Asymptotics of minimal deterministic finite automata recognizing a finite binary language

AofA 2020

Andrew Elvey Price, Wenjie Fang, and Michael Wallner

Institut Denis-Poisson, Université de Tours, France

September, 2020

What is a DFA?

Deterministic finite automata (DFA)

DFA on alphabet $\{a, b\}$

Graph with

- two outgoing edges from each node (state), labelled a and b
- An initial state q_0
- A set F of *final states* (coloured green).

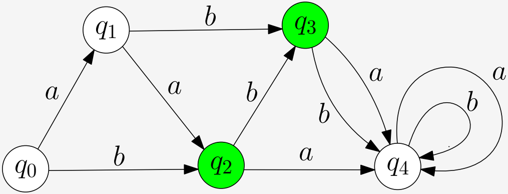


Figure: A DFA.

Deterministic finite automata (DFA)

DFA on alphabet $\{a, b\}$

Graph with

- two outgoing edges from each node (state), labelled a and b
- An initial state q_0
- A set F of *final states* (coloured green).

Properties

- **Language:** the set of accepted words
- **Minimal:** no DFA with fewer states accepts the same language
- **Acyclic:** no cycles (except loops at unique sink)

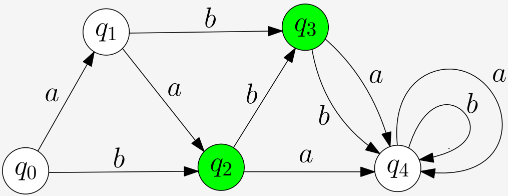
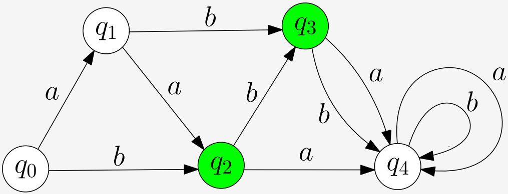


Figure: A DFA. This is the minimal DFA recognising the language $\{aa, aab, ab, b, bb\}$.

Counting minimal acyclic DFAs

This work: Asymptotics of the numbers m_n of minimal, acyclic DFAs on a binary alphabet with $n+1$ nodes.

- Studied by Domaratzki, Kisman, Shallit and Liskovets between 2002 and 2006
- Best bounds were out by an exponential factor
- We gave upper and lower bounds differing by a $\Theta(n^{1/4})$ factor, by relating the DFAs to compacted trees.



Main result

Main result – A stretched exponential appears

Theorem

The number m_n of minimal DFAs recognising a finite binary for $n \to \infty$

$$m_n = \Theta\left(n!\, 8^n e^{3a_1 n^{1/3}} n^{7/8}\right),$$

where $a_1 \approx -2.3381$ is the largest root of the Airy function
$\mathrm{Ai}(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt.$

Main result – A stretched exponential appears

Theorem

The number m_n of minimal DFAs recognising a finite binary for $n \to \infty$

$$m_n = \Theta\left(n!\, 8^n e^{3a_1 n^{1/3}} n^{7/8}\right),$$

where $a_1 \approx -2.3381$ is the largest root of the Airy function $\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt.$

Conjecture

Experimentally we find

$$m_n \sim \gamma n! 8^n e^{3a_1 n^{1/3}} n^{7/8},$$

where

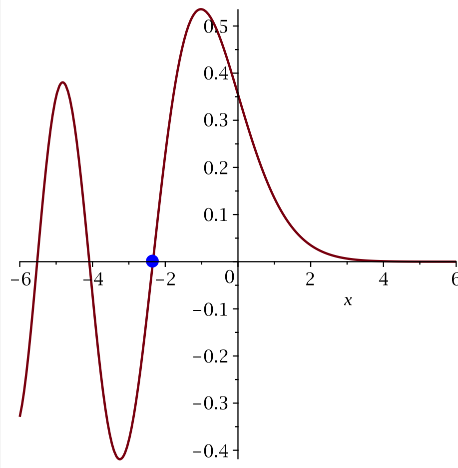
$$\gamma \approx 76.438160702.$$

What is the Airy function?

Properties

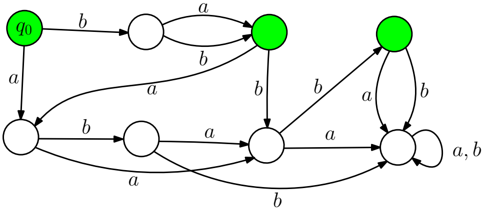
- $\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt$
- Largest root $a_1 \approx -2.3381$
- $\lim_{x\to\infty} \mathrm{Ai}(x) = 0$
- Also defined by $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$

- [Banderier, Flajolet, Schaeffer, Soria 2001]: Random Maps
- [Flajolet, Louchard 2001]: Brownian excursion area

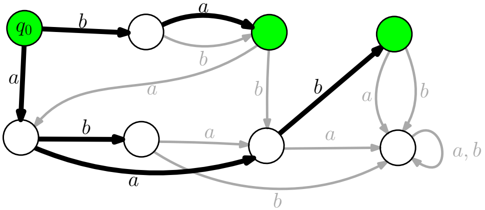


Bijection to decorated paths

Bijection to decorated paths

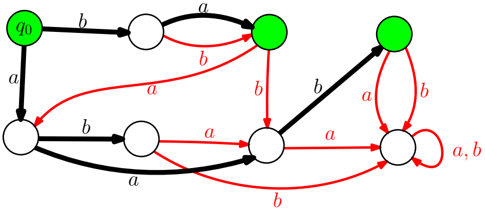


Bijection to decorated paths



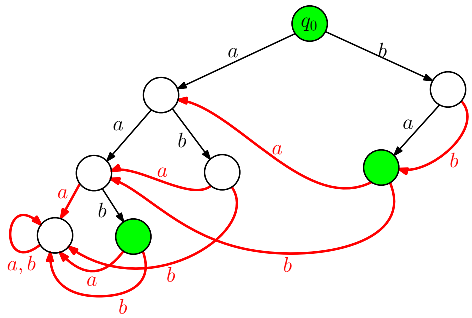
- Highlight spanning tree given by depth first search (ignoring the sink)
- I.e., Black path to each vertex is first in lexicographic order
- Colour other edges red
- Draw as a binary tree with *a* edges pointing left and *b* edges pointing right

Bijection to decorated paths



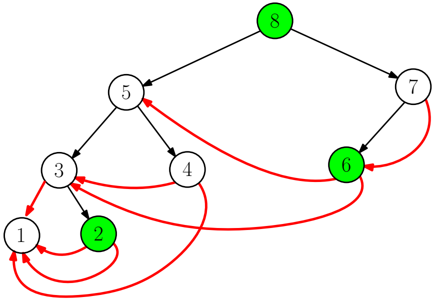
- Highlight spanning tree given by depth first search (ignoring the sink)
- I.e., Black path to each vertex is first in lexicographic order
- Colour other edges red
- Draw as a binary tree with *a* edges pointing left and *b* edges pointing right

