Greedy maximal independent sets via local limits

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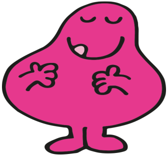
Tel Aviv University

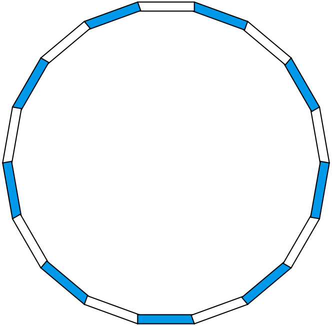
The 31st International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (AofA2020)

September 2020



Joint work with Michael Krivelevich, Tamás Mészáros and Clara Shikhelman

Parking cars on a cycle



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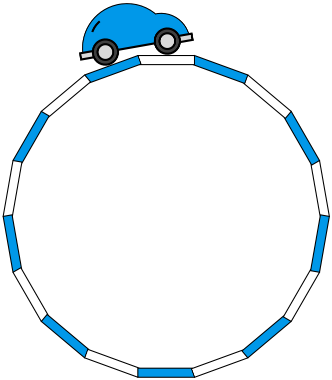

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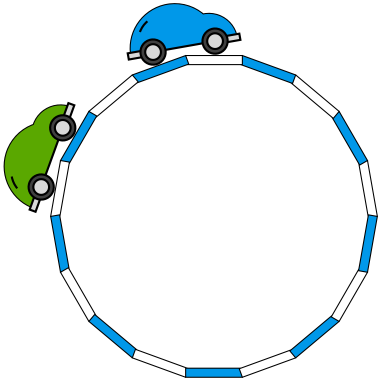

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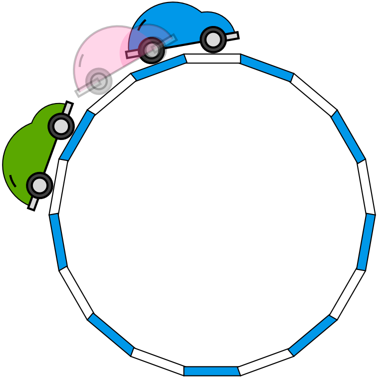

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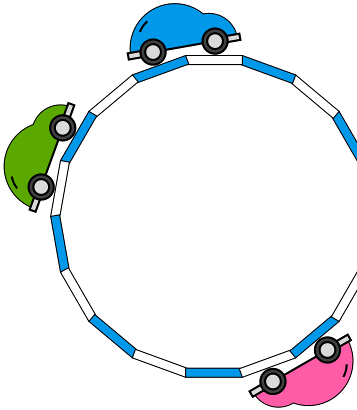

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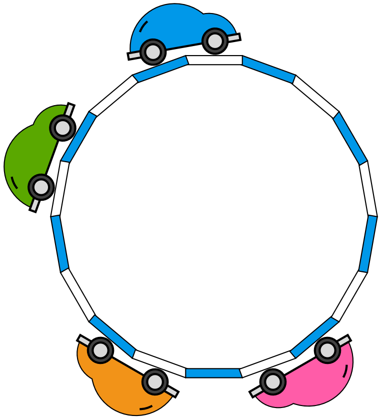

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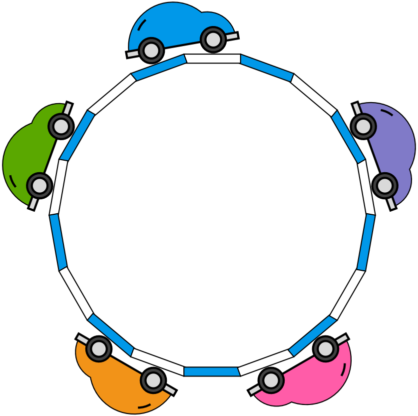

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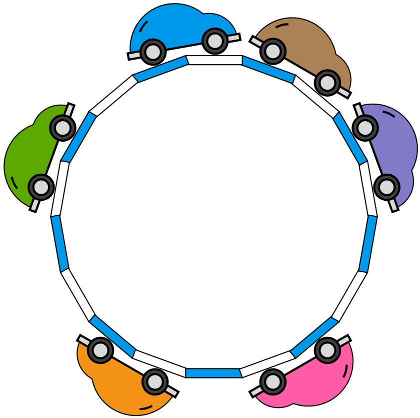

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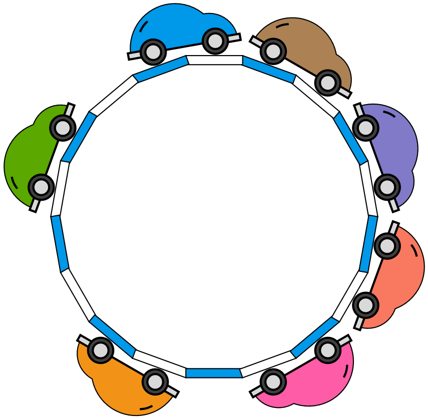

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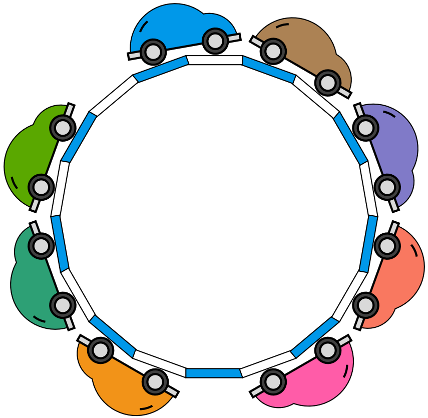

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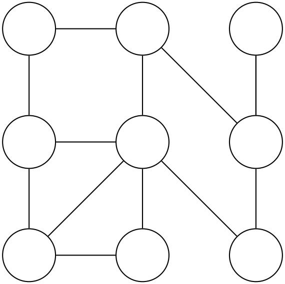

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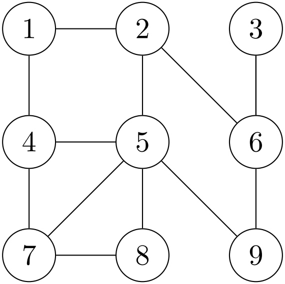

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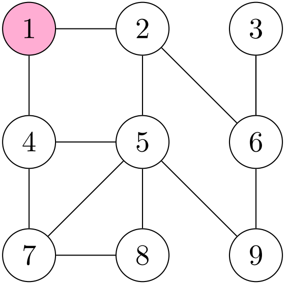

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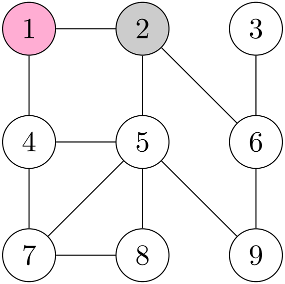

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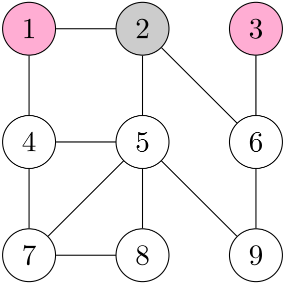

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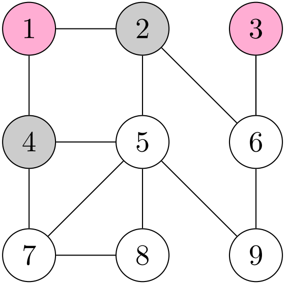

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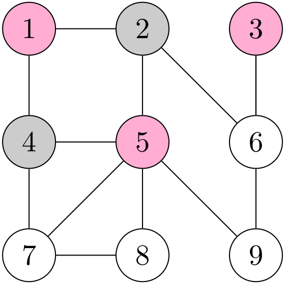

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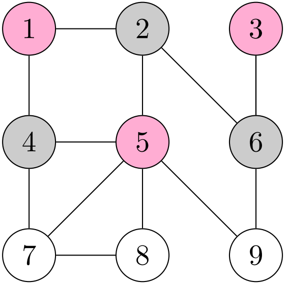

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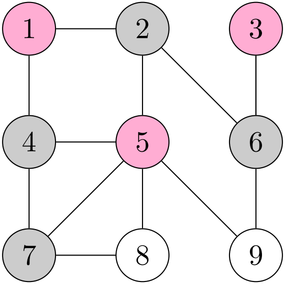

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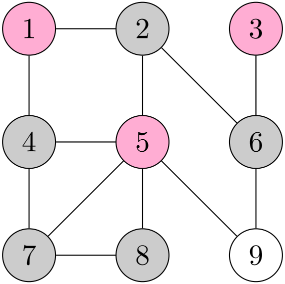

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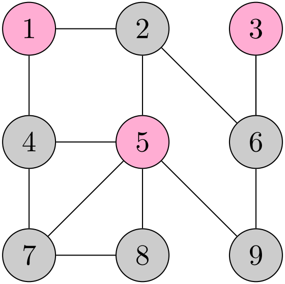

