Irregular $\mathbf{d_n}$-process is distinguishable from uniform random $\mathbf{d_n}$-graph

Erlang Surya

Georgia Institute of Technology

Joint work with Mike Molloy and Lutz Warnke

Context and Overview

Model: Random d-process G_d^P

- Start with an empty graph on n vertices
- In each step: add one random edge so that max-degree stays $\leq d$

- Natural random greedy algorithm to generate d-regular graph

Main Question: Wormald (1999)

How similar are d-process G_d^P and uniform random d-regular graph G_d?

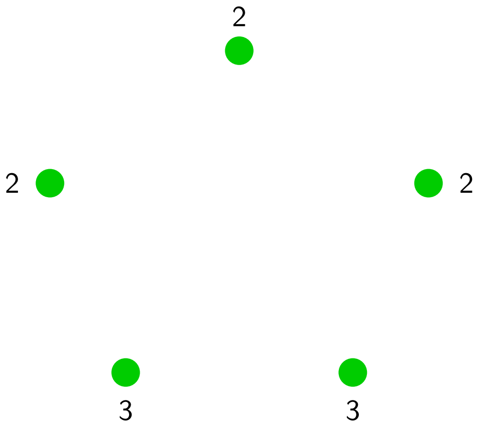
- Wormald conjectured they are similar
- Unclear how to approach

This Talk: Variant for degree sequences $\mathbf{d_n}$

Process $G_{\mathbf{d_n}}^P$ not similar to uniform $G_{\mathbf{d_n}}$ for irregular $\mathbf{d_n}$