Bijection to decorated paths



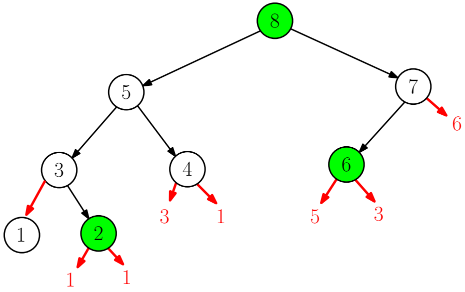
- Highlight spanning tree given by depth first search (ignoring the sink)
- I.e., Black path to each vertex is first in lexicographic order
- Colour other edges red
- Draw as a binary tree with *a* edges pointing left and *b* edges pointing right

Bijection to decorated paths



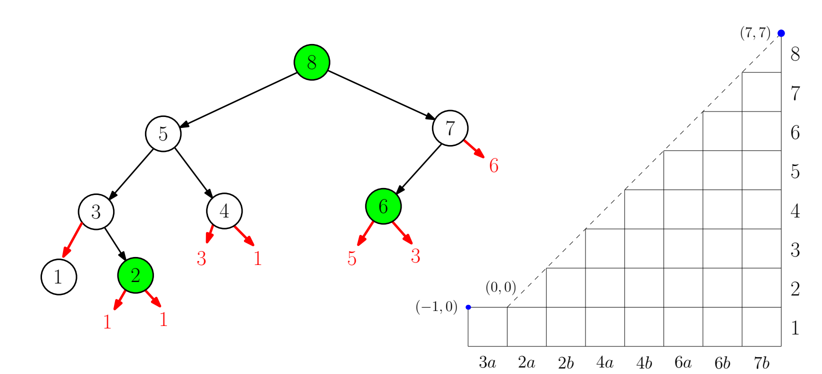
- Label nodes in post-order. By construction red edges point from a larger number to a smaller number
- → Label pointers

Bijection to decorated paths

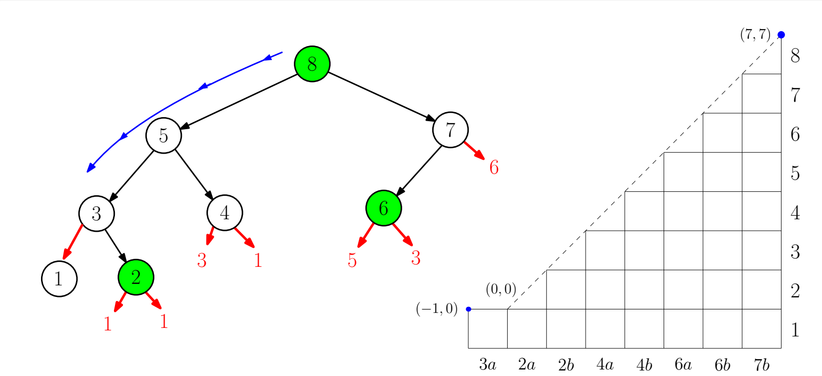


- Label nodes in post-order. By construction red edges point from a larger number to a smaller number
- $\rightarrow$ Label pointers

Bijection to decorated paths



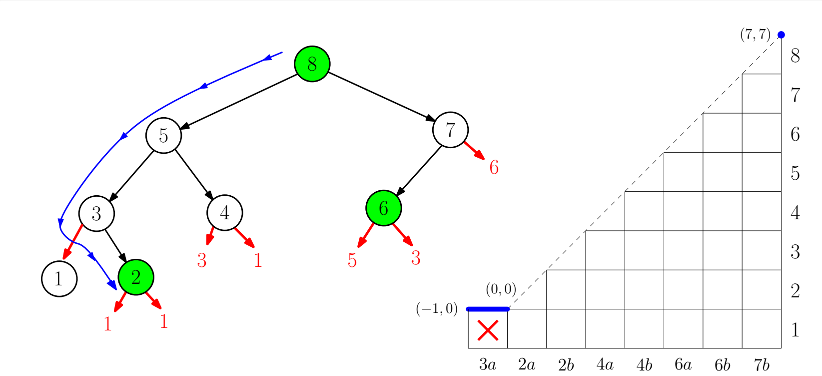
Bijection to decorated paths



When the tree traversal...

- goes up: add up step with colour matching the corresponding node.
- passes a pointer:
  - add horizontal step
  - mark box corresponding to pointer label

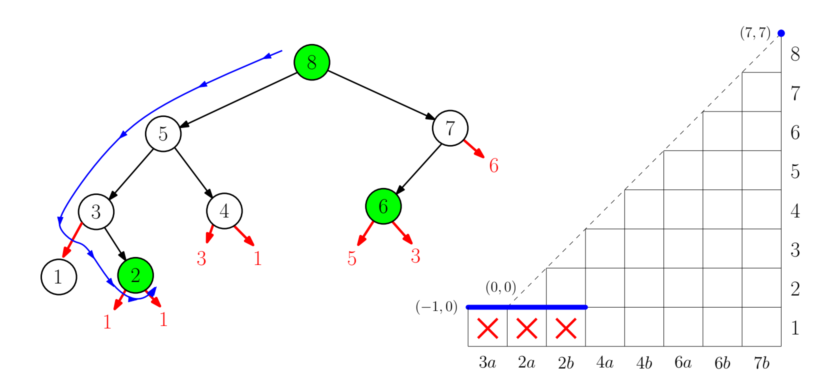
Bijection to decorated paths



When the tree traversal...

- goes up: add up step with colour matching the corresponding node.
- passes a pointer:
  - add horizontal step
  - mark box corresponding to pointer label

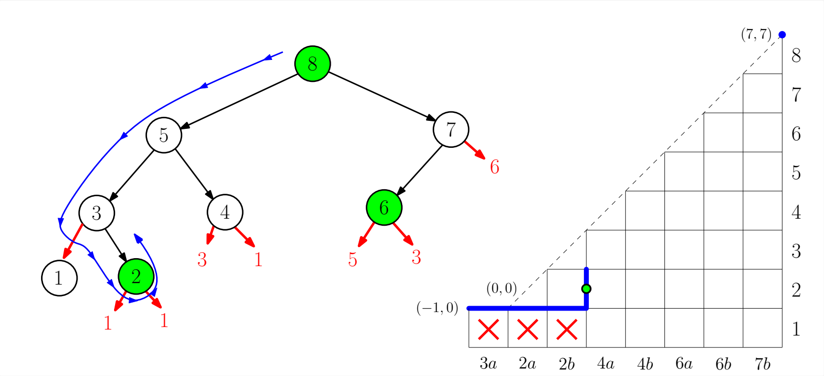
Bijection to decorated paths



When the tree traversal...

