Packing nearly optimal Ramsey $R(3, t)$ graphs

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Joint work with Lutz Warnke

Theme of this talk

Topic of this talk

Construct pseudo-random triangle-free subgraphs of dense graphs

- Previous results only make such construction in complete graphs
- Construct via polynomial time randomized algorithm
  - Self-stabilization mechanism built into algorithm to control errors
- Approximately decompose complete graph into such Δ-free graphs
  - Solve a Ramsey theory conjecture by Fox, Liebenau, Person, Szabo et al

Review of previous results

Erdős (1961) + Spencer (1977) + Krivelevich (1994)

All find an n-vertex graph $G \subseteq K_n$ such that
G is Δ-free with independence number $\alpha(G) \leq C\sqrt{n}\log n$

- Construct G in the binomial random graph $G_{n,p}$

Kim (1995) + Bohman (2008): one nearly optimal $R(3,t)$ graph

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- Construct G by (semi-random variation of) Δ-*free process*: greedily add random edges that do not create a Δ

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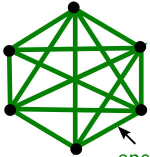

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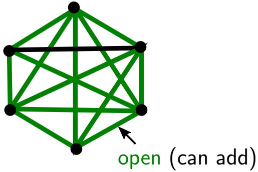

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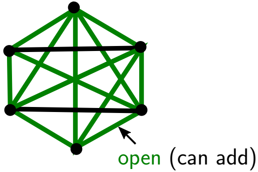

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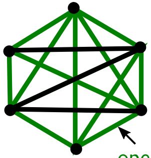

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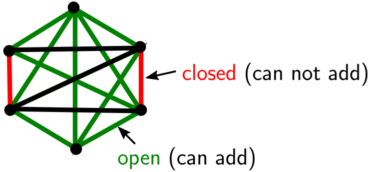

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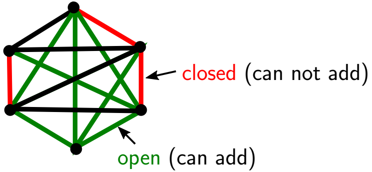

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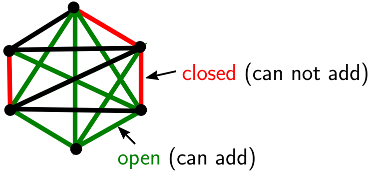
- Construct G by (semi-random variation of) Δ-*free process*:
  greedily add random edges that do not create a Δ
  Semi-random variation: add many random edges in each step



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* Tight up to the constant: Ajtai-Komlós-Szemerédi (1980)

* Lead to the right order of magnitude of Ramsey number $R(3,t)$

  - Kim received Fulkerson Prize in 1997

Main Result: nearly optimal $R(3, t)$ graphs

Kim (1995) + Bohman (2008): one nearly optimal $R(3, t)$ graph

Both find an n-vertex graph $G \subseteq K_n$ such that
G is Δ-free with independence number $\alpha(G) \leq C\sqrt{n \log n}$

- Using (semi-random variation of) Δ-*free process*:
  greedily add random edges that do not create a Δ

G., Warnke (2020): almost packing of nearly optimal $R(3, t)$ graphs

Given $\varepsilon > 0$, we find edge-disjoint graphs $(G_i)_{i \in \mathcal{I}}$ with $G_i \subseteq K_n$ such that
(a) each G_i is n-vertex Δ-free with $\alpha(G_i) \leq C_\varepsilon \sqrt{n \log n}$
(b) the union of the G_i contains $\geq (1 - \varepsilon)\binom{n}{2}$ edges

- Using simple *polynomial-time randomized algorithm*:
  sequentially choose G_i via semi-random variation of Δ-free process
  - Start with $H_0 = K_n$
  - Find $G_i \subseteq H_i$ and set $H_{i+1} = H_i \setminus G_i$ and repeat

Glimpse of the proof

Main-Technical-Result: find pseudo-random Δ-free subgraph $G \subseteq H$

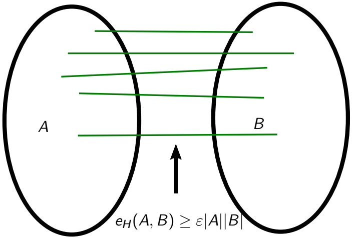
Let $\varrho := \sqrt{\beta(\log n)/n}$ and $s := C_\varepsilon \sqrt{n \log n}$. If $H \subseteq K_n$ is such that

$$e_H(A, B) \geq \varepsilon\, |A||B|$$

for all disjoint sets A, B of size s, then we can find Δ-free $G \subseteq H$ with

$$e_G(A, B) = (1 \pm \delta)\, \varrho\, e_H(A, B)$$

for all disjoint A, B of size s.



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- Pseudo-randomness of G_i ensures all local parts of H_i behave similarly

Implies packing result:

- Start with $H_0 = K_n$
- Sequentially choose $G_i \subseteq H_i$ and set $H_{i+1} = H_i \setminus G_i$

$$e_{H_i}(A, B) = \big(1 - (1 \pm \delta)\varrho\big)^i |A||B|$$

- Stop when $e_{H_I}(A, B) \approx \varepsilon |A||B|$ holds

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Proof based on semi-random variation of Δ-free process:

- Do *not* require degree/codegree regularity of H
- 'Self-stabilization' mechanism built into process (to control errors)
- Tools: Bounded-Differences-Ineq. and Upper-Tail-Ineq. of Warnke

Summary

G., Warnke (2020): almost packing of nearly optimal $R(3,t)$ graphs

Given $\varepsilon > 0$, we find edge-disjoint graphs $(G_i)_{i \in \mathcal{I}}$ with $G_i \subseteq K_n$ such that
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Remarks

- Can find the $(G_i)_{i \in \mathcal{I}}$ via polynomial time randomized algorithm
- Applications in Ramsey theory: solve a conjecture of Fox, Liebenau, Person, Szabo et al

Open problem

Other applications of 'self-stabilization' in design of randomized algorithms?

Reference

He Guo, Lutz Warnke, *Packing nearly optimal Ramsey $R(3,t)$ graphs*, Combinatorica **40**, 63–103 (2020)