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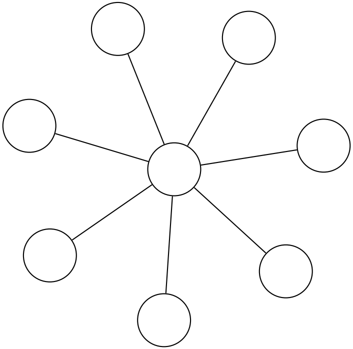

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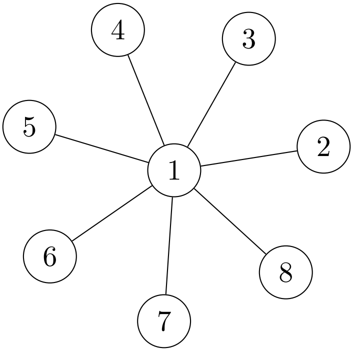

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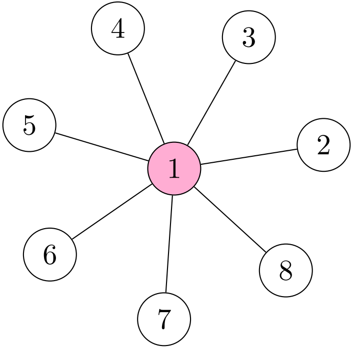

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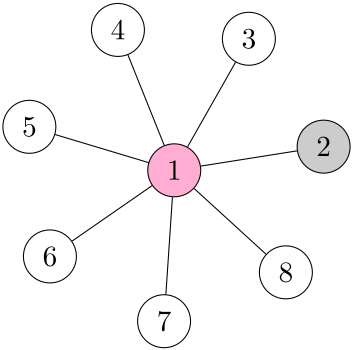

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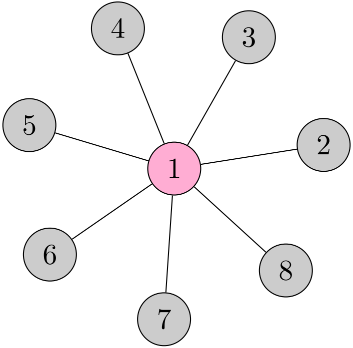

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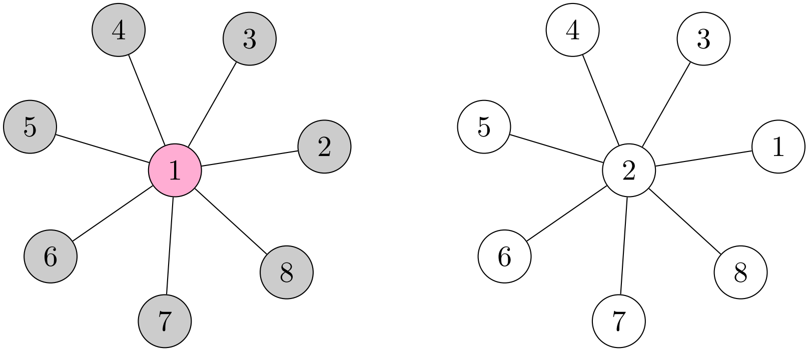

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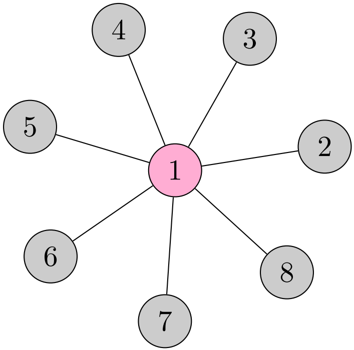

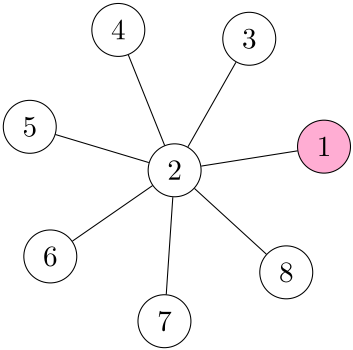

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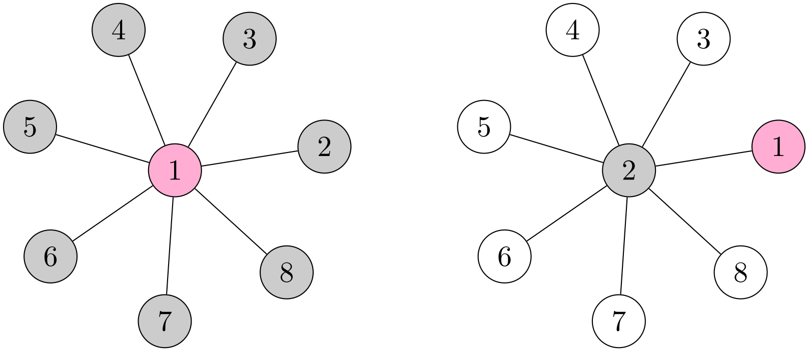

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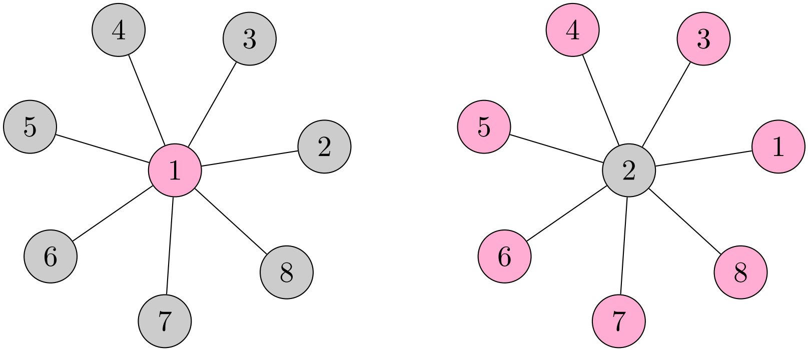

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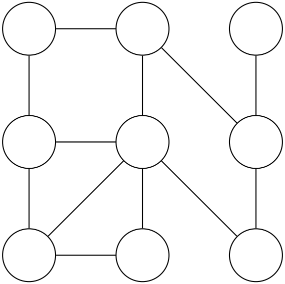

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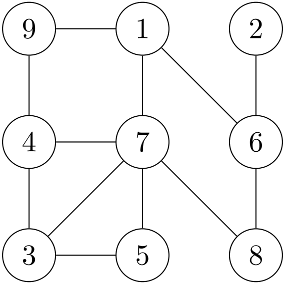

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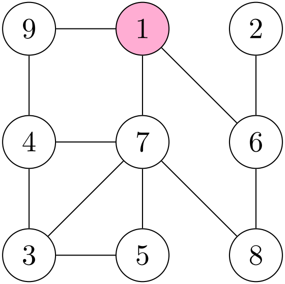

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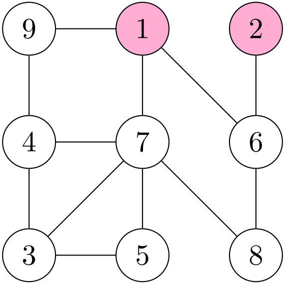

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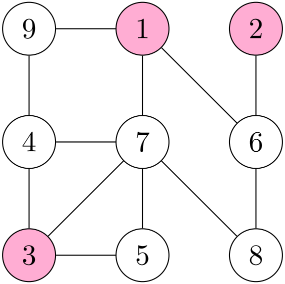

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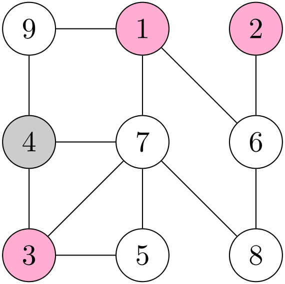

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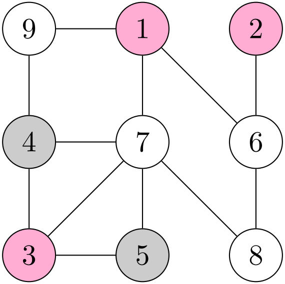

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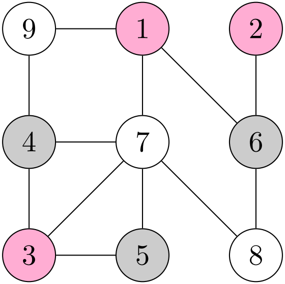

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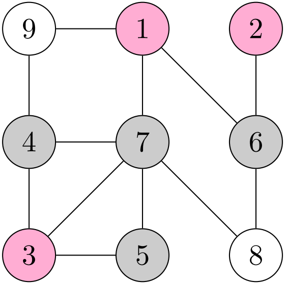