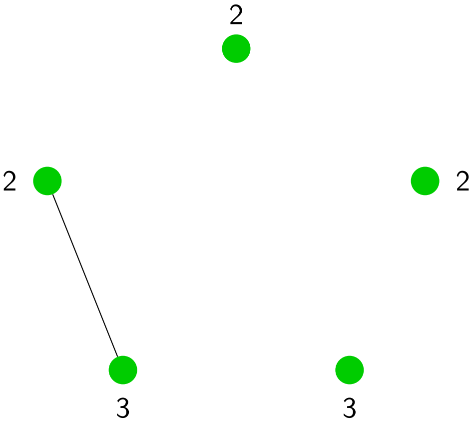
Generalization to Degree Sequences

Example: Process with degree sequence $\mathbf{d_5} = (2, 2, 2, 3, 3)$



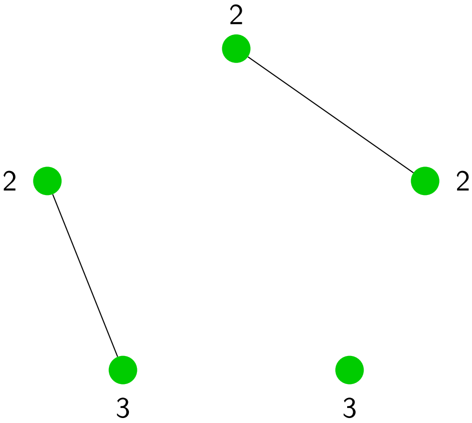
Generalization to Degree Sequences

Example: Process with degree sequence $\mathbf{d_5} = (2, 2, 2, 3, 3)$



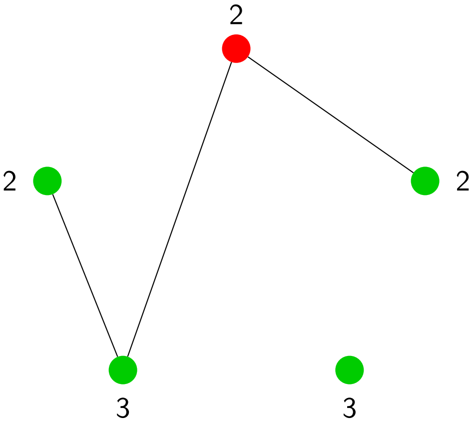
Generalization to Degree Sequences

Example: Process with degree sequence $\mathbf{d_5} = (2, 2, 2, 3, 3)$



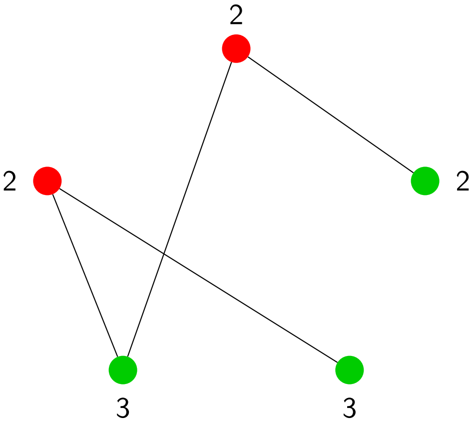
Generalization to Degree Sequences

Example: Process with degree sequence $\mathbf{d_5} = (2, 2, 2, 3, 3)$



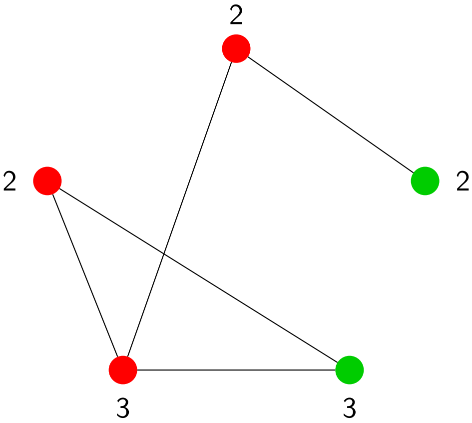
Generalization to Degree Sequences

Example: Process with degree sequence $\mathbf{d_5} = (2, 2, 2, 3, 3)$



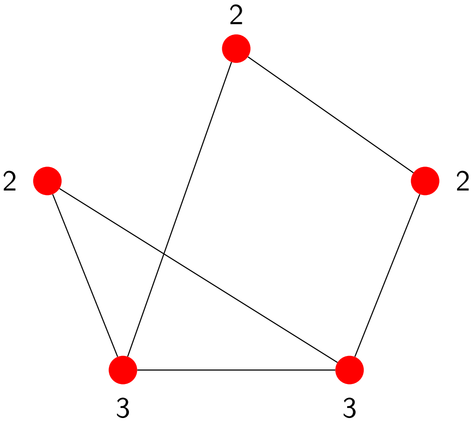
Generalization to Degree Sequences

Example: Process with degree sequence $\mathbf{d_5} = (2, 2, 2, 3, 3)$



Generalization to Degree Sequences

Example: Process with degree sequence $\mathbf{d_5} = (2, 2, 2, 3, 3)$



Main Result: $\mathbf{d_n}$-process and uniform model differ

$\mathbf{d_n}$ *not almost regular* : no degree appears $\geq 0.99n$ times in $\mathbf{d_n}$

Molloy, S., Warnke (2021+)

If the bounded degree sequence $\mathbf{d_n}$ is not almost regular, then can whp distinguish $\mathbf{d_n}$-process $G^P_{\mathbf{d_n}}$ and uniform random $\mathbf{d_n}$-graph $G_{\mathbf{d_n}}$

Today: Assume $\#$ degree 1 vertex $\in [0.01n, 0.99n]$

- **Proof Idea:** *Show discrepancy in edge statistic*
  - Number of 1-1 edges whp differ
- **Proof Technique:** *Switching* applied to $\mathbf{d_n}$-process $G^P_{\mathbf{d_n}}$
  - Usually only applied to uniform models (not stochastic processes)

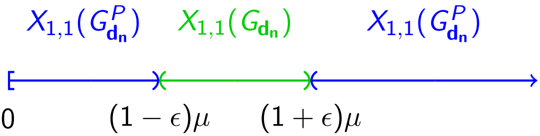
Main Technical Result: Discrepancy in Edge Statistic

