More Models of Walks Avoiding a Quadrant

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The problem

Question
How many walks of length $n$ starting from $(0,0)$ avoid the quadrant?

- We fix the starting point $(0,0)$,
- a step set $S \subseteq \{-1, 0, 1\}^2 \setminus \{(0,0)\}$ of small steps, and
- the three-quadrant cone $C = \{(i,j) : i \geq 0 \text{ or } j \geq 0\}$. 
Real-life applications

More seriously ...

- it is a model for many discrete objects in
  - combinatorics, statistical physics
  - probability theory, queueing theory
  - ...
How many interesting models are there?

- $S \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\} \Rightarrow 2^8 = 256$ models
- However, some are equivalent
  - to a model of walks in the full or half-space ($\Rightarrow$ algebraic)
  - to another model in the collection (diagonal symmetry)

- We are left with 74 interesting models (79 in the quarter plane)
The 74 interesting models in the three-quadrant cone
Interesting questions

- Closed form/asymptotics for the number $c(n)$ of walks of length $n$?
- Closed form/asymptotics for the number $c_{i,j}(n)$ of walks ending at $(i,j)$?
- The generating functions and their nature?

$$C(t) = \sum_{n \geq 0} c(n) t^n, \quad C(x, y; t) = \sum_{(i,j) \in C} \sum_{n \geq 0} c_{i,j}(n) t^n x^i y^j$$

- Can we express these series?
- Are they rational/algebraic/D-finite?
A hierarchy of formal power series

The formal power series $C(t)$ is

- **rational** if it can be written as
  \[ C(t) = \frac{P(t)}{Q(t)}, \]
  where $P(t)$ and $Q(t)$ are polynomials in $t$.

- **algebraic** (over $\mathbb{Q}(t)$) if it satisfies a (non-trivial) polynomial equation
  \[ P(t, C(t)) = 0. \]

- **D-finite** if it satisfies a (non-trivial) linear differential equation with polynomial coefficients:
  \[ p_k(t) C^{(k)}(t) + \cdots + p_0(t) C(t) = 0. \]

Why is it important to be D-finite?

- Nice and effective closure properties (sum, product, differentiation, \ldots )
- Fast algorithms to compute coefficients
- Asymptotics of coefficients
**Solved cases**

**D-finite** \( C(x, y; t) \)

1. \[ \]
   [Bousquet-Mélou 16]

2. \[ \]
   [Bousquet-Mélou 16]

3. \[ \]
   [Raschel, Trotignon 19]

4. \[ \]
   [Raschel, Trotignon 19]

5. \[ \]
   [Raschel, Trotignon 19]

6. \[ \]
   King [This talk!]

**D-finite excursions** \( \sum_{n \geq 0} c_{0,0}(n)t^n \)

7. \[ \]
   [Budd 20]

8. \[ \]
   [Budd 20]

9. \[ \]
   [Elvey-Price 20]

10. \[ \]
    [Elvey-Price 20]

**Non-D-finite**

- 51 models [Mustapha 19]
The taxonomy so far

- non-D-finite
- excursions D-finite
- D-finite
- D-finite (algebraic)
The group of the walk for the king

- From now on we use $\bar{x} := \frac{1}{x}$ and $\bar{y} := \frac{1}{y}$
- The step polynomial encodes the possible steps
  \[ S(x, y) = x + xy + y + \bar{x}y + \bar{x} + \bar{x}\bar{y} + \bar{y} + x\bar{y}. \]
- $S(x, y)$ is left unchanged by the rational transformations
  \[ \Phi : (x, y) \mapsto (\bar{x}, y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, \bar{y}). \]
- They are involutions and generate a finite dihedral group $G$:

\[
\begin{array}{cccc}
\Phi & (x, y) & \rightarrow & (\bar{x}, y) \\
\Psi & (\bar{x}, y) & \rightarrow & (x, \bar{y}) \\
\Phi & (x, \bar{y}) & \rightarrow & (x, y) \\
\Psi & (x, y) & \rightarrow & (\bar{x}, y)
\end{array}
\]