- goes up: add up step with colour matching the corresponding node.
- passes a pointer:
  - add horizontal step
  - mark box corresponding to pointer label

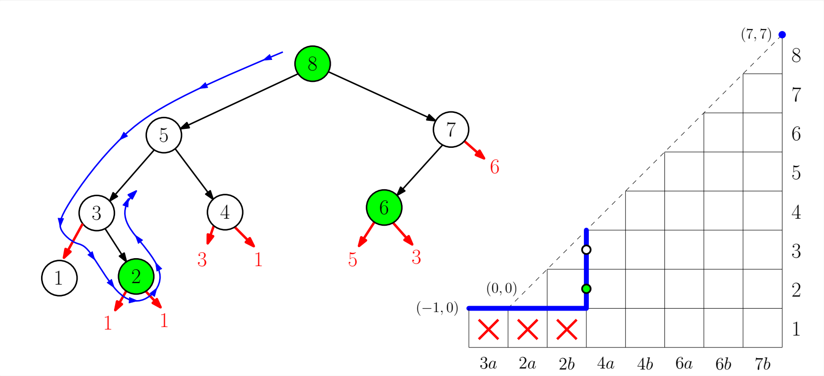
Bijection to decorated paths



When the tree traversal...

- goes up: add up step with colour matching the corresponding node.
- passes a pointer:
  - add horizontal step
  - mark box corresponding to pointer label

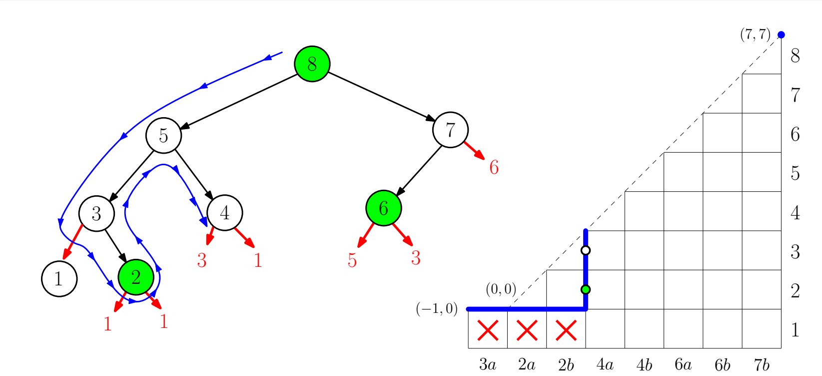
Bijection to decorated paths



When the tree traversal...

- goes up: add up step with colour matching the corresponding node.
- passes a pointer:
  - add horizontal step
  - mark box corresponding to pointer label

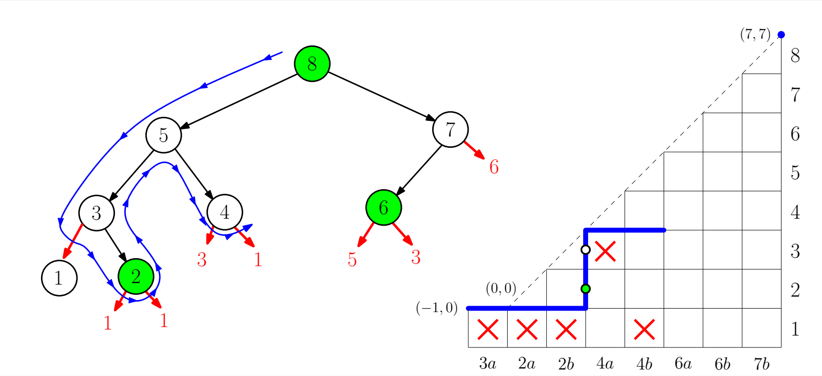
Bijection to decorated paths



When the tree traversal...

- goes up: add up step with colour matching the corresponding node.
- passes a pointer:
  - add horizontal step
  - mark box corresponding to pointer label

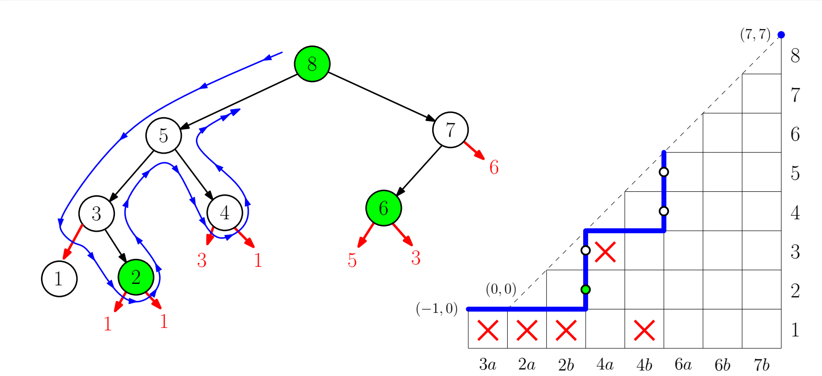
Bijection to decorated paths



When the tree traversal...

- goes up: add up step with colour matching the corresponding node.
- passes a pointer:
  - add horizontal step
  - mark box corresponding to pointer label

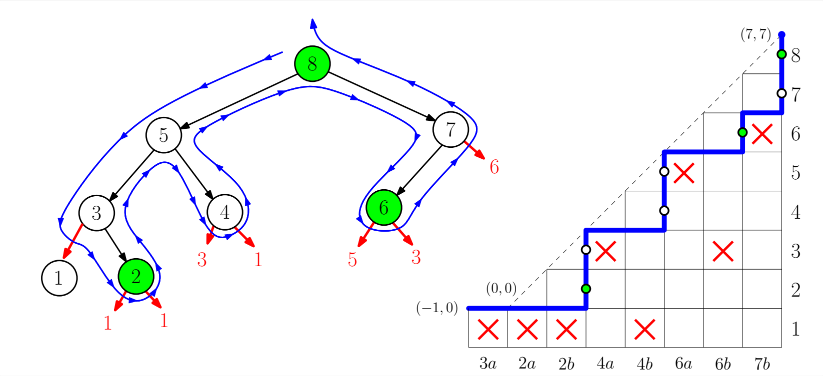
Bijection to decorated paths



When the tree traversal...

- goes up: add up step with colour matching the corresponding node.
- passes a pointer:
  - add horizontal step
  - mark box corresponding to pointer label

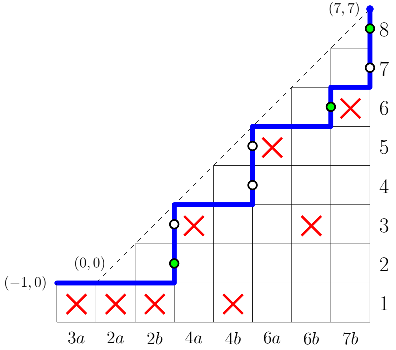
Bijection to decorated paths



When the tree traversal...

