The Giant Component and 2-Core in Sparse Random Outerplanar Graphs

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(Joint work with Mihyun Kang)

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TU Graz

DOCTORAL PROGRAM DISCRETE MATHEMATICS
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Overview

- Random planar graphs
- Emergence of the giant component
  - Random planar graphs
  - Random outerplanar graphs
- Core–kernel approach for planar graphs
- Core approach for outerplanar graphs

Random planar graph $P(n, m)$

- Pick a planar graph with n vertices and m edges (uniformly at random).

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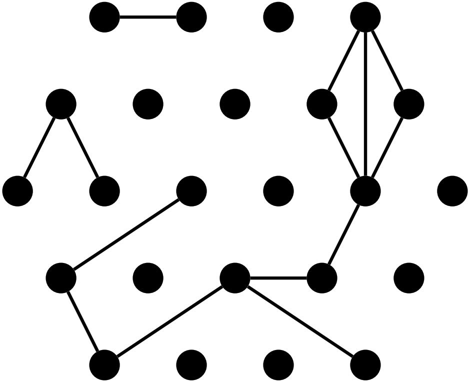
$n = 24 \qquad m = 14$



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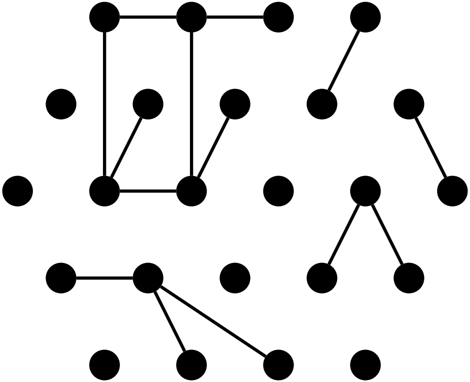

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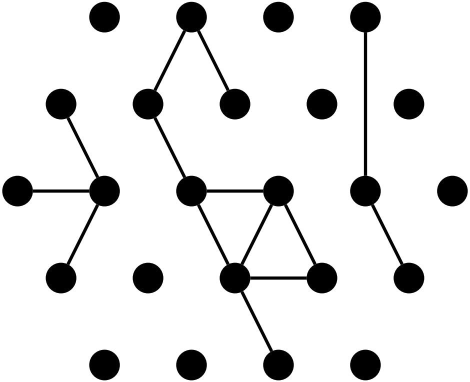

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- $n \to \infty$

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- How does the component structure change when m varies?

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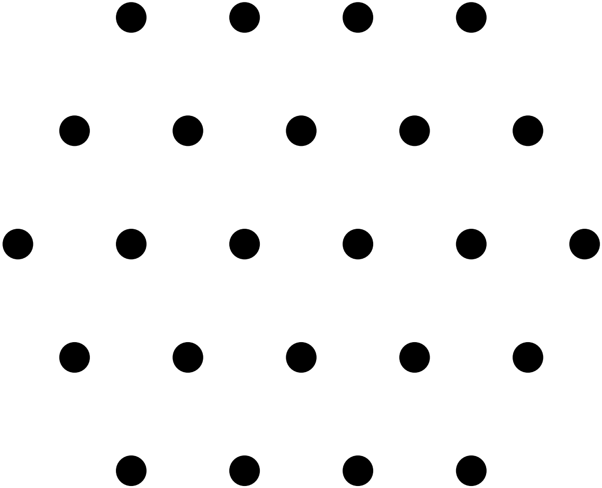

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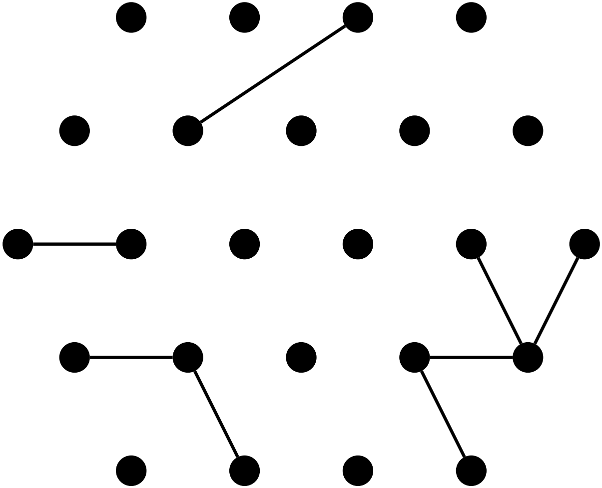

