UNIVERSITY OF TWENTE.

Probabilistic Analysis of Optimization Problems on Sparse Random Shortest Path Metrics

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Joint work with Bodo Manthey

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Optimization in practice

- Large scale optimization problems are hard to solve within reasonable time.
- Often heuristics are used to provide (non-optimal) solutions.
- Big gap between theoretical and actual performance!

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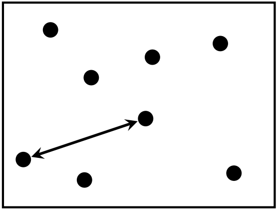
- Large scale optimization problems are hard to solve within reasonable time.
- Often heuristics are used to provide (non-optimal) solutions.
- Big gap between theoretical and actual performance!
- Some examples of worst case approximation ratios:
  - Greedy for Minimum-weight Perfect Matching: $O(n^{\log_2(3/2)}) \approx O(n^{0.58})$
  - Nearest Neighbor (greedy) for TSP: $O(\log(n))$
  - 2-Opt (local search) for TSP: $O(\sqrt{n})$
  - etc.

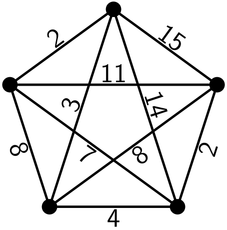
Optimization in practice

- Large scale optimization problems are hard to solve within reasonable time.
- Often heuristics are used to provide (non-optimal) solutions.
- Big gap between theoretical and actual performance!
- Probabilistic analysis and other 'beyond worst-case analysis' methods are nowadays used for analysis of the performance of these heuristics.
- Interested in $\mathbb{E}\left[\frac{ALG}{OPT}\right]$ (instead of $\frac{\mathbb{E}[ALG]}{\mathbb{E}[OPT]}$).

Random (Metric) Spaces



random in $[0,1]^2$



independent edge lengths

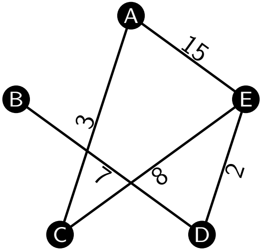
Framework for Random Metric Spaces

- We look at different models for random metric spaces.
- We study them and analyse the performance of heuristics on them.
- Goal:
  - help choosing the right heuristic for a given problem;
  - facilitate the design of better heuristics.

Random Shortest Path Metrics

- Graph $G = (V, E)$ (on n vertices)
- Random 'edge weights' $w(e)$ for all edges $e \in E$
- Distances $d(u, v)$ given by the shortest u, v-path w.r.t. weights, for all vertices $u, v \in V$
  - $d(v, v) = 0$ for all $v \in V$
  - Symmetry: $d(u, v) = d(v, u)$ for all $u, v \in V$
  - Triangle inequality: $d(u, v) \leq d(u, s) + d(s, v)$ for all $u, s, v \in V$

Random Shortest Path Metrics – Example



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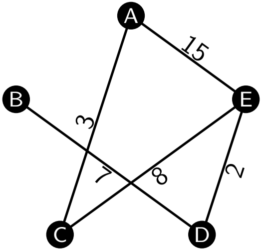
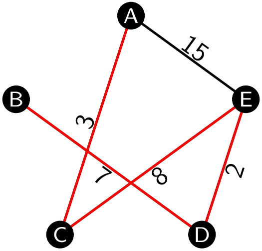


d	A	B	C	D	E
A	0				
B		0			
C			0		
D				0	
E					0

Random Shortest Path Metrics – Example



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A	0				
B		0			
C			0		
D				0	
E					0

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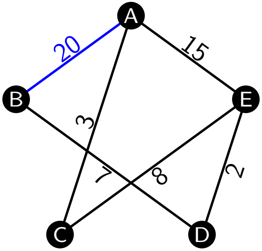
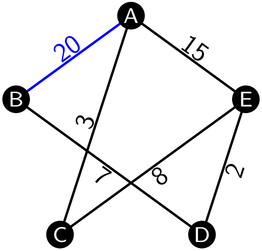


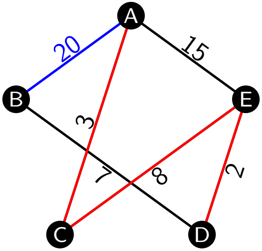
d	A	B	C	D	E
A	0	20			
B		0			
C			0		
D				0	
E					0