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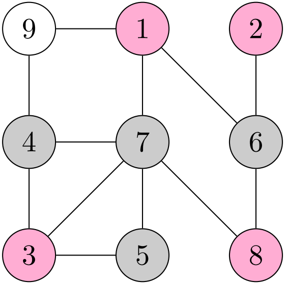

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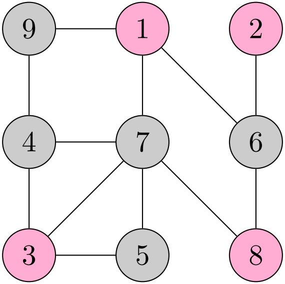

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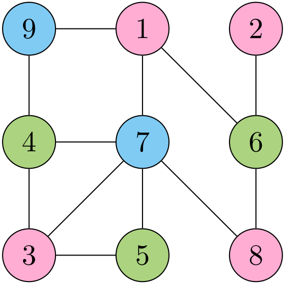

Greedy independence ratio — previous work

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BJL '17, BJM '17	random graphs with given degree sequence

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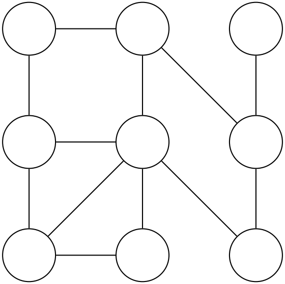

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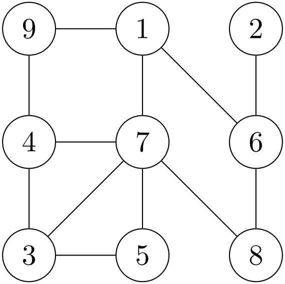

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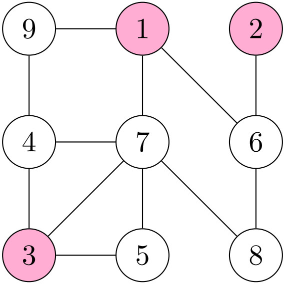

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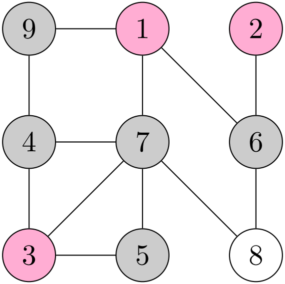

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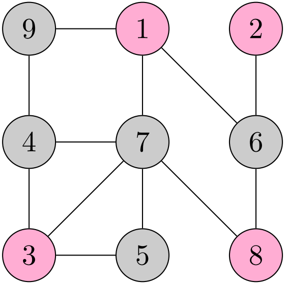

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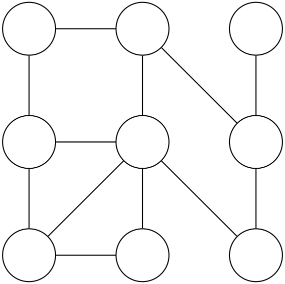

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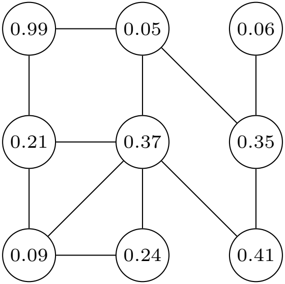

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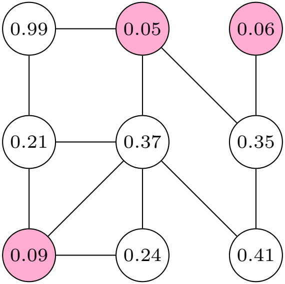

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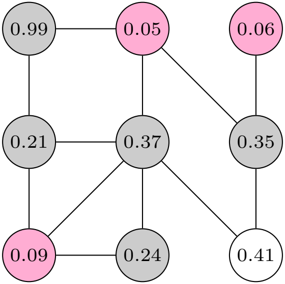

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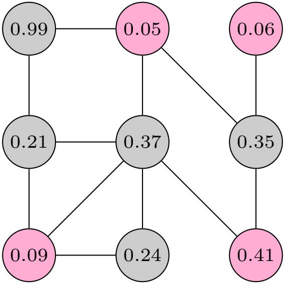

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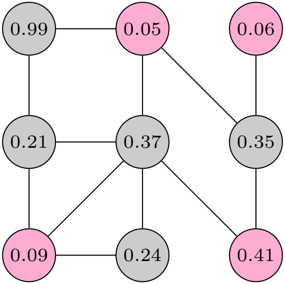

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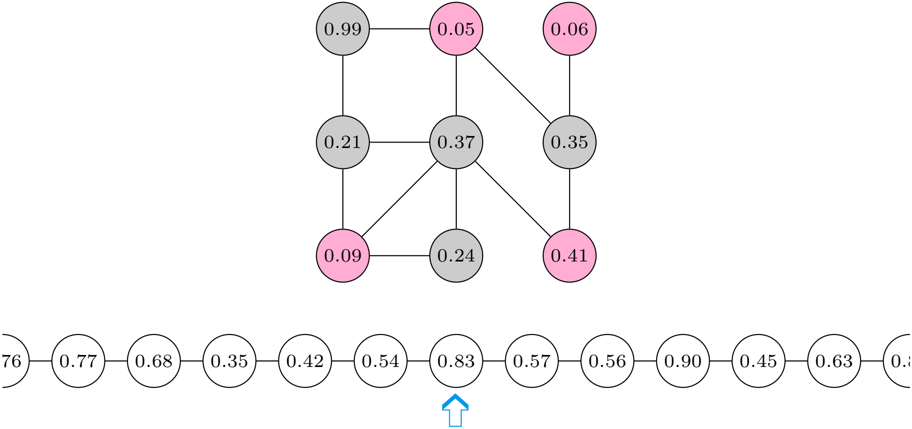

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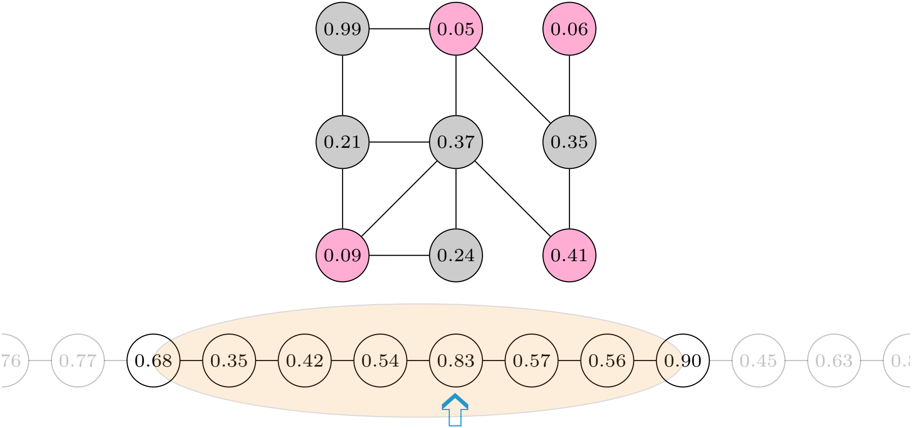

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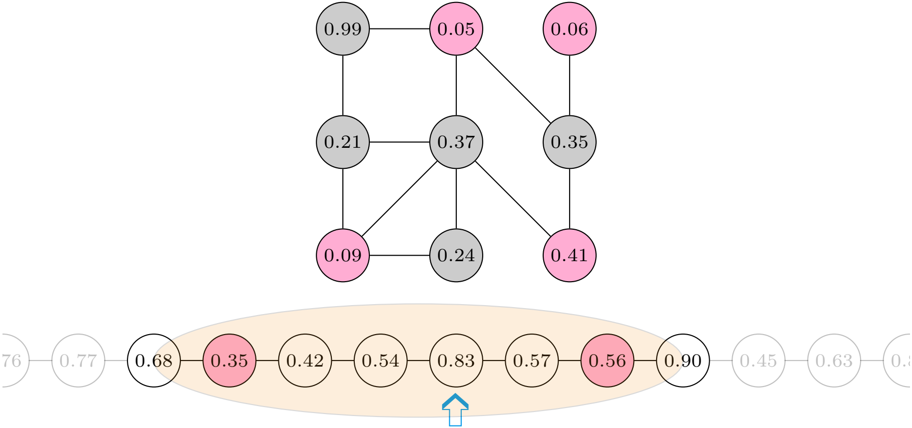