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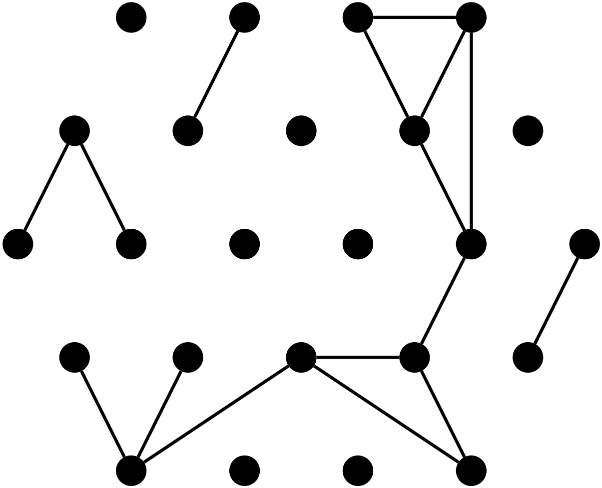
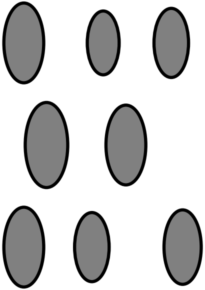


Random planar graph $P(n, m)$

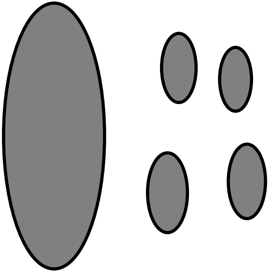
Theorem [Kang, Łuczak, 2012]

The giant component emerges at

$$m = n/2 + O\left(n^{2/3}\right).$$



$m \leq n/2 + O\left(n^{2/3}\right)$

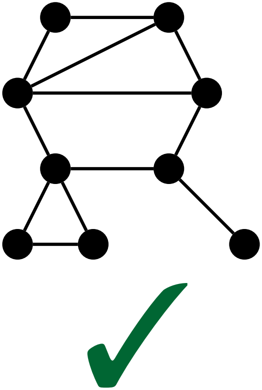


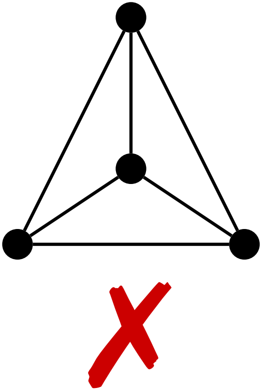
$m \geq n/2 + \omega\left(n^{2/3}\right)$

Random outerplanar graphs

Outerplanar graph

has a planar drawing, such that all vertices belong to the outer face.





Random outerplanar graphs

Theorem [Kang-M. 2020]

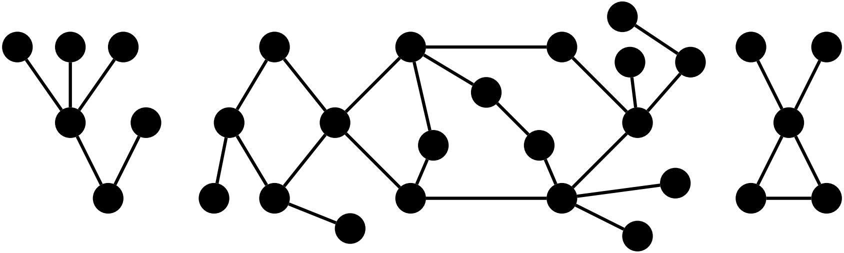
Outerplanar graphs feature a similar phase transition as planar graphs, i.e. the giant component emerges at

$$m = n/2 + O\left(n^{2/3}\right).$$

Core–kernel approach for planar graphs

Decomposition

- *Complex part*: union of components with at least two cycles



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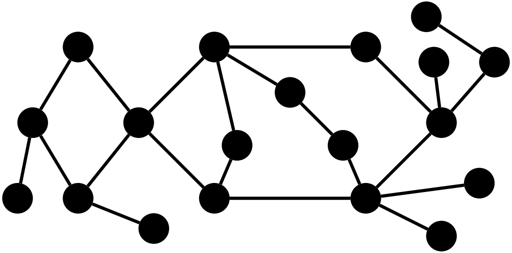