Random Shortest Path Metrics – Example



d	A	B	C	D	E
A	0	20	3		
B		0			
C			0		
D				0	
E					0

Random Shortest Path Metrics – Example



d	A	B	C	D	E
A	0	20	3		
B		0			
C			0		
D				0	
E					0

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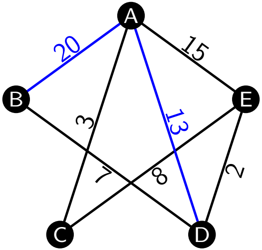

d	A	B	C	D	E
A	0	20	3	13	
B		0			
C			0		
D				0	
E					0

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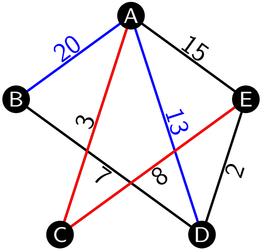

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C			0		
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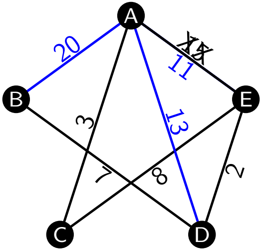
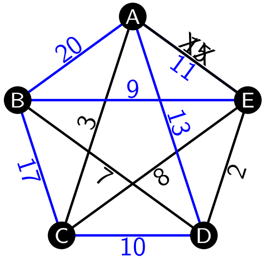


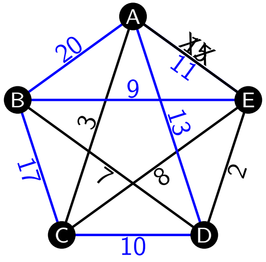
d	A	B	C	D	E
A	0	20	3	13	11
B		0			
C			0		
D				0	
E					0

Random Shortest Path Metrics – Example



d	A	B	C	D	E
A	0	20	3	13	11
B	20	0	17	7	9
C	3	17	0	10	8
D	13	7	10	0	2
E	11	9	8	2	0

Random Shortest Path Metrics – Example



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- Edge weights from (standard)exponential distribution

$\Rightarrow$ ‘memorylessness property’:
$\mathbb{P}(X > s + t \mid X > t) = \mathbb{P}(X > s)$ for all $s, t \geq 0$.

$\Rightarrow$ ‘minimum property’:
$X_1, \ldots, X_k \sim \mathrm{Exp}(1) \quad \Rightarrow \quad \min(X_i) \sim \mathrm{Exp}(k)$.

Random Shortest Path Metrics (RSPM)

- Graph $G = (V, E)$ (on n vertices)
- Random 'edge weights' $w(e)$ for all edges $e \in E$
- Distances $d(u, v)$ given by the shortest u, v-path w.r.t. weights, for all vertices $u, v \in V$
- Also known as First Passage Percolation (FPP)
- A widely studied model, but (until recently) not used for probabilistic analysis

Related results

- Probabilistic analysis using RSPM on complete graphs proposed by Karp & Steele (1985)

Theorem (Bringmann, Engels, Manthey, Rao 2013)

On RSPM generated from *complete graphs*, the following heuristics have expected approximation ratio $O(1)$:

- Greedy for Minimum-Distance Perfect Matching;
- Nearest Neighbor Heuristic for TSP;
- Insertion Heuristics for TSP (for any insertion rule R).

Also a ‘trivial’ $O(\log(n))$ approximation ratio for 2-opt for TSP, open question whether this can be improved.

Related results

- Recent efforts to adapt the model to a more realistic one.

Theorem (K., Manthey, Visser 2019)

On RSPM generated from *(dense) Erdős–Rényi random graphs*, the following heuristics have expected approximation ratio $O(1)$:

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Next step: RSPM generated from sparse graphs.

- Start from grid graphs, because most studied in FPP.

Main Result

Theorem (K., Manthey 2020)

On RSPM generated from *square grid graphs*, the following heuristics have expected approximation ratio $O(1)$:

- Greedy for Minimum-Distance Perfect Matching;*
- Nearest Neighbor Heuristic for TSP;*
- Insertion Heuristics for TSP (for any insertion rule R);*
- 2-opt for TSP (for any choice of the improvements).†

* Also for RSPM generated from a certain wide class of sparse graphs.