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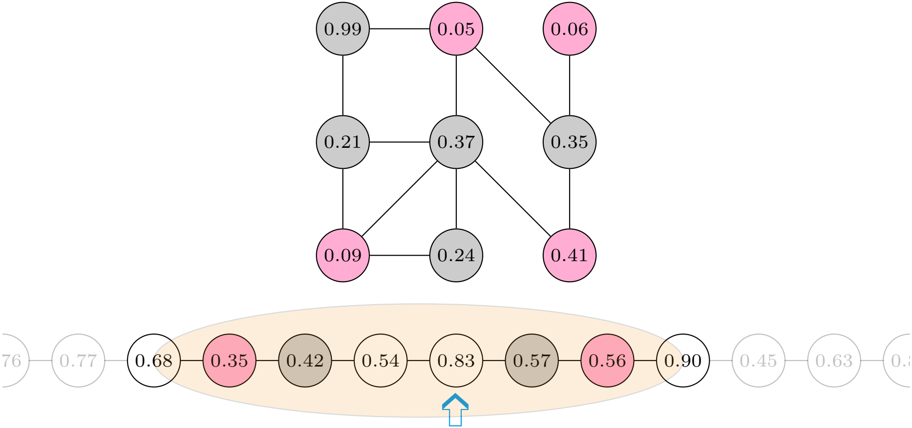

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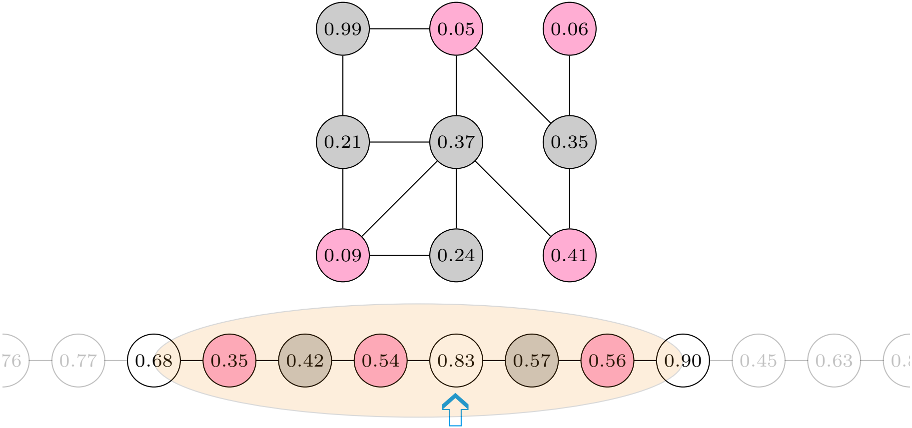

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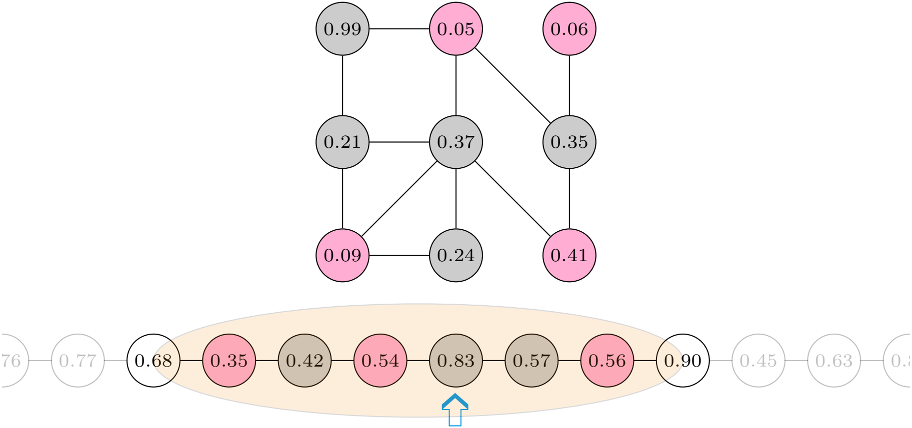

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- Decay of correlation $\implies$ $\iota(G_n) \sim \mathbb{E}[\iota(G_n)]$ a.a.s.

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Local limits (a.k.a. Benjamini–Schramm Limits)

We say that a (random) graph sequence G_n *converges locally* to a (random) rooted graph (U, ρ) if for every $r \geq 0$ the ball $B_{G_n}(\rho_n, r)$ converges in distribution to $B_U(\rho, r)$, where ρ_n is a uniform vertex of G_n.

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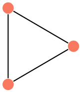

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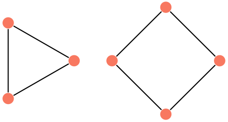

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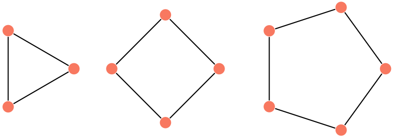

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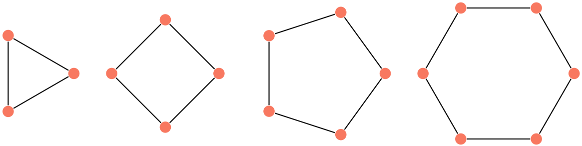

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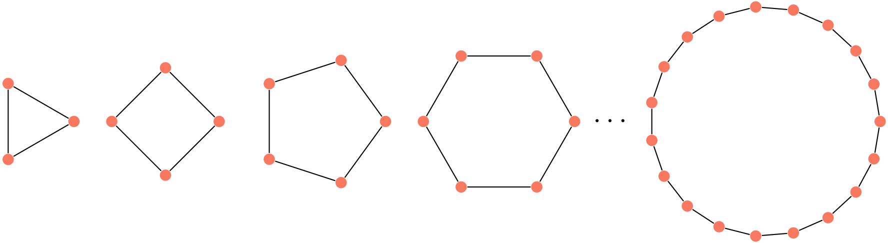

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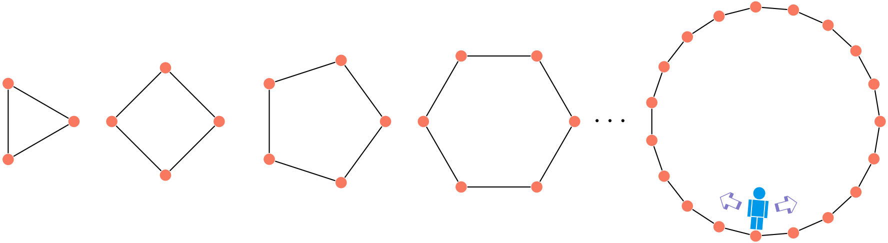

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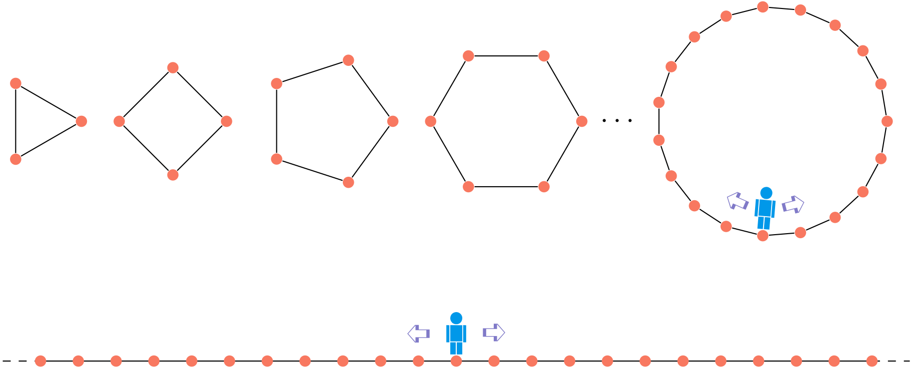

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Examples

- $P_n, C_n \xrightarrow{\text{loc}} \mathbb{Z}$
- $[n]^d \xrightarrow{\text{loc}} \mathbb{Z}^d$
- $G(n, d/n) \xrightarrow{\text{loc}} \mathcal{T}_d$, a Galton–Watson Pois$(d)$ tree
- $G_{n,d} \xrightarrow{\text{loc}}$ the d-regular tree
- Uniform random tree $T_n \xrightarrow{\text{loc}} \hat{\mathcal{T}}_1$, a size-biased GW Pois$(1)$ tree
- Finite d-ary balanced tree $\xrightarrow{\text{loc}}$ the canopy tree