- goes up: add up step with colour matching the corresponding node.
- passes a pointer:
  - add horizontal step
  - mark box corresponding to pointer label

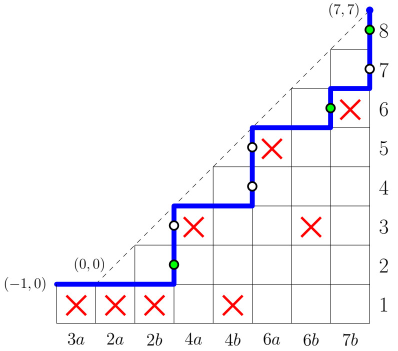
Decorated paths



- Path starts at $(-1, 0)$ and ends at (n, n)
- Path stays below diagonal (after first step)
- One box is marked below each horizontal step
- Each vertical step is coloured white or green

By the bijection: The number of these paths is the number d_n of acyclic DFAs with $n + 1$ nodes.

Decorated paths



Recurrence: Denote by $a_{n,m}$ the number of paths ending at (n, m).

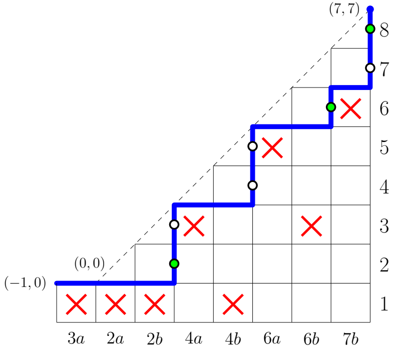
$$a_{n,m} = 2a_{n,m-1} + (m+1)a_{n-1,m}, \qquad \text{for } n \geq m$$

$$a_{-1,0} = 1.$$

By the bijection: $d_n = a_{n,n}$ is the number of acyclic DFAs with $n+1$ nodes.

What about minimality?

Decorated paths



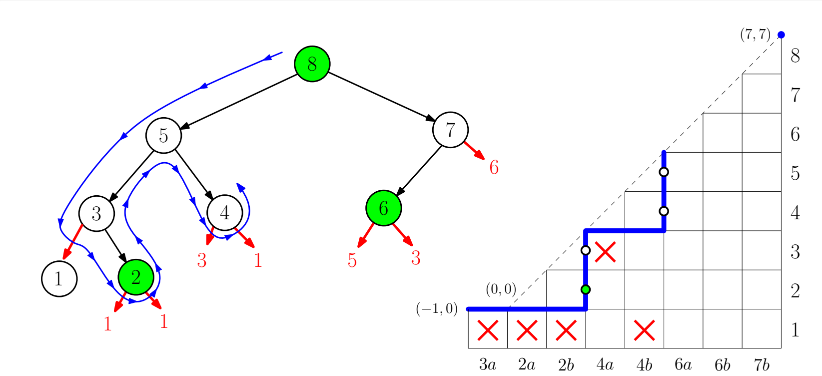
Recurrence: Denote by $a_{n,m}$ the number of paths ending at (n, m).

$$a_{n,m} = 2a_{n,m-1} + (m+1)a_{n-1,m}, \qquad \text{for } n \geq m$$
$$a_{-1,0} = 1.$$

By the bijection: $d_n = a_{n,n}$ is the number of acyclic DFAs with $n + 1$ nodes.

What about minimality?

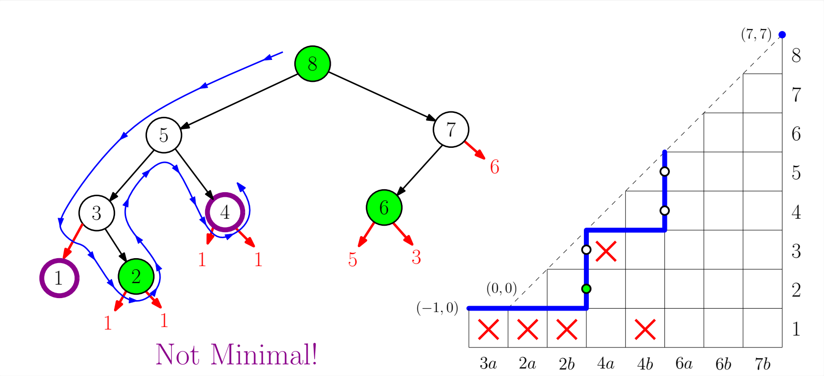
Minimal acyclic DFAs



For the DFA to be minimal, no state can be equivalent to a previous state:

- only possible if the new node is a leaf.
- If leaf is labelled $m + 1$, then m choices of pointer labels and state colour must be avoided.
- Leaf corresponds to $\rightarrow \rightarrow \uparrow$ in path.

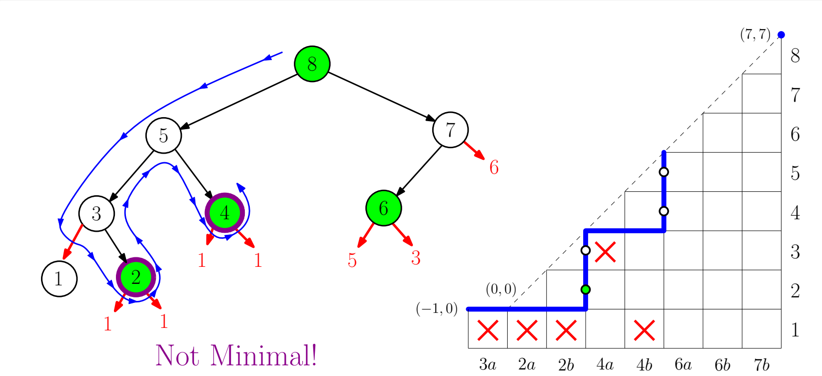
Minimal acyclic DFAs



For the DFA to be minimal, no state can be equivalent to a previous state:

- only possible if the new node is a leaf.
- If leaf is labelled $m + 1$, then m choices of pointer labels and state colour must be avoided.
- Leaf corresponds to $\rightarrow \rightarrow \uparrow$ in path.

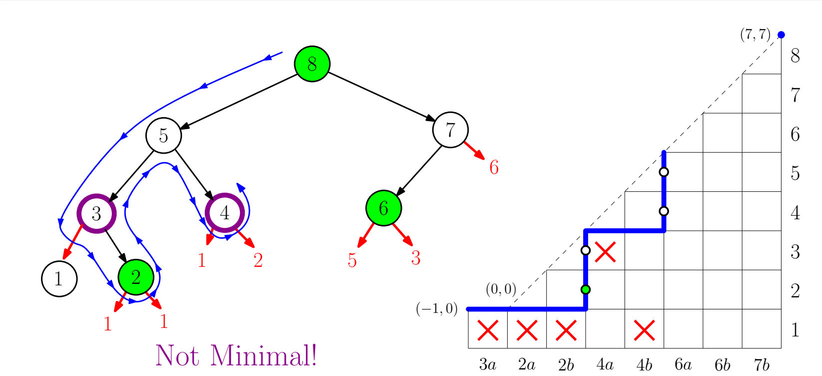
Minimal acyclic DFAs



For the DFA to be minimal, no state can be equivalent to a previous state:

- only possible if the new node is a leaf.
- If leaf is labelled $m+1$, then m choices of pointer labels and state colour must be avoided.
- Leaf corresponds to $\rightarrow \rightarrow \uparrow$ in path.