† Also for RSPM generated from arbitrary sparse graphs.

Main Result

Theorem (K., Manthey 2020)

On RSPM generated from *square grid graphs*, the following heuristics have expected approximation ratio $O(1)$:

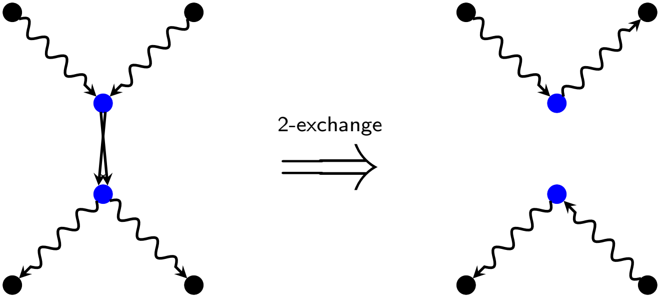
- Greedy for Minimum-Distance Perfect Matching;*
- Nearest Neighbor Heuristic for TSP;*
- Insertion Heuristics for TSP (for any insertion rule R);*
- 2-opt for TSP (for any choice of the improvements).†

- Remainder of this presentation:
  - Idea for the 2-opt result;
  - Quick sketch of the 'road' to the greedy matching result.

Idea for the 2-opt result

Observation

Consider the shortest paths corresponding to an arbitrary 2-optimal solution. Then, every edge of G is used at most twice (once per direction).



Idea for the 2-opt result

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Consider the shortest paths corresponding to an arbitrary 2-optimal solution. Then, every edge of G is used at most twice (once per direction).

- Any 2-optimal solution has length at most twice the sum of all edge weights, so $\mathbb{E}[WLO] \leq O(n)$.
- Any TSP solution uses at least $n-1$ different edge weights, so $\mathbb{E}[TSP] \geq \Omega(n)$.

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- Any TSP solution uses at least $n-1$ different edge weights, so $\mathbb{E}[TSP] \geq \Omega(n)$.

- $\mathbb{E}\left[\frac{WLO}{TSP}\right] = O(1)$

RSPM (general graphs) – Structural properties

Theorem (Davis, Prieditis 1993)

Let G be a complete graph and let $\tau_k(v)$ denote the distance to the k-th closest vertex from v. Then, for any k and v,

$$\tau_k(v) \sim \sum_{i=1}^{k-1} \mathrm{Exp}(i \cdot (n - i)).$$

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Generalization

Suppose that $|\delta(U)| \geq f(|U|)$ for some function $f(\cdot)$ and all $U \subseteq V$. Then, for any $k \in [n]$ and any $v \in V$,

$$\tau_k(v) \precsim \sum_{i=1}^{k-1} \mathrm{Exp}(f(i)).$$

RSPM (grid graphs) – cut sizes

Theorem (Bollobás, Leader 1991)

Let G be a finite square grid graph on $n = N^2$ vertices. Then, for any $U \subseteq V$:

$$|\delta(U)| \geq \begin{cases} 2\sqrt{|U|} & \text{if } |U| \leq n/4, \\ \sqrt{n} & \text{if } n/4 \leq |U| \leq 3n/4, \\ 2\sqrt{n - |U|} & \text{if } |U| \geq 3n/4. \end{cases}$$

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Remark

All results that follow can be generalized to any family of graphs that satisfies $|\delta(U)| \geq \Omega(|U|^{\varepsilon})$ for all $U \subseteq V$ with $|U| \leq cn$ (where $\varepsilon, c \in (0,1)$ are constants).

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Corollary

Let $\tau_k(v)$ denote the distance to the k-th closest vertex from v. Then, for any $k \leq n/4$ and any $v \in V$,

$$\tau_k(v) \precsim \sum_{i=1}^{k-1} \mathrm{Exp}(2\sqrt{i}).$$

RSPM (general graphs) – 'toolbox'

Theorem (Clustering)

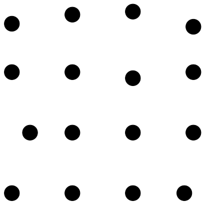
Let $\Delta > 0$. If we partition the instance into clusters of diameter at most 4Δ, then the expected number of clusters needed is $O(1 + n/\Delta^2)$.