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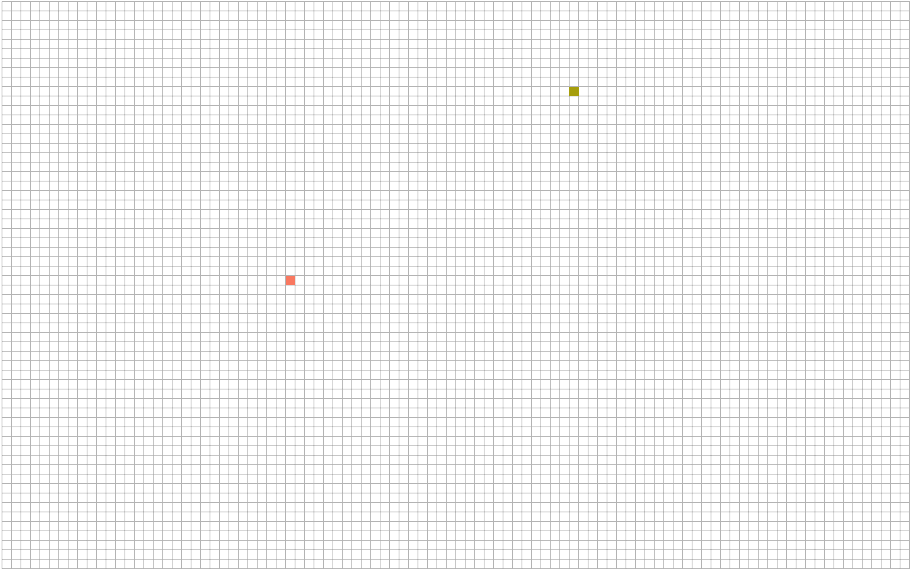

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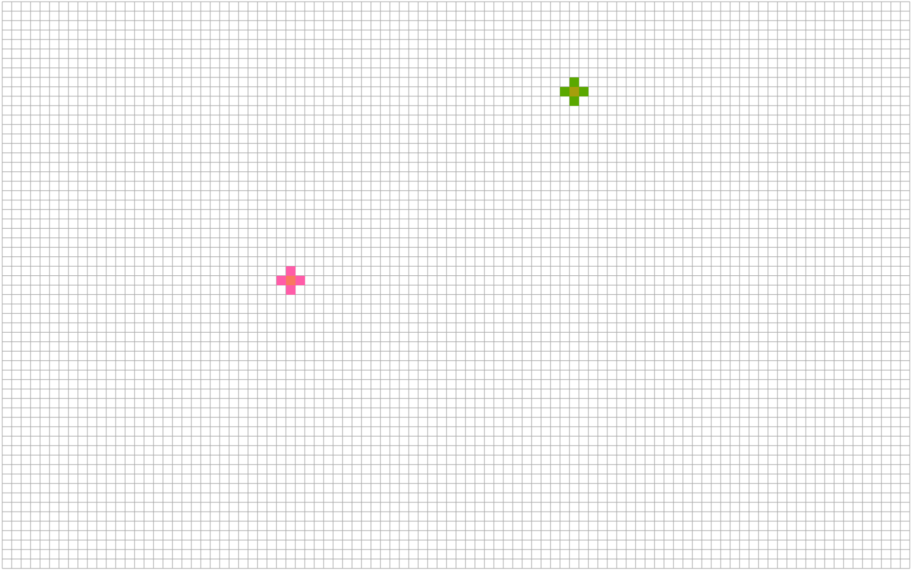

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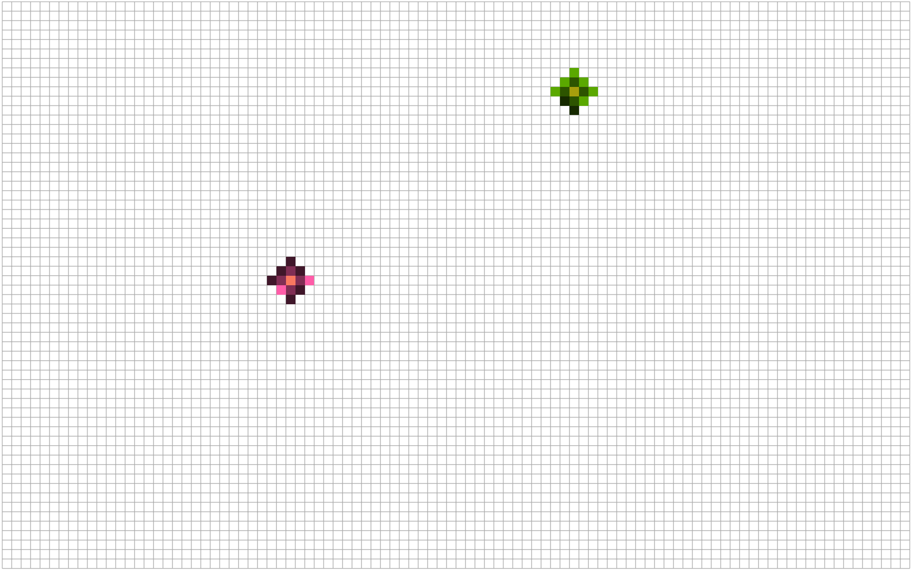

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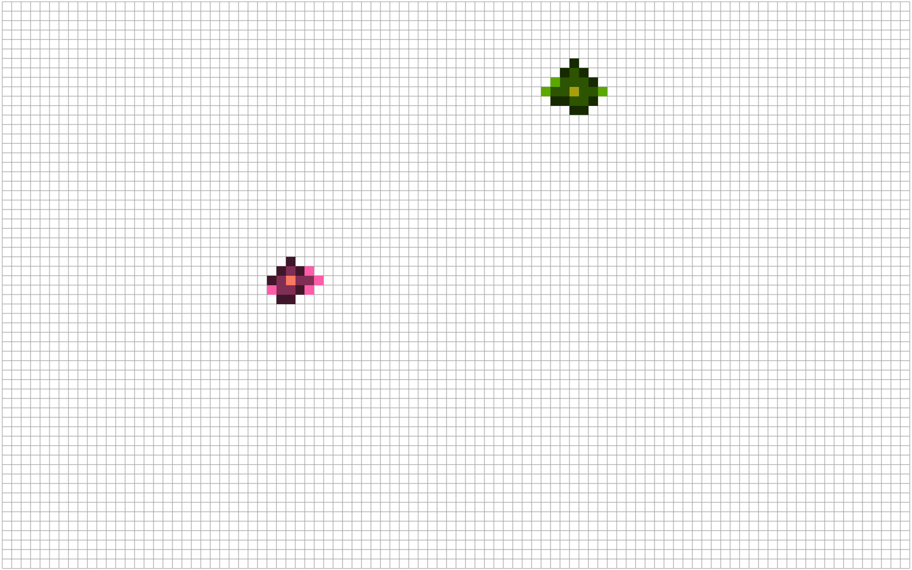

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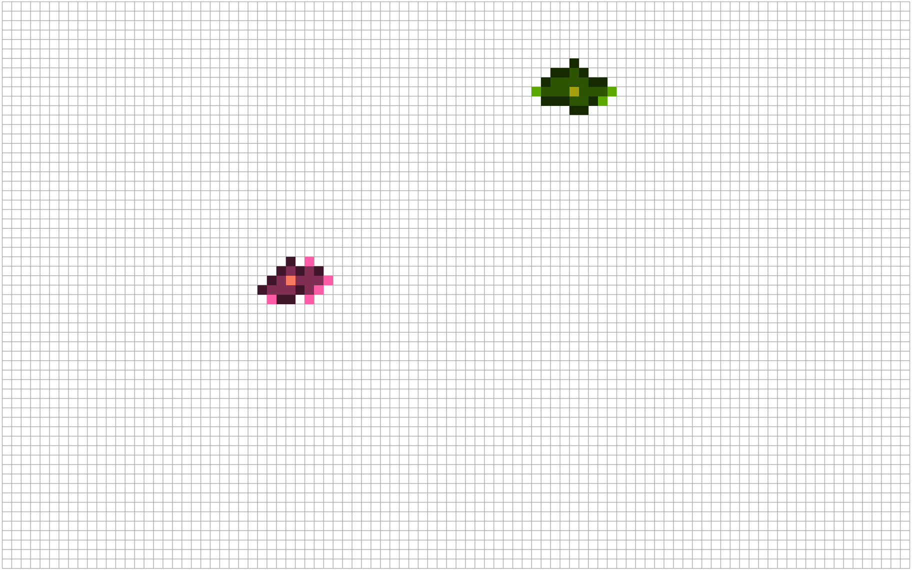

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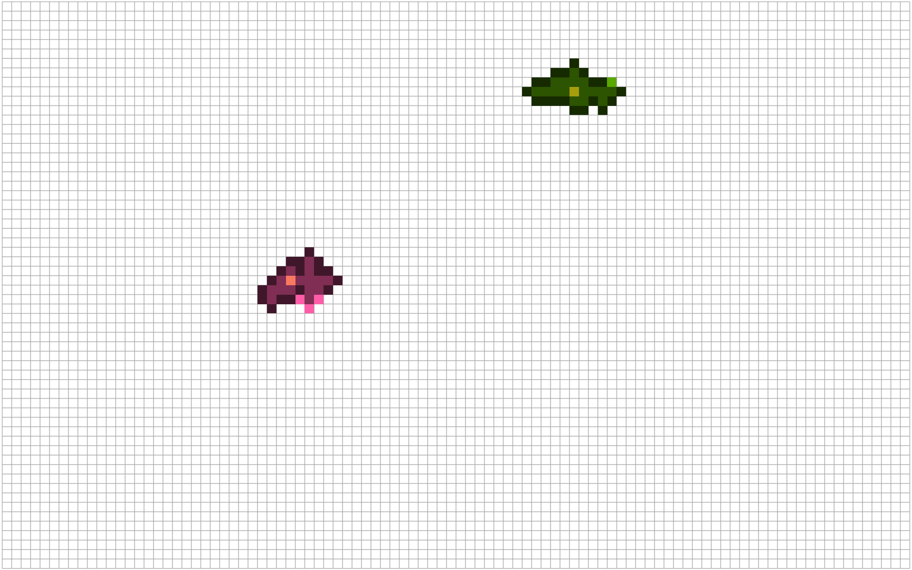