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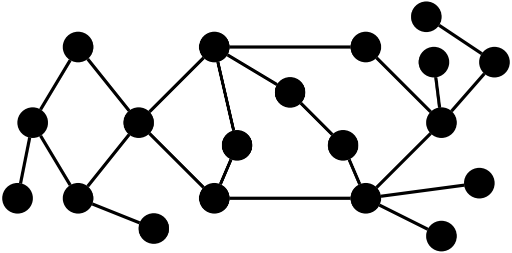

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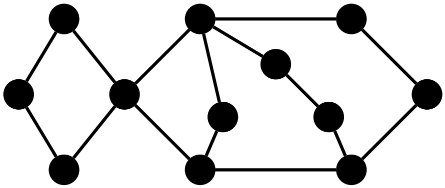

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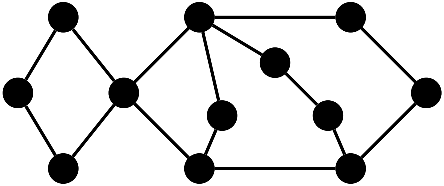
- *Complex part*: union of components with at least two cycles
- *Core*: maximal subgraph of minimum degree at least two
- *Kernel*: replace paths consisting of vertices of degree two by edges



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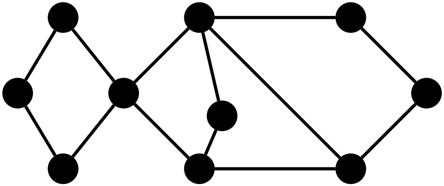

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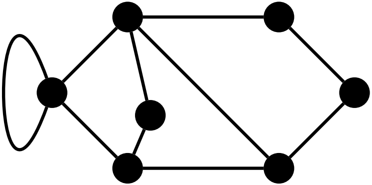

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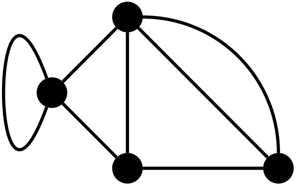
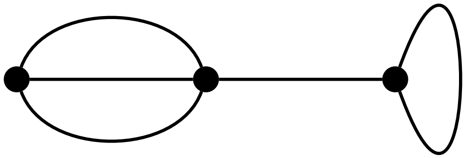


Core–kernel approach for planar graphs

Construction

- Start with a kernel



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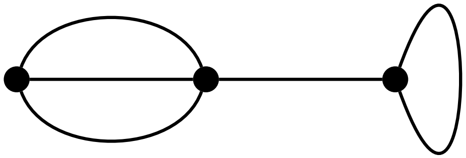
- Start with a kernel
- Subdivide edges to obtain core



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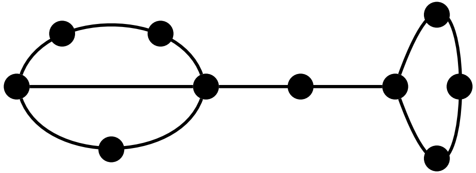
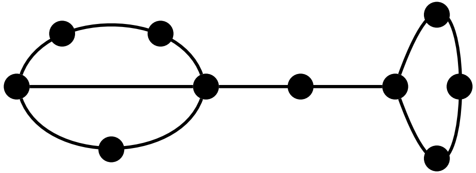


Core–kernel approach for planar graphs

Construction

- Start with a kernel
- Subdivide edges to obtain core
- Attach rooted trees to get complex part



