is greater by a factor of $\frac{4n^2}{\gamma^4(2g+1)(2g+2)}$. 
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