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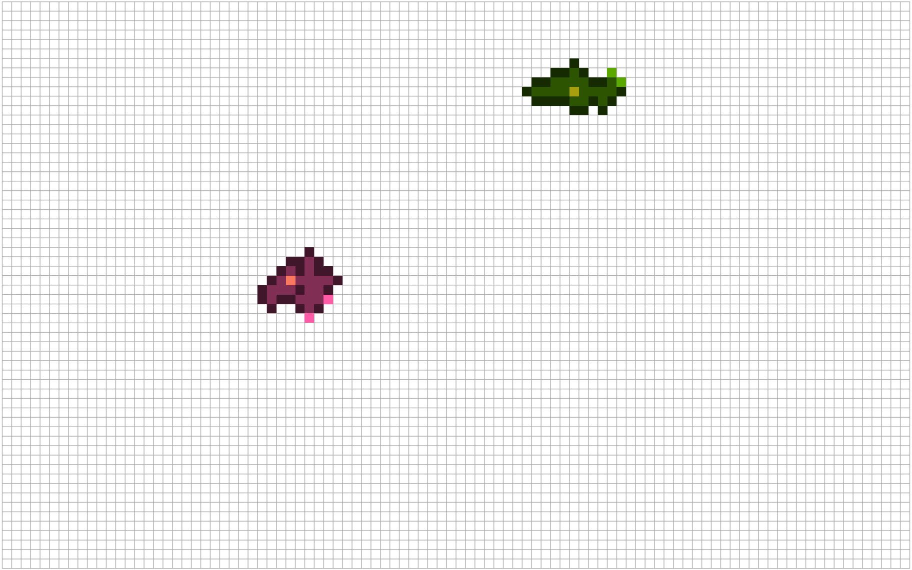

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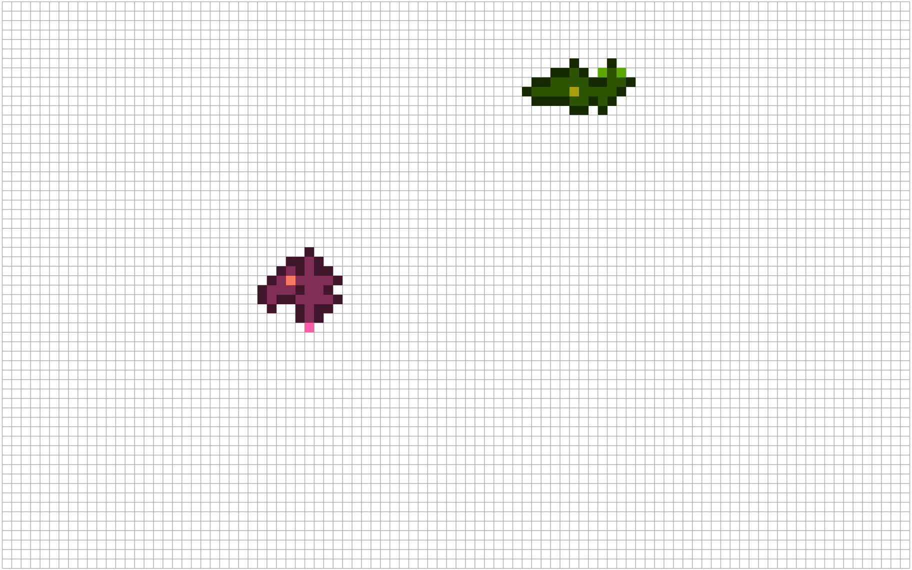

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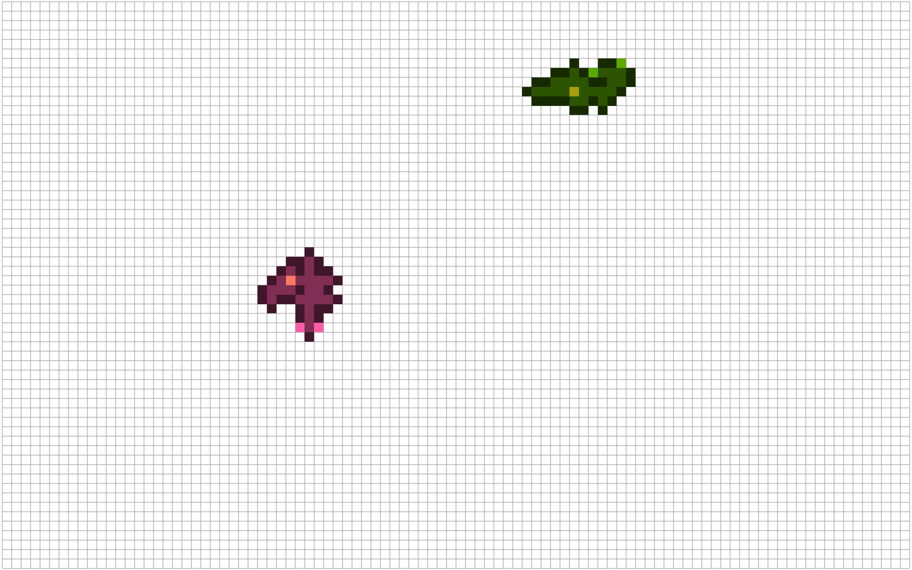

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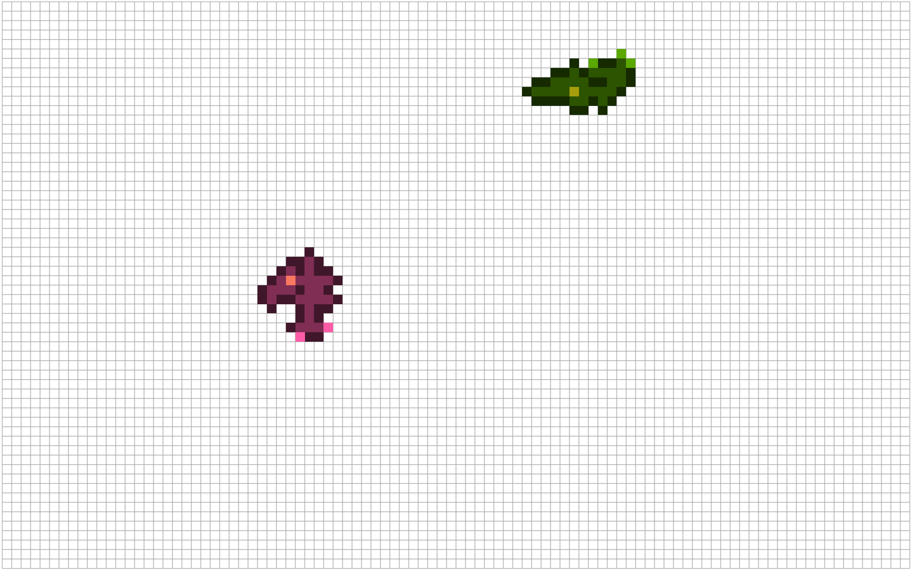

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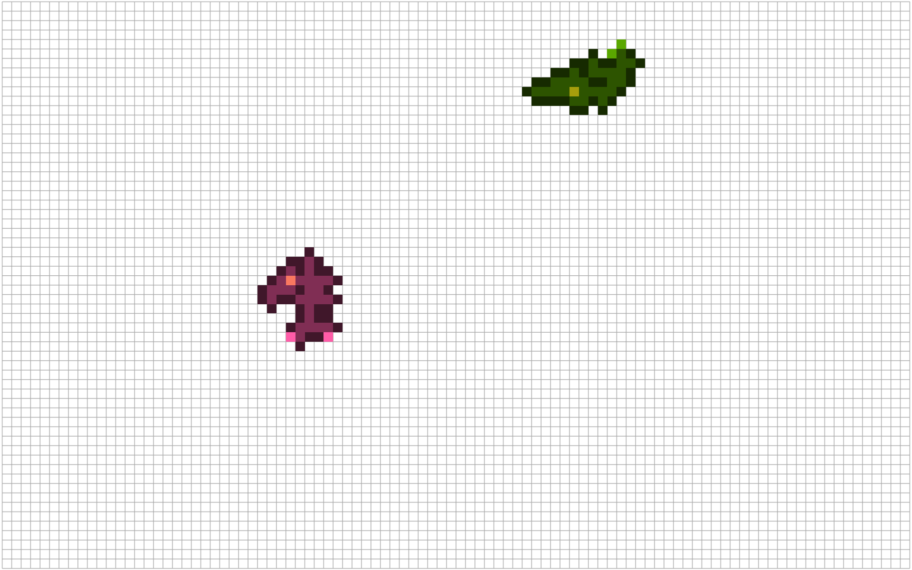