Lemma (Tail bound for $\Delta_{\max}$)

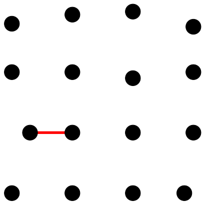
Let $\Delta_{\max} := \max_{u,v} d(u, v)$. Then for $x \geq 9\sqrt{n}$ we have

$$\mathbb{P}(\Delta_{\max} \geq x) = ne^{-x}.$$

Greedy Matching



Greedy Matching



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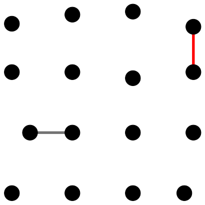

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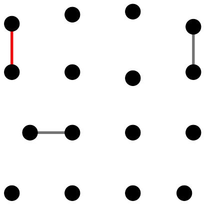

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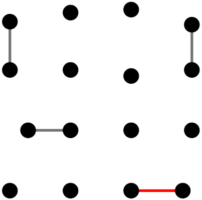

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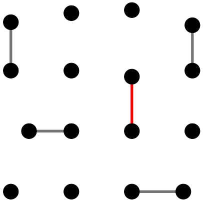

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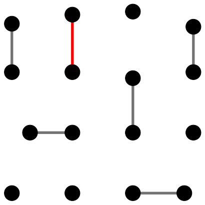

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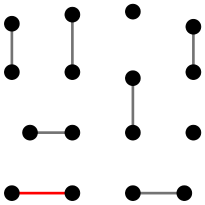

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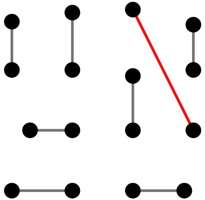

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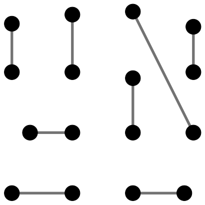

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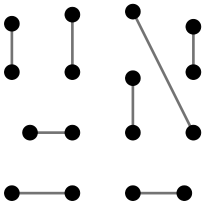


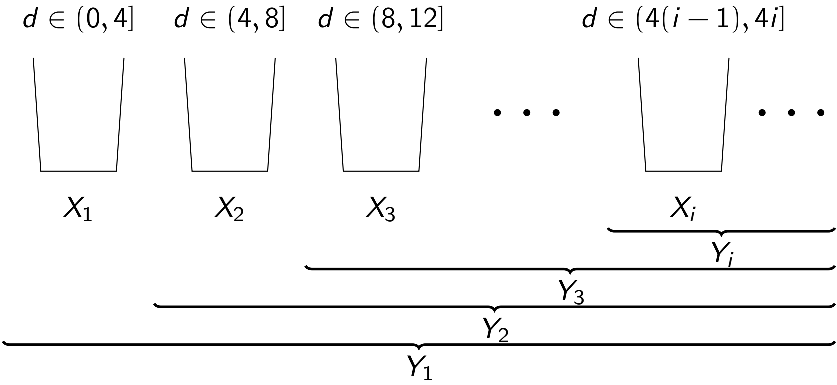
Lemma

Greedy outputs a matching with expected costs at most $O(n)$.

Theorem

Greedy has an expected approximation ratio of $O(1)$ on RSPM generated from grid graphs.

Proof idea



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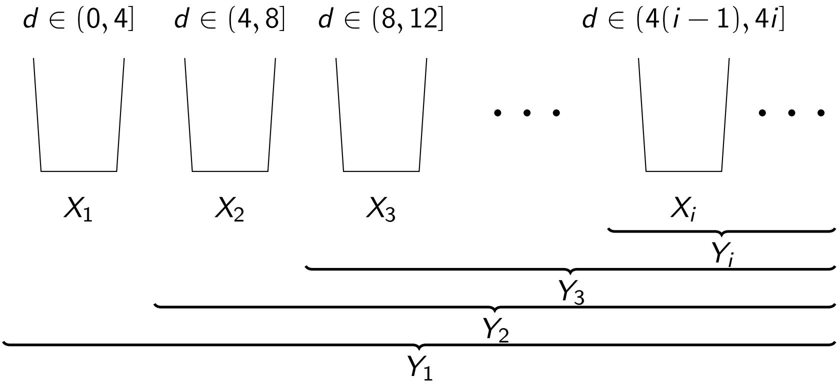

- $\mathbb{E}[GR] \leq \sum_{i=1}^{\infty} 4i \cdot \mathbb{E}[X_i] = \sum_{i=1}^{\infty} 4 \cdot \mathbb{E}[Y_i]$.
- $Y_1 = n/2$.