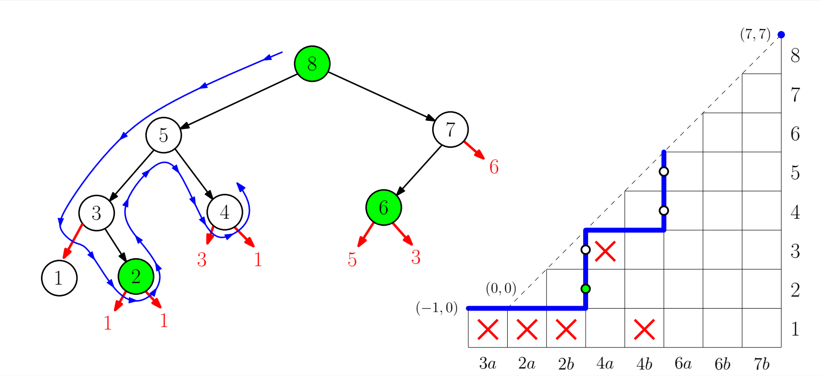
Minimal acyclic DFAs



For the DFA to be minimal, no state can be equivalent to a previous state:

- only possible if the new node is a leaf.
- If leaf is labelled $m+1$, then m choices of pointer labels and state colour must be avoided.
- Leaf corresponds to $\rightarrow \rightarrow \uparrow$ in path.

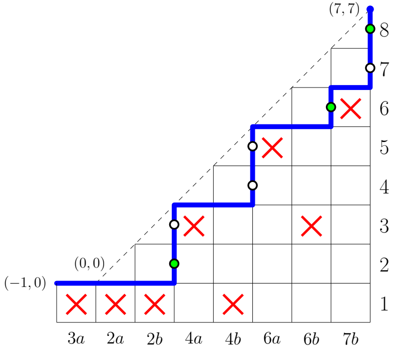
Minimal acyclic DFAs



For the DFA to be minimal, no state can be equivalent to a previous state:

- only possible if the new node is a leaf.
- If leaf is labelled $m+1$, then m choices of pointer labels and state colour must be avoided.
- Leaf corresponds to $\rightarrow\rightarrow\uparrow$ in path.

Recurrence for minimal DFAs



Recurrence: Denote by $b_{n,m}$ the number of paths ending at (n, m).

$$b_{n,m} = 2b_{n,m-1} + (m+1)b_{n-1,m} - mb_{n-2,m-1}, \qquad \text{for } n \geq m,$$

$$b_{-1,0} = 1.$$

By the bijection: $m_n = b_{n,n}$ is the number of minimal acyclic DFAs with $n+1$ nodes.

Transforming recurrence for minimal DFAs

Recurrence: Denote by $b_{n,m}$ the number of paths ending at (n, m).

$$b_{n,m} = 2b_{n,m-1} + (m+1)b_{n-1,m} - mb_{n-2,m-1}, \qquad \text{for } n \geq m,$$
$$b_{-1,0} = 1.$$

By the bijection: $m_n = b_{n,n}$ is the number of minimal acyclic DFAs with $n+1$ nodes.

Transformation: Define $e_{n,m}$ by

$$e_{n+m,n-m} = \frac{1}{n!2^{m-1}} b_{n,m}.$$

New recurrence:

$$e_{n,m} = \frac{n-m+2}{n+m} e_{n-1,m-1} + e_{n-1,m+1} - \frac{n-m}{(n+m)(n+m-2)} e_{n-3,m-1}.$$

Now $m_n = n!2^{n-1} e_{2n,0}$.

Weights are now closer to 1, and steps (now ↗ and ↘) always increase n.

Transforming recurrence for minimal DFAs

Recurrence: Denote by $b_{n,m}$ the number of paths ending at (n, m).

$$b_{n,m} = 2b_{n,m-1} + (m+1)b_{n-1,m} - mb_{n-2,m-1}, \qquad \text{for } n \geq m,$$
$$b_{-1,0} = 1.$$

By the bijection: $m_n = b_{n,n}$ is the number of minimal acyclic DFAs with $n+1$ nodes.

Transformation: Define $e_{n,m}$ by

$$e_{n+m,n-m} = \frac{1}{n!2^{m-1}} b_{n,m}.$$

New recurrence:

$$e_{n,m} = \frac{n-m+2}{n+m} e_{n-1,m-1} + e_{n-1,m+1} - \frac{n-m}{(n+m)(n+m-2)} e_{n-3,m-1}.$$

Now $m_n = n!2^{n-1} e_{2n,0}$.

Weights are now closer to 1, and steps (now $\nearrow$ and $\searrow$) always increase n.

Transforming recurrence for minimal DFAs

Recurrence: Denote by $b_{n,m}$ the number of paths ending at (n,m).

$$b_{n,m} = 2b_{n,m-1} + (m+1)b_{n-1,m} - mb_{n-2,m-1}, \qquad \text{for } n \geq m,$$
$$b_{-1,0} = 1.$$

By the bijection: $m_n = b_{n,n}$ is the number of minimal acyclic DFAs with $n+1$ nodes.

Transformation: Define $e_{n,m}$ by

$$e_{n+m,n-m} = \frac{1}{n!2^{m-1}} b_{n,m}.$$

New recurrence:

$$e_{n,m} = \frac{n-m+2}{n+m} e_{n-1,m-1} + e_{n-1,m+1} - \frac{n-m}{(n+m)(n+m-2)} e_{n-3,m-1}.$$

Now $m_n = n!2^{n-1} e_{2n,0}$.

Weights are now closer to 1, and steps (now $\nearrow$ and $\searrow$) always increase n.

Heuristics

Heuristics

We want to understand $e_{n,m}$ for large n.

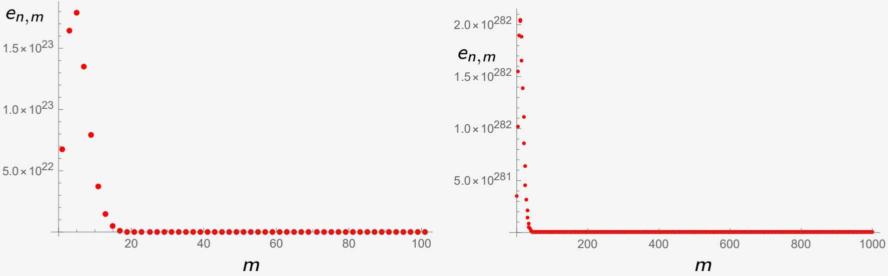


Figure: Plots of $e_{n,m}$ against $m+1$. **Left:** $n = 100$, **Right:** $n = 1000$

- Let's zoom in to the left (small m) where interesting things are happening.
- It seems to be converging to something...

Guess: $e_{n,m} \approx h(n) f\left(\frac{m+1}{g(n)}\right)$. Moreover, we guess $g(n) = \sqrt[3]{n}$.