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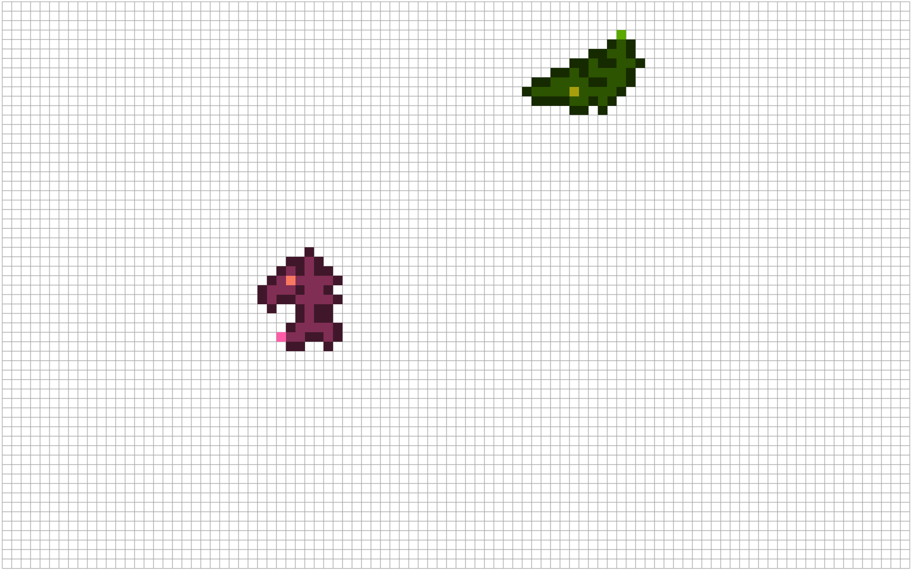

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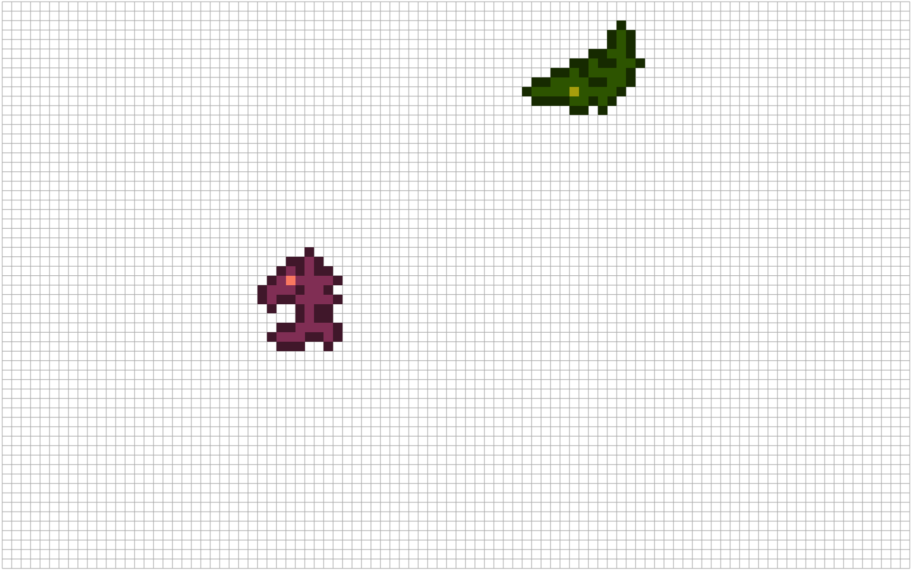

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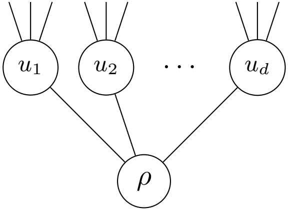

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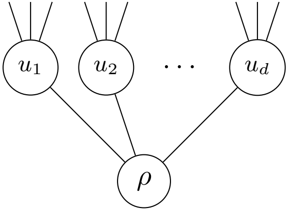

Assuming the children of ρ are roots to independent subtrees, and conditioning on the label of ρ, children of the *past* are roots to independent processes.

Systems of ordinary differential equations

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Thus, if y is a unique solution of

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Let (U, ρ) be a (simple) multitype branching process.

$$\begin{aligned} y_k(x) &= \mathbb{P}(\rho \in \mathbf{I}(U[\mathcal{P}_\rho]) \wedge \sigma_\rho < x \mid \tau = k) \\ &= x \cdot \mathbb{P}(\rho \in \mathbf{I}(U[\mathcal{P}_\rho]) \mid \sigma_\rho < x, \tau = k) \\ &= \int_0^x \mathbb{P}(\rho \in \mathbf{I}(U[\mathcal{P}_\rho]) \mid \sigma_\rho = z, \tau = k) dz \\ y_k'(x) &= \mathbb{P}(\rho \in \mathbf{I}(U[\mathcal{P}_\rho]) \mid \sigma_\rho = x, \tau = k) \end{aligned}$$

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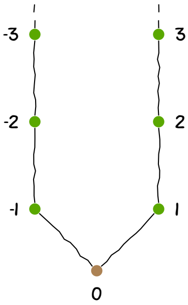

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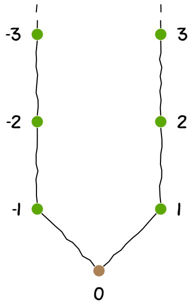

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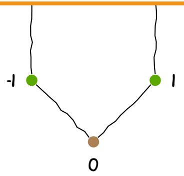

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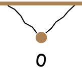

Application: binomial random graphs

Easy fact: $G(n, d/n)$ converges locally to the Pois(d) branching process.

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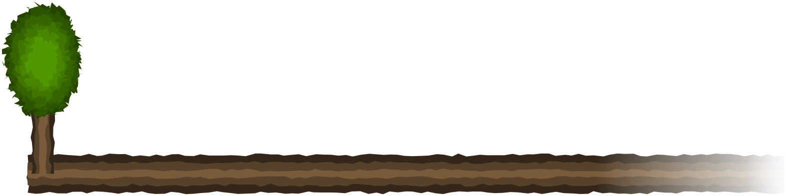

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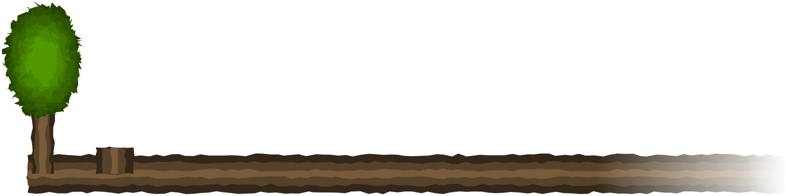

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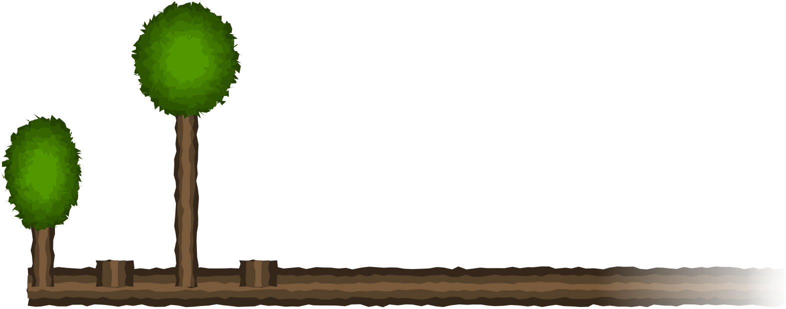

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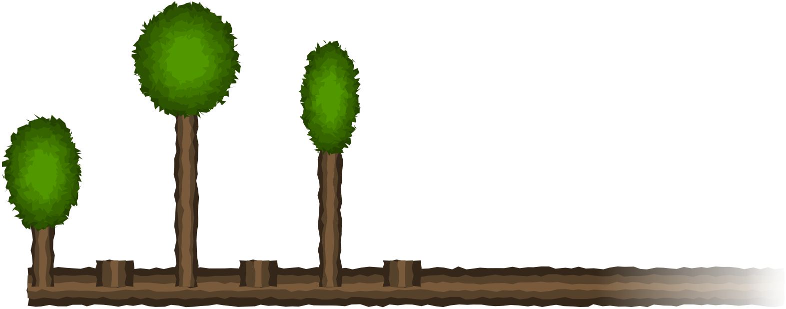