$X_{1,1}(G) = \#$ of edges with endpoints of degree 1 in G

Can distinguish both models via $X_{1,1}$

There exists μ and $\epsilon = \epsilon(\Delta) > 0$ such that with high probability

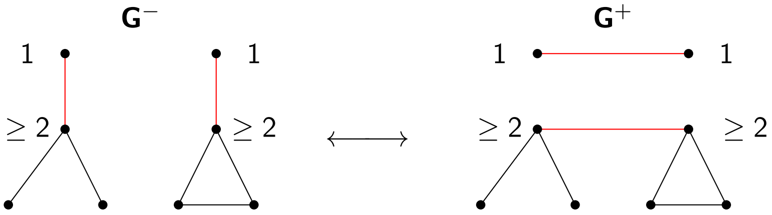
$$X_{1,1}(G_{\mathbf{d_n}}) \in [(1-\epsilon)\mu, (1+\epsilon)\mu] \quad \text{and} \quad X_{1,1}(G^P_{\mathbf{d_n}}) \notin [(1-\epsilon)\mu, (1+\epsilon)\mu]$$



- **Concentration of** $X_{1,1}(G_{\mathbf{d_n}})$: standard via configuration model
- **Understand** $X_{1,1}(G^P_{\mathbf{d_n}})$: *switching* ($\longrightarrow$ *Rest of the talk*)

Switching: Change # of 1-1 edges by exactly one

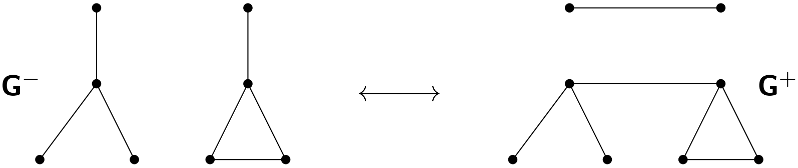
Definition via Example:



- **Goal**: compare $\mathbb{P}(G^P_{\mathbf{d_n}} = G^+)$ and $\mathbb{P}(G^P_{\mathbf{d_n}} = G^-)$
- **Standard switching for uniform models:**
  - Compare size of different graph-classes based on # of 1-1 edges
- **Problem for stochastic processes:**
  - Order of edges matters (as no longer uniform)
- **Solution**: Expand based on order σ of edges of G

$$\mathbb{P}(G^P_{\mathbf{d_n}} = G) = \sum_{\sigma} \mathbb{P}(\mathbf{d_n}\text{-process returns } \sigma)$$

How Switching Affect $\mathbf{d_n}$-process Probabilities



Switching Lemma (for probabilities)

$$\frac{\mathbb{P}(G^P_{\mathbf{d_n}} = G^+)}{\mathbb{P}(G^P_{\mathbf{d_n}} = G^-)} \geq 1 + \epsilon' \qquad \text{where } \epsilon' > 0 \text{ depends on } \Delta$$

Proof Idea: understand how switching affects $\mathbb{P}(\mathbf{d_n}\text{-process returns } \sigma)$

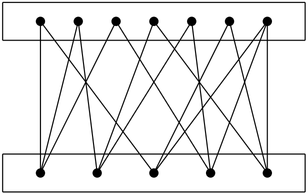
$$\mathbb{P}(G^P_{\mathbf{d_n}} = G) = \sum_{\sigma} \mathbb{P}(\mathbf{d_n}\text{-process returns } \sigma)$$

Surprisingly tractable: tractable by expanding out probabilities

Switching: Graph Count Based on $X_{1,1}$

Notation: $G \in \mathbf{d_n}$ if G has degree sequence $\mathbf{d_n}$

Auxiliary Graph: by adding edge between G^+, G^-:



