Prague Dimension of Random Graphs

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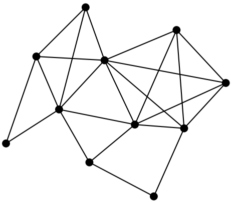
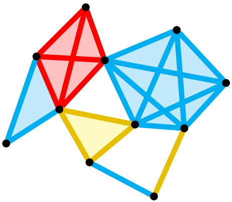
Graph Parameters of Interest

The Prague Dimension $\dim_p(G)$

$\dim_p(G) :=$ the least k s.t. $\exists$ a k-colorable clique edge-covering of $\overline{G}$.

- Introduced by Nešetřil, Pultr, and Rödl in the late 1970s.
- Many equivalent definitions. The clique-based one is useful for us.
- For brief, we introduce the *clique color number* $cc'(G) := \dim_p(\overline{G})$.





Main Question

Conjecture (Füredi-Kantor)

With high probability, $\dim_p(G_{n,p}) = \Theta\left(\frac{n}{\log n}\right)$ for constant $p \in (0, 1)$.

- Suffices to show $cc'(G_{n,p}) = \Theta\left(\frac{n}{\log n}\right)$ whp, as

$$\dim_p(G_{n,p}) \overset{d}{=} cc'(G_{n,1-p}).$$

- Lower bound is simple:

$$cc'(G_{n,p}) \geq \frac{\Delta(G_{n,p})}{\omega(G_{n,p}) - 1} = \Theta\left(\frac{n}{\log n}\right).$$

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- **Difficulty:** For upper bound, we need to color/cover $G_{n,p}$ with mostly cliques of size $\Theta(\log n)$.

Main Result and Proof Components

Main Theorem (Guo, Patton, Warnke 2020^+)

With high probability, $cc'(G_{n,p}) = \Theta\left(\frac{n}{\log n}\right)$ for constant $p \in (0,1)$.

Two Main Challenges for Upper Bound:

1. Need a clique covering $\mathcal{C}$ of $G_{n,p}$ with most cliques of size $\Theta(\log n)$.

2. Need to color $\mathcal{C}$ with $O(n/\log n)$ colors.

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1. Need a clique covering $\mathcal{C}$ of $G_{n,p}$ with most cliques of size $\Theta(\log n)$.
   - Can use a "nibble" algorithm.
   - Gives efficient covering

     $$\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_I,$$

     where each $\mathcal{C}_i$ is uniform.
2. Need to color $\mathcal{C}$ with $O(n/\log n)$ colors.

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   $$\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_I,$$

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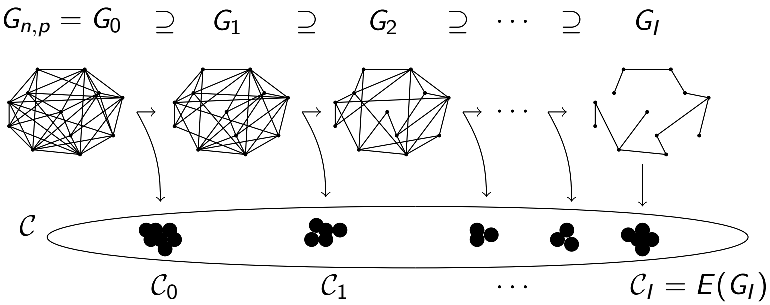
2. Need to color $\mathcal{C}$ with $O(n/\log n)$ colors.
   - We can check that $\sum_i \Delta(\mathcal{C}_i) = O(n/\log n)$.
   - Want to say $\chi'(\mathcal{C}_i) = O(\Delta(\mathcal{C}_i))$, so we can use

   $$\chi'(\mathcal{C}) \leq \sum_i \chi'(\mathcal{C}_i) \leq \sum_i O(\Delta(\mathcal{C}_i)) = O(n/\log n).$$

Part 1: Clique Selection via Semi-Random Algorithm

Clique covering generated as follows.



- G_{i+1} generated by removing a random set $\mathcal{C}_i$ of cliques from G_i.
- Take all remaining edges in $\mathcal{C}_I = E(G_I)$, and set

$$\mathcal{C} = \mathcal{C}_0 \cup \cdots \cup \mathcal{C}_I.$$

Part 2: A Hypergraph Coloring Problem

Proof Strategy

$$\chi'(\mathcal{C}) \leq \sum_{0 \leq i \leq I} \chi'(\mathcal{C}_i) \overset{?}{\leq} \sum_{0 \leq i \leq I} O(\Delta(\mathcal{C}_i)) \leq O\left(\frac{n}{\log n}\right).$$

We want $\chi'(C_i) \leq O(\Delta(\mathcal{C}_i))$ to complete our proof.

- For $i = I$: $\chi'(\mathcal{C}_I) \leq 2\Delta(\mathcal{C}_I)$ by Vizing's Theorem.
- For $i < I$: We want a Pippenger-Spencer like result.

Theorem (Pippenger-Spencer, 1989)

For constant $k \geq 2$ and $\epsilon > 0$, any hypergraph $\mathcal{H}$ that is k-uniform, is sufficiently regular, and has small codegree, satisfies

$$\chi'(\mathcal{H}) \leq (1+\epsilon)\Delta(\mathcal{H}).$$

Part 2: A Hypergraph Coloring Problem

Proof Strategy

$$\chi'(\mathcal{C}) \leq \sum_{0 \leq i \leq I} \chi'(\mathcal{C}_i) \overset{?}{\leq} \sum_{0 \leq i \leq I} O(\Delta(\mathcal{C}_i)) \leq O\left(\frac{n}{\log n}\right).$$

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Problem

Pippenger-Spencer only applies to hypergraphs with constant uniformity k.