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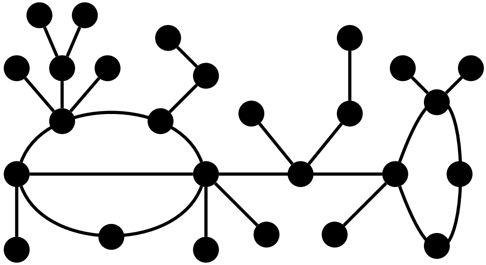
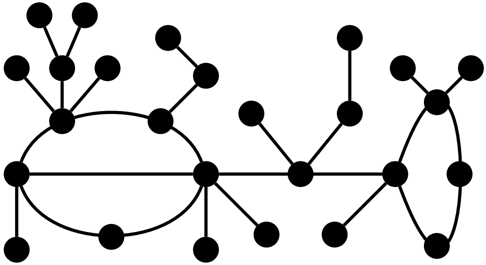


Core–kernel approach for planar graphs

Construction

- Start with a kernel
- Subdivide edges to obtain core
- Attach rooted trees to get complex part
- Add trees and unicyclic components



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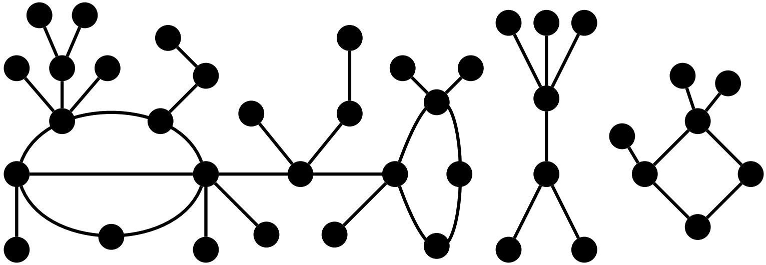

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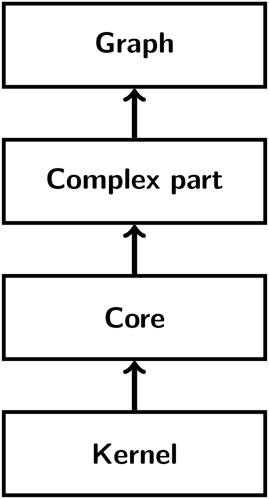

Core–kernel approach for planar graphs

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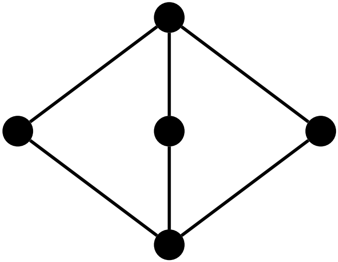
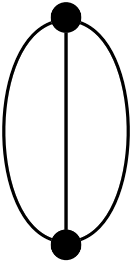
- BUT: A non-outerplanar graph can have an outerplanar kernel.



$K_{2,3}$



kernel of $K_{2,3}$

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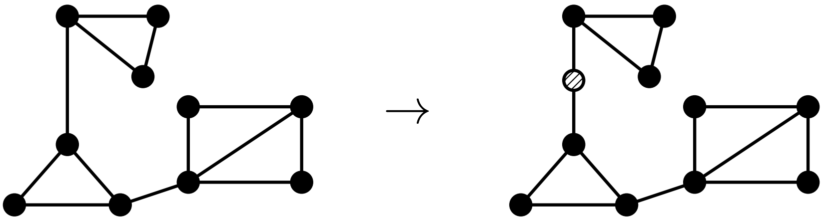

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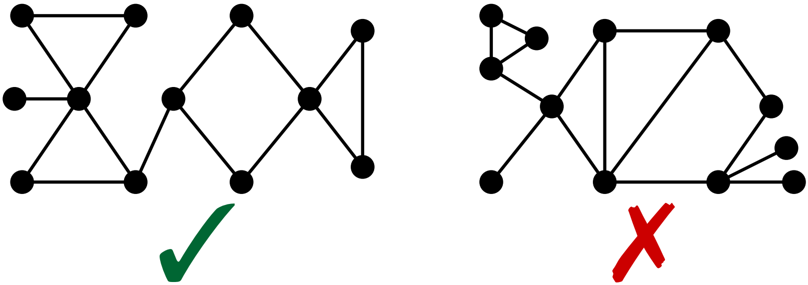

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- Proof idea for planar graphs: Core–kernel approach
  - Key fact: a graph is planar $\Longleftrightarrow$ its kernel is planar
- Proof idea for outerplanar graphs: Core approach
  - Direct analysis of the core (without using the kernel)