Heuristics

We want to understand $e_{n,m}$ for large n.

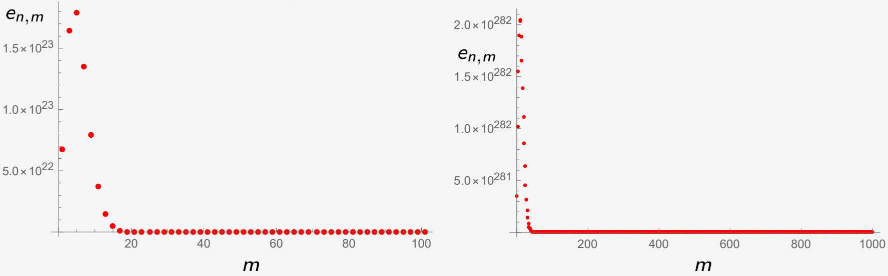


Figure: Plots of $e_{n,m}$ against $m+1$. **Left:** $n = 100$, **Right:** $n = 1000$

- Let's zoom in to the left (small m) where interesting things are happening.
- It seems to be converging to something...

Guess: $e_{n,m} \approx h(n) f\left(\frac{m+1}{g(n)}\right)$. Moreover, we guess $g(n) = \sqrt[3]{n}$.

Heuristics

We want to understand $e_{n,m}$ for large n.

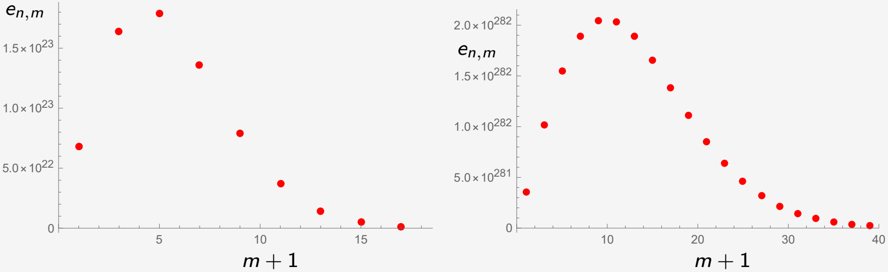


Figure: Plots of $e_{n,m}$ against $m+1$. **Left:** $n = 100$, **Right:** $n = 1000$

- Let's zoom in to the left (small m) where interesting things are happening.
- It seems to be converging to something...

Guess: $e_{n,m} \approx h(n) f\left(\frac{m+1}{g(n)}\right)$. Moreover, we guess $g(n) = \sqrt[3]{n}$.

Heuristics

We want to understand $e_{n,m}$ for large n.

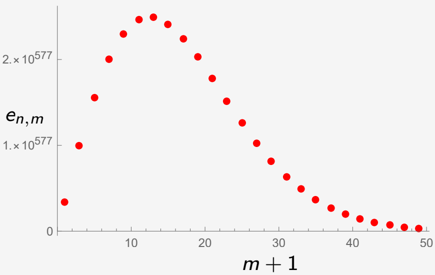


Figure: **Left:** Plot of $e_{n,m}$ against $m+1$ for $n = 2000$. **Right:** Limiting function $f(x)$

- Let's zoom in to the left (small m) where interesting things are happening.
- It seems to be converging to something...

Guess: $e_{n,m} \approx h(n) f\left(\frac{m+1}{g(n)}\right)$. Moreover, we guess $g(n) = \sqrt[3]{n}$.

Heuristics

We want to understand $e_{n,m}$ for large n.

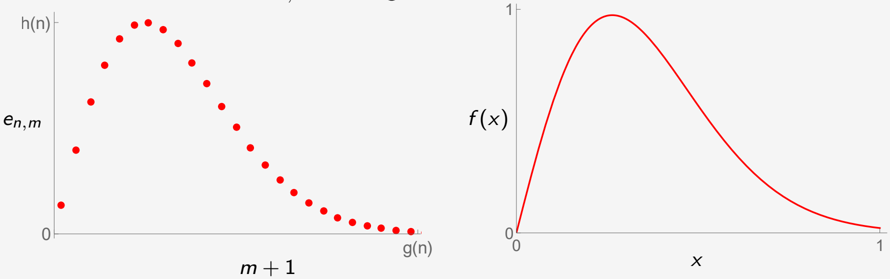


Figure: **Left:** Plot of $e_{n,m}$ against $m+1$ for $n = 2000$. **Right:** Limiting function $f(x)$

- Let's zoom in to the left (small m) where interesting things are happening.
- It seems to be converging to something...

Guess: $e_{n,m} \approx h(n) f\left(\frac{m+1}{g(n)}\right)$. Moreover, we guess $g(n) = \sqrt[3]{n}$.

Heuristics

We want to understand $e_{n,m}$ for large n.

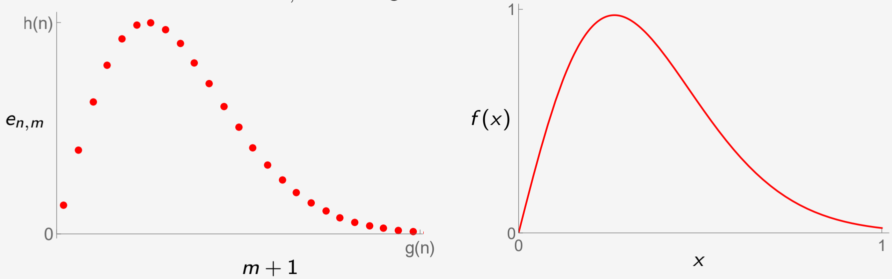


Figure: **Left:** Plot of $e_{n,m}$ against $m+1$ for $n = 2000$. **Right:** Limiting function $f(x)$

- Let's zoom in to the left (small m) where interesting things are happening.
- It seems to be converging to something...

Guess: $e_{n,m} \approx h(n) f\left(\frac{m+1}{\sqrt[3]{n}}\right)$. Moreover, we guess $g(n) = \sqrt[3]{n}$.

Heuristic analysis of weighted paths

Recurrence:

$$e_{n,m} = \frac{n-m+2}{n+m} e_{n-1,m-1} + e_{n-1,m+1} - \frac{n-m}{(n+m)(n+m-2)} e_{n-3,m-1}.$$

Guess: $e_{n,m} \approx h(n) f\left(\frac{m+1}{\sqrt[3]{n}}\right).$

Substitute into recurrence and set $m = \kappa\sqrt[3]{n} - 1$:

$$s_n := \frac{h(n)}{h(n-1)} \approx 2 + \frac{f''(\kappa) - 2\kappa f(\kappa)}{f(\kappa)} n^{-2/3} + O(n^{-1})$$

Solution (assuming equality above):

$$s_n = 2 + cn^{-2/3} + O(n^{-1}) \quad \Rightarrow \quad h(n) \approx 2^n e^{\frac{3c}{2} n^{1/3}}$$

$$f''(\kappa) = (2\kappa + c) f(\kappa) \quad \Rightarrow \quad f(\kappa) = \mathrm{Ai}(2^{-2/3}(2\kappa + c))$$

Where c is constant.