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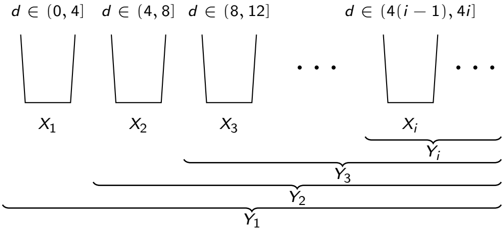
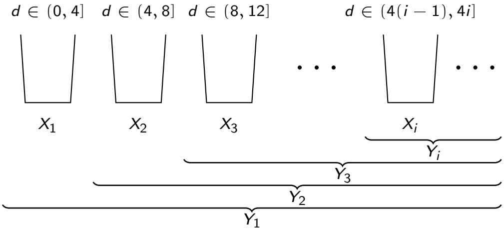
- $\mathbb{E}[GR] \leq \sum_{i=1}^{\infty} 4i \cdot \mathbb{E}[X_i] = \sum_{i=1}^{\infty} 4 \cdot \mathbb{E}[Y_i]$.



Proof idea

- $\mathbb{E}[GR] \leq \sum_{i=1}^{\infty} 4i \cdot \mathbb{E}[X_i] = \sum_{i=1}^{\infty} 4 \cdot \mathbb{E}[Y_i]$.
- A partitioning in clusters of diameter $\leq 4(i-1)$ needs $\leq O(1 + n/(i-1)^2)$ clusters.



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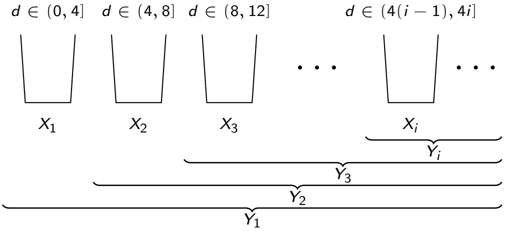
- A partitioning in clusters of diameter $\leq 4(i-1)$ needs $\leq O(1 + n/(i-1)^2)$ clusters.
- When 'Greedy' reaches bin i, at most $O(1 + n/(i-1)^2)$ unmatched vertices remain.



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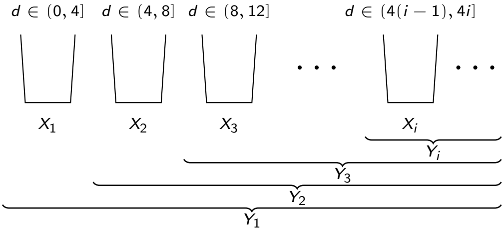
- When 'Greedy' reaches bin i, at most $O(1 + n/(i-1)^2)$ unmatched vertices remain.
- So $\mathbb{E}[Y_i] \leq O(1 + n/(i-1)^2)$ for $i > 1$.



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- When 'Greedy' reaches bin i, at most $O(1 + n/(i-1)^2)$ unmatched vertices remain.
- So $\mathbb{E}[Y_i] \leq O(1 + n/(i-1)^2)$ for $i > 1$.
- For 'large' i we have $\mathbb{E}[Y_i] \leq n \cdot \mathbb{P}(\Delta_{\max} \geq 4(i-1)) \leq n^2 e^{-4(i-1)}$.

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- A partitioning in clusters of diameter $\leq 4(i-1)$ needs $\leq O(1 + n/(i-1)^2)$ clusters.
- When 'Greedy' reaches bin i, at most $O(1 + n/(i-1)^2)$ unmatched vertices remain.
- So $\mathbb{E}[Y_i] \leq O(1 + n/(i-1)^2)$ for $i > 1$.
- For 'large' i we have $\mathbb{E}[Y_i] \leq n \cdot \mathbb{P}(\Delta_{\max} \geq 4(i-1)) \leq n^2 e^{-4(i-1)}$.
- Summing over all i yields $\mathbb{E}[GR] \leq O(n)$.

Open problems

- RSPM on arbitrary sparse graphs?
- Only using a subset of the vertices?
- ‘Hybrid heuristics’?
- ...