$\longrightarrow G_{\ell+1} = \{G \in \mathbf{d_n} : X_{1,1}(G) = \ell + 1\}$

$\longrightarrow G_{\ell} = \{G \in \mathbf{d_n} : X_{1,1}(G) = \ell\}$

Key Point: Auxiliary graph is roughly regular when $\ell \approx \mu$

Switching lemma then implies:

$$\frac{\mathbb{P}(G^P_{\mathbf{d_n}} \in G_{\ell+1})}{\mathbb{P}(G^P_{\mathbf{d_n}} \in G_{\ell})} \geq 1 + \epsilon'$$

Proof of Main Theorem (Sketch)

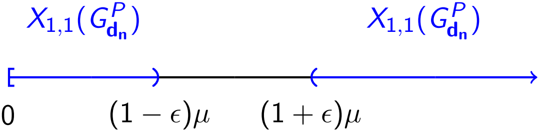
Definition: $\mathcal{N}_z = \{G \in \mathbf{d_n} : |X_{1,1}(G) - \mu| \leq z\}$.

Key Point implies (for $z \leq 2\epsilon\mu$)

$$\frac{\mathbb{P}[G^P_{\mathbf{d_n}} \in \mathcal{N}_z]}{\mathbb{P}[G^P_{\mathbf{d_n}} \in \mathcal{N}_{z+1}]} \leq \frac{\sum_{\mu-z\leq\ell\leq\mu+z} \mathbb{P}(G^P_{\mathbf{d_n}} \in G_\ell)}{\sum_{\mu-z\leq\ell\leq\mu+z} \mathbb{P}(G^P_{\mathbf{d_n}} \in G_{\ell+1})} \leq \frac{1}{1+\epsilon'}$$

$$\mathbb{P}[G^P_{\mathbf{d_n}} \in \mathcal{N}_{\epsilon\mu}] \leq \frac{\mathbb{P}[G^P_{\mathbf{d_n}} \in \mathcal{N}_{\epsilon\mu}]}{\mathbb{P}[G^P_{\mathbf{d_n}} \in \mathcal{N}_{2\epsilon\mu}]} = \prod_{z=\epsilon\mu}^{2\epsilon\mu-1} \frac{\mathbb{P}[G^P_{\mathbf{d_n}} \in \mathcal{N}_z]}{\mathbb{P}[G^P_{\mathbf{d_n}} \in \mathcal{N}_{z+1}]} \leq \frac{1}{(1+\epsilon')^{\epsilon\mu}} \to 0$$

Conclusion: whp have

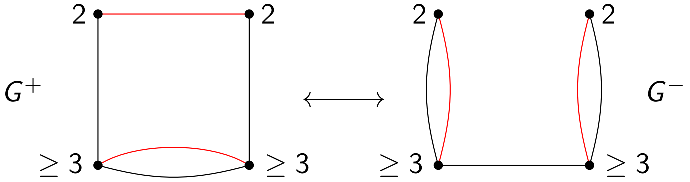


General case: More complicated

Analogous Switching Lemma (for probabilities) fails

$$\frac{\mathbb{P}(G^P_{\mathbf{d_n}} = G^+)}{\mathbb{P}(G^P_{\mathbf{d_n}} = G^-)} \geq 1 + \epsilon'$$

Counterexample with minimum degree $\delta = 2$:



$P \approx 0.25$ $\qquad$ $P \approx 0.28$

Summary

$\mathbf{d_n}$-process and uniform model $G_{\mathbf{d_n}}$ differ

If the bounded degree sequence $\mathbf{d_n}$ is not almost regular, then can whp distinguish $\mathbf{d_n}$-process $G^P_{\mathbf{d_n}}$ and random $\mathbf{d_n}$-graph $G_{\mathbf{d_n}}$

- Wormald's conjecture does not extend to irregular case
- Proof technique: use switching for stochastic process

Questions

- Other applications of switching to stochastic process?
- Wormald's conjecture for d-regular random graph process?