Then $f(0) = 0$ implies $c = 2^{2/3} a_1$, where $a_1 \approx -2.338$ satisfies $\mathrm{Ai}(a_1) = 0$.

Heuristic analysis of weighted paths

Recurrence:

$$e_{n,m} = \frac{n-m+2}{n+m} e_{n-1,m-1} + e_{n-1,m+1} - \frac{n-m}{(n+m)(n+m-2)} e_{n-3,m-1}.$$

Guess: $e_{n,m} \approx h(n) f\left(\frac{m+1}{\sqrt[3]{n}}\right).$

Substitute into recurrence and set $m = \kappa\sqrt[3]{n} - 1$:

$$s_n := \frac{h(n)}{h(n-1)} \approx 2 + \frac{f''(\kappa) - 2\kappa f(\kappa)}{f(\kappa)} n^{-2/3} + O(n^{-1})$$

Solution (assuming equality above):

$$s_n = 2 + cn^{-2/3} + O(n^{-1}) \quad \Rightarrow \quad h(n) \approx 2^n e^{\frac{3c}{2} n^{1/3}}$$

$$f''(\kappa) = (2\kappa + c) f(\kappa) \quad \Rightarrow \quad f(\kappa) = \mathrm{Ai}(2^{-2/3}(2\kappa + c))$$

Where c is constant.

Then $f(0) = 0$ implies $c = 2^{2/3} a_1$, where $a_1 \approx -2.338$ satisfies $\mathrm{Ai}(a_1) = 0$.

Heuristic analysis of weighted paths

Recurrence:

$$e_{n,m} = \frac{n-m+2}{n+m} e_{n-1,m-1} + e_{n-1,m+1} - \frac{n-m}{(n+m)(n+m-2)} e_{n-3,m-1}.$$

Guess: $e_{n,m} \approx h(n) f\left(\frac{m+1}{\sqrt[3]{n}}\right).$

Substitute into recurrence and set $m = \kappa\sqrt[3]{n} - 1$:

$$s_n := \frac{h(n)}{h(n-1)} \approx 2 + \frac{f''(\kappa) - 2\kappa f(\kappa)}{f(\kappa)} n^{-2/3} + O(n^{-1})$$

Solution (assuming equality above):

$$s_n = 2 + cn^{-2/3} + O(n^{-1}) \qquad \Rightarrow \qquad h(n) \approx 2^n e^{\frac{3c}{2} n^{1/3}}$$

$$f''(\kappa) = (2\kappa + c) f(\kappa) \qquad \Rightarrow \qquad f(\kappa) = \mathrm{Ai}(2^{-2/3}(2\kappa + c))$$

Where c is constant.

Then $f(0) = 0$ implies $c = 2^{2/3} a_1$, where $a_1 \approx -2.338$ satisfies $\mathrm{Ai}(a_1) = 0$.

Heuristic analysis of weighted paths

Recurrence:

$$e_{n,m} = \frac{n-m+2}{n+m} e_{n-1,m-1} + e_{n-1,m+1} - \frac{n-m}{(n+m)(n+m-2)} e_{n-3,m-1}.$$

Guess: $e_{n,m} \approx h(n) f\left(\frac{m+1}{\sqrt[3]{n}}\right).$

Substitute into recurrence and set $m = \kappa\sqrt[3]{n} - 1$:

$$s_n := \frac{h(n)}{h(n-1)} \approx 2 + \frac{f''(\kappa) - 2\kappa f(\kappa)}{f(\kappa)} n^{-2/3} + O(n^{-1})$$

Solution (assuming equality above):

$$s_n = 2 + cn^{-2/3} + O(n^{-1}) \quad \Rightarrow \quad h(n) \approx 2^n e^{\frac{3c}{2} n^{1/3}}$$

$$f''(\kappa) = (2\kappa + c) f(\kappa) \quad \Rightarrow \quad f(\kappa) = \mathrm{Ai}(2^{-2/3}(2\kappa + c))$$

Where c is constant and Ai is the Airy function.

Then $f(0) = 0$ implies $c = 2^{2/3} a_1$, where $a_1 \approx -2.338$ satisfies $\mathrm{Ai}(a_1) = 0$.

Heuristic analysis of weighted paths

Recurrence:

$$e_{n,m} = \frac{n-m+2}{n+m} e_{n-1,m-1} + e_{n-1,m+1} - \frac{n-m}{(n+m)(n+m-2)} e_{n-3,m-1}.$$

Guess: $e_{n,m} \approx h(n) f\left(\frac{m+1}{\sqrt[3]{n}}\right).$

Substitute into recurrence and set $m = \kappa\sqrt[3]{n} - 1$:

$$s_n := \frac{h(n)}{h(n-1)} \approx 2 + \frac{f''(\kappa) - 2\kappa f(\kappa)}{f(\kappa)} n^{-2/3} + O(n^{-1})$$

Solution (assuming equality above):

$$s_n = 2 + cn^{-2/3} + O(n^{-1}) \quad \Rightarrow \quad h(n) \approx 2^n e^{\frac{3c}{2} n^{1/3}}$$

$$f''(\kappa) = (2\kappa + c) f(\kappa) \quad \Rightarrow \quad f(\kappa) = \mathrm{Ai}(2^{-2/3}(2\kappa + c))$$

Where c is constant and Ai is the Airy function.
Then $f(0) = 0$ implies $c = 2^{2/3} a_1$, where $a_1 \approx -2.338$ satisfies $\mathrm{Ai}(a_1) = 0$.

Refined heuristic analysis of weighted paths

Let $a_1 \approx -2.3381$ be the largest root of the Airy function Ai.
First guess:

$$e_{n,m} \approx h(n) f\left(\frac{m+1}{\sqrt[3]{n}}\right),$$

yields estimates

$$h(n) \approx 2^n e^{3a_1 (n/2)^{1/3}}$$
$$f(\kappa) = \mathrm{Ai}(2^{1/3}\kappa + a_1)$$

Refined guess:

$$e_{n,m} \approx h(n)\left(f_0\left(\frac{m+1}{\sqrt[3]{n}}\right) + n^{-1/3} f_1\left(\frac{m+1}{\sqrt[3]{n}}\right)\right),$$

yields estimates

$$h(n) \sim const \cdot 2^n e^{3a_1 (n/2)^{1/3}} n^{29/24}$$
$$f_0(\kappa) = \mathrm{Ai}(2^{1/3}\kappa + a_1)$$

This way we conjecture the asymptotic form for acyclic minimal DFAs:

$$m_n = 2^{n-1} n! e_{2n,0} = \Theta\left(n! 8^n e^{3a_1 n^{1/3}} n^{7/8}\right)$$

Refined heuristic analysis of weighted paths

Let $a_1 \approx -2.3381$ be the largest root of the Airy function Ai.