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Solution

Exploit that $\mathcal{C}_i$ is a *random* set of cliques. Can extend to $k = O(\log n)$.

Our Hypergraph Coloring Result

Chromatic Index of Random Subhypergraphs (Guo, Patton, Warnke)

Let $\mathcal{H}$ be a k-uniform hypergraph that satisfies

- *Edge uniformity*: $2 \leq k \leq b \log n$,
- *Approximately regular*: $\deg_{\mathcal{H}}(v) = (1 \pm n^{-\sigma})D$,
- *Small codegree*: $\deg_{\mathcal{H}}(u, v) \leq n^{-\sigma} D$.

$\mathcal{H}_m :=$ random subhypergraph of $\mathcal{H}$ containing $n^{1+\sigma} \leq m \ll e(\mathcal{H})$ edges. Then whp,

$$\chi'(\mathcal{H}_m) \leq (1+\delta)\Delta(\mathcal{H}_m) \qquad \text{for } \delta \approx b/\sigma.$$

Key Point: Can allow for edges of size $O(\log n)$.

Corollary

$$\chi'(\mathcal{H}_m) \leq \begin{cases} (1+\epsilon)\Delta(\mathcal{H}_m) & \text{if } k = o(\log n). \\ O(\Delta(\mathcal{H}_m)) & \text{if } k = O(\log n). \quad \longleftarrow \text{(What we use.)} \end{cases}$$

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Algorithmic proof

- Natural random greedy alg. colors $E(\mathcal{H}_m)$ using $\lfloor (1+\delta)\frac{km}{n} \rfloor$ colors

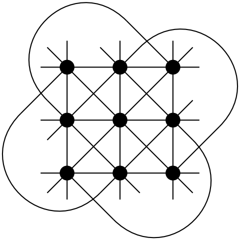
$$\Delta(\mathcal{H}_m) \approx \Delta(\mathcal{H})\frac{m}{e(\mathcal{H})} \approx D\frac{m}{nD/k} = \frac{km}{n}$$

- Analysis based on differential equation method

Hypergraph Coloring Algorithm Example

Random greedy hypergraph coloring algorithm

1. Let $Q := \{1, \ldots, q\}$ be the set of possible colors for $q = \lfloor (1+\delta)\frac{km}{n} \rfloor$
2. For step $1 \leq i \leq m$:
    1. Sample an edge $e \in E(\mathcal{H})$ uniformly at random
    2. Color e by a available color in Q uniformly at random



Let $Q := \{R, B, G, Y\}$

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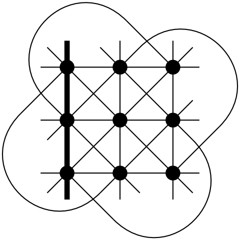

Available colors for the selected edge: $\{R, B, G, Y\}$

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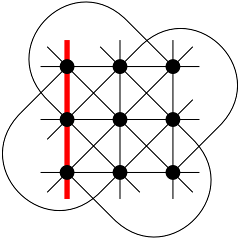

Color the selected edge by R

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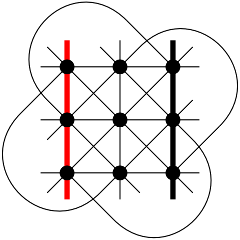

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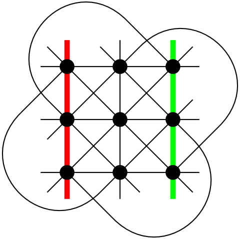

Color the selected edge by G

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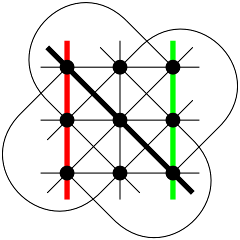

Available colors for the selected edge: $\{B, Y\}$

Hypergraph Coloring Algorithm Example

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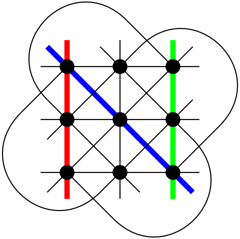

Color the selected edge by B

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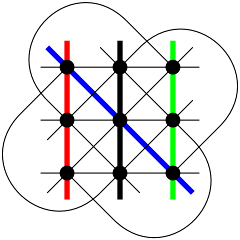

Available colors for the selected edge: $\{R, G, Y\}$

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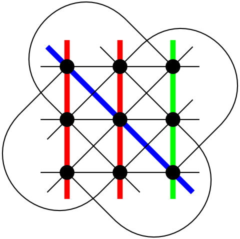

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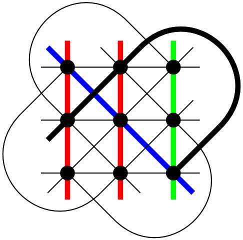

Available colors for the selected edge: $\{Y\}$

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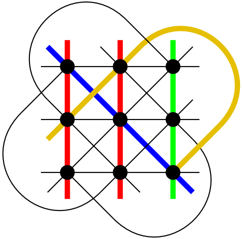

Color the selected edge by Y

Summary

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Then whp, $\chi'(\mathcal{H}_m) \leq (1 + \delta)\Delta(\mathcal{H}_m)$ for $\delta \approx b/\sigma$.

- Extends Pippenger-Spencer theorem for random subhypergraphs.
- Verifies conjecture of Füredi-Kantor that $\dim_p(G_{n,p}) = \Theta\left(\frac{n}{\log n}\right)$ whp.

Open Problems

- Does the same hold for general hypergraphs?
- Can the $k = O(\log n)$ be relaxed for random hypergraphs?