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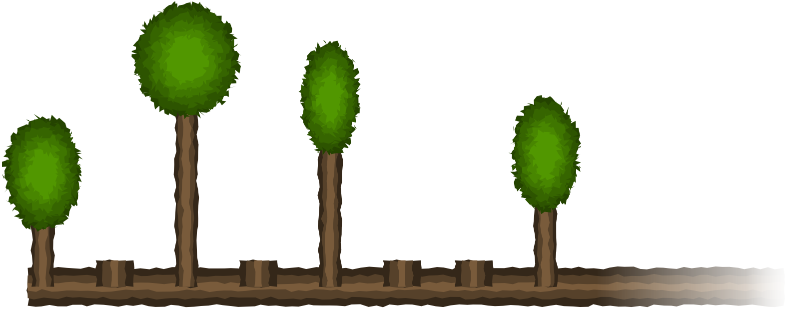

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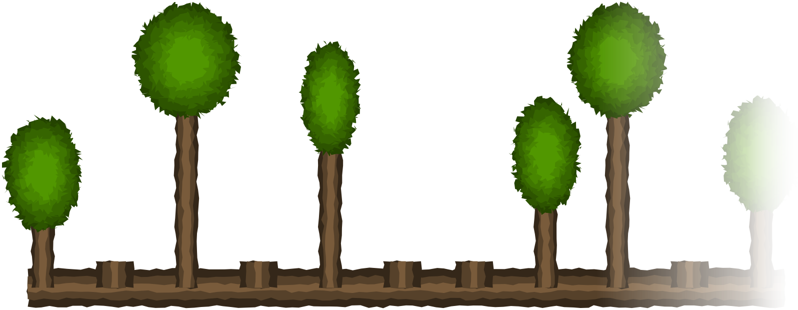

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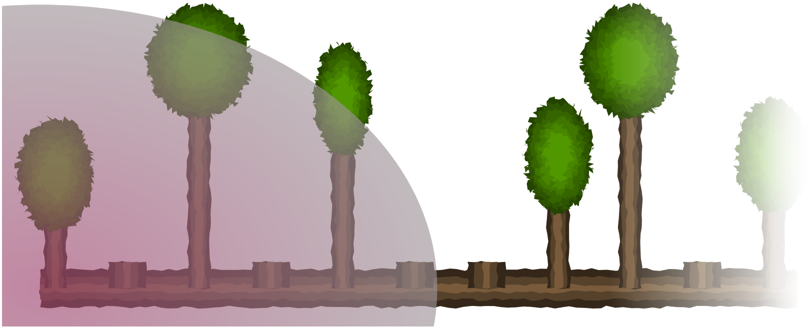

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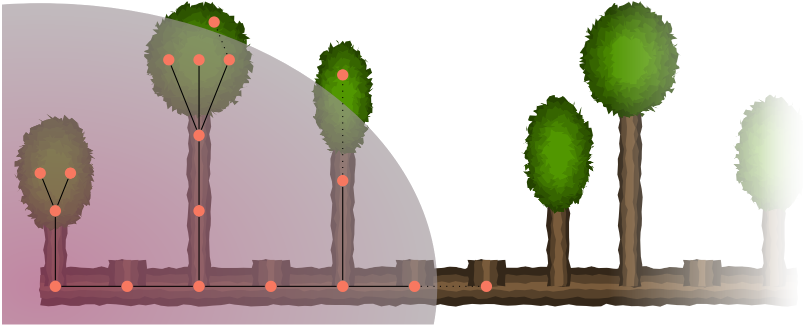

Application: uniform random trees

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$$\alpha(T_n)/n \to W_0(1) \approx 0.56714...$$

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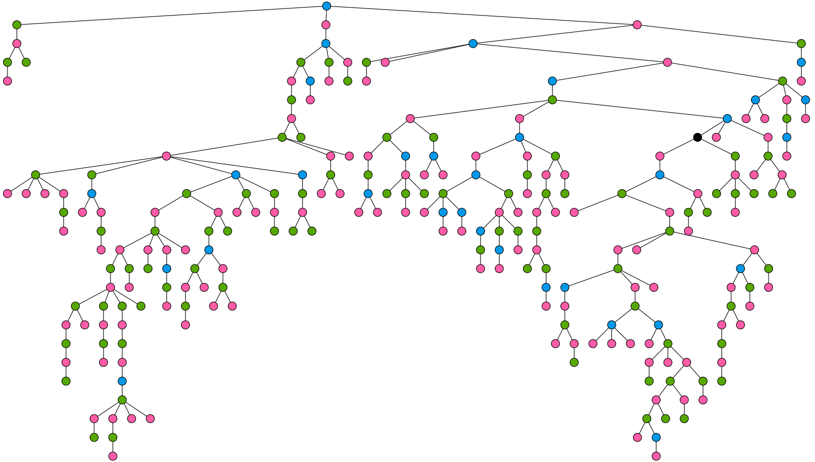

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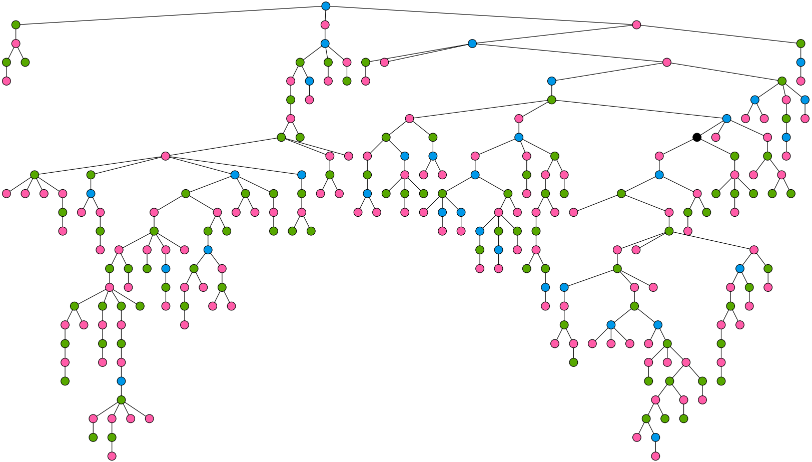
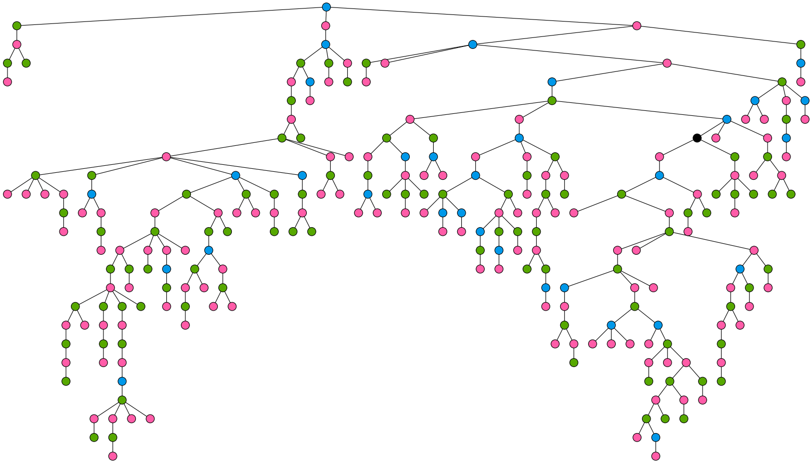


red: 125 (50%), green: 92 ($\approx$ 37%), blue: 32 ($\approx$ 13%), black: 1

Simulations don't lie (but I do)



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Greedy independence ratio — results

Flory '39, Page '59 $\iota(P_n) \to \frac{1}{2}(1 - e^{-2})$

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	(same for functional digraphs)	✷

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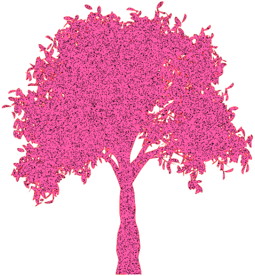

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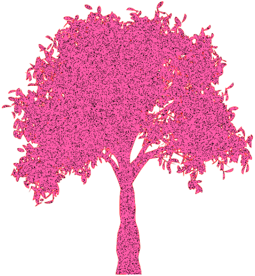
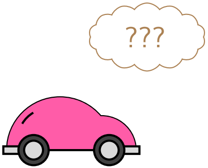


Theorem (Krivelevich, Mészáros, M., Shikhelman '20)

If T is a tree on n vertices, then $\mathbb{E}[\iota(P_n)] \leq \mathbb{E}[\iota(T)]$.

What's next?

- Graph sequences that are not locally tree-like
- Better/other local rules
- Other colours



Thank You!