First guess:

$$e_{n,m} \approx h(n) f\left(\frac{m+1}{\sqrt[3]{n}}\right),$$

yields estimates

$$h(n) \approx 2^n e^{3a_1 (n/2)^{1/3}}$$

$$f(\kappa) = \mathrm{Ai}(2^{1/3}\kappa + a_1)$$

Refined guess:

$$e_{n,m} \approx h(n)\left(f_0\left(\frac{m+1}{\sqrt[3]{n}}\right) + n^{-1/3} f_1\left(\frac{m+1}{\sqrt[3]{n}}\right)\right),$$

yields estimates

$$h(n) \sim const \cdot 2^n e^{3a_1 (n/2)^{1/3}} n^{29/24}$$

$$f_0(\kappa) = \mathrm{Ai}(2^{1/3}\kappa + a_1)$$

This way we conjecture the asymptotic form for acyclic minimal DFAs:

$$m_n = 2^{n-1} n! e_{2n,0} = \Theta\left(n! 8^n e^{3a_1 n^{1/3}} n^{7/8}\right)$$

Refined heuristic analysis of weighted paths

Let $a_1 \approx -2.3381$ be the largest root of the Airy function Ai.

First guess:

$$e_{n,m} \approx h(n) f\left(\frac{m+1}{\sqrt[3]{n}}\right),$$

yields estimates

$$h(n) \approx 2^n e^{3a_1(n/2)^{1/3}}$$

$$f(\kappa) = \mathrm{Ai}(2^{1/3}\kappa + a_1)$$

Refined guess:

$$e_{n,m} \approx h(n)\left(f_0\left(\frac{m+1}{\sqrt[3]{n}}\right) + n^{-1/3} f_1\left(\frac{m+1}{\sqrt[3]{n}}\right)\right),$$

yields estimates

$$h(n) \sim const \cdot 2^n e^{3a_1(n/2)^{1/3}} n^{29/24}$$

$$f_0(\kappa) = \mathrm{Ai}(2^{1/3}\kappa + a_1)$$

This way we conjecture the asymptotic form for acyclic minimal DFAs:

$$m_n = 2^{n-1} n! e_{2n,0} = \Theta\left(n! 8^n e^{3a_1 n^{1/3}} n^{7/8}\right)$$

Inductive proof

Proof method

Recall:

$$e_{n,m} = \frac{n-m+2}{n+m} e_{n-1,m-1} + e_{n-1,m+1} - \frac{n-m}{(n+m)(n+m-2)} e_{n-3,m-1}$$

Number of minimal acyclic DFAs is $m_n = 2^{n-1} n! e_{2n,0}$.

Method:

Find sequences $A_{n,k}$ and $B_{n,k}$ with the same asymptotic form, such that

$$A_{n,k} \leq e_{n,k} \leq B_{n,k},$$

for all k and all n large enough.

Proof method

Recall:

$$e_{n,m} = \frac{n-m+2}{n+m} e_{n-1,m-1} + e_{n-1,m+1} - \frac{n-m}{(n+m)(n+m-2)} e_{n-3,m-1}$$

Number of minimal acyclic DFAs is $m_n = 2^{n-1} n! e_{2n,0}$.

Method:
Find sequences $A_{n,k}$ and $B_{n,k}$ with the same asymptotic form, such that

$$A_{n,k} \leq e_{n,k} \leq B_{n,k},$$

for all k and all n large enough.

How to find them?

1. Use heuristics
2. Fiddle until they satisfy the recurrence of $e_{n,k}$ with the equalities replaced by inequalities:

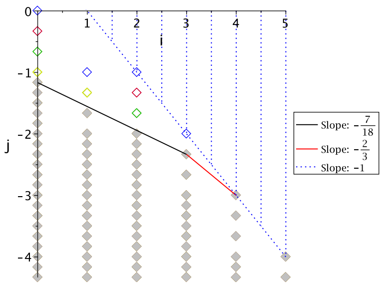
$$= \quad \longrightarrow \quad \leq \text{ and } \geq$$

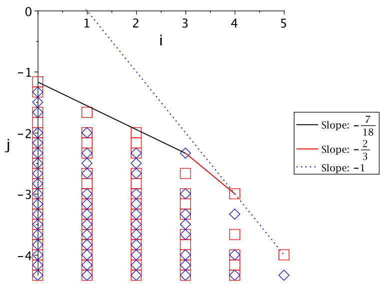
3. Prove $A_{n,k} \leq e_{n,k} \leq B_{n,k}$ by induction.

Technicalities

Lots of technicalities:

- Before induction, we have to remove the negative term from the recurrence, but we have to do so precisely for asymptotics to stay the same.
- We only prove bounds for small *m*; we prove that large *m* terms don't matter
- The lower bound is negative for very large *m*, so we have to be careful with induction
- We only prove the bounds for sufficiently large *n*, but this only makes a difference to the constant term. Proof involves colourful Newton polygons:





Summary

Enumeration of minimal acyclic DFAs

1. Bijection to decorated paths
2. Recurrence for decorated paths
3. Heuristic analysis of recurrence
4. Inductive proof using heuristics

Lower bound:

$$m_n \geq \gamma_1 \, n! 8^n e^{3a_1 n^{1/3}} n^{7/8},$$

for some constant $\gamma_1 > 0$.

Summary

Enumeration of minimal acyclic DFAs

1. Bijection to decorated paths
2. Recurrence for decorated paths
3. Heuristic analysis of recurrence
4. Inductive proof using heuristics

Lower bound:

$$m_n \geq \gamma_1 \, n! 8^n e^{3a_1 n^{1/3}} n^{7/8},$$

for some constant $\gamma_1 > 0$.

Upper bound (similar proof):

$$m_n \leq \gamma_2 \, n! 8^n e^{3a_1 n^{1/3}} n^{7/8},$$

for some constant $\gamma_2 > 0$.

The end

Theorem

The number of minimal DFAs recognizing a finite binary language satisfies for $n \to \infty$

$$m_n = \Theta\left(n!\, 8^n e^{3a_1 n^{1/3}} n^{7/8}\right),$$

where $a_1 \approx -2.3381$ *is the largest root of the Airy function* $\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt.$

Further problems:

- Determining the constant term, or at least proving that one exists.
- How does the statistic *number of states in DFA* for a finite binary language interact with other natural statistics, like number of words? length of longest word? etc.
- For the method: Does anyone have a tricky recurrence to try?

The end

Theorem

The number of minimal DFAs recognizing a finite binary language satisfies for $n \to \infty$

$$m_n = \Theta\left(n!\, 8^n e^{3a_1 n^{1/3}} n^{7/8}\right),$$

where $a_1 \approx -2.3381$ *is the largest root of the Airy function* $\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt.$

Further problems:

- Determining the constant term, or at least proving that one exists.
- How does the statistic *number of states in DFA* for a finite binary language interact with other natural statistics, like number of words? length of longest word? etc.
- For the method: Does anyone have a tricky recurrence to try?