- The group can be defined for any model with small steps!
The quarter plane

Quarter plane

\[ Q = \{(i,j) : i \geq 0 \text{ and } j \geq 0\}. \]

Generating function

\[ Q(x, y; t) = \sum_{i,j \geq 0} \sum_{n \geq 0} q_{i,j}(n) t^n. \]

Theorem [Bousquet-Mélou, Mishna 10], [Bostan, Kauers 10], [Kurkova, Raschel 12], [Mishna, Rechnitzer 07], [Melczer, Mishna 13], [and more!]

The series \(Q(x, y; t)\) is D-finite if and only if \(G\) is finite.
An algebraicity phenomenon for the king

Theorem

The generating function \( C(x, y; t) \equiv C(x, y) \), of king walks starting from \((0,0)\) that are confined to \(C\), satisfies

\[
C(x, y) = A(x, y) + \frac{1}{3} \left( Q(x, y) - \bar{x}^2 Q(\bar{x}, y) - \bar{y}^2 Q(x, \bar{y}) \right),
\]

where \( A(x, y) \) is algebraic of degree 216 over \( \mathbb{Q}(x, y, t) \).

This series satisfies

\[
K(x, y)A(x, y) = \frac{2 + \bar{x}^2 + \bar{y}^2}{3} - t\bar{y}(x + 1 + \bar{x})A_-(\bar{x})
\]

\[
- t\bar{x}(y + 1 + \bar{y})A_-(\bar{y}) - t\bar{x}\bar{y}A_{0,0},
\]

where \( A_-(x) \in \mathbb{Q}[x][[t]] \) is algebraic of degree 72 over \( \mathbb{Q}(x, t) \) and \( A_{0,0} \in \mathbb{Q}[[t]] \) is algebraic of degree 24 over \( \mathbb{Q}(t) \).

Such a phenomenon already proved for \( + \) and \( \times \) in [Bousquet-Mélou 16].

Asymptotics

**Corollary**

The number $c_{0,0}(n)$ of $n$-step king walks confined to $C$ and ending at the origin, and the number $c(n)$ of walks of $C$ ending anywhere satisfy for $n \to \infty$:

$$c_{0,0}(n) \sim \left( \frac{2^{29} K}{3^7} \right)^{1/3} \frac{\Gamma(2/3)}{\pi} \frac{8^n}{n^{5/3}},$$

$$c(n) \sim \left( \frac{2^{32} K}{3^7} \right)^{1/6} \frac{1}{\Gamma(2/3)} \frac{8^n}{n^{1/3}},$$

where $K$ is the unique real root of

$$101^6 K^3 - 601275603 K^2 + 92811 K - 1.$$
A functional equation

Step by step construction \( (S(x, y) = x + xy + y + \bar{x}y + \bar{x} + \bar{x}\bar{y} + \bar{y} + x\bar{y}) \):

\[
C(x, y) = 1 + tS(x, y)C(x, y) - t\bar{x}(\bar{y} + 1 + y)C_-(\bar{y}) - t\bar{y}(\bar{x} + 1 + x)C_-(\bar{x}) - t\bar{x}\bar{y}C_{0,0}
\]

with

\[
C_-(x) = \sum_{i > 0, n \geq 0} c_{-i, 0}(n)x^i t^n \in \bar{x} \quad \text{and} \quad C_{0,0} = \sum_{n \geq 0} c_{0,0}(n)x^i t^n \in \bar{x}.
\]
A functional equation

Step by step construction \((S(x, y) = x + xy + y + \bar{x}y + \bar{x} + \bar{x}\bar{y} + \bar{y} + x\bar{y})\):

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\[ C_-(x) = \sum_{i > 0, n \geq 0} c_{-i, 0}(n)x^i t^n \in \bar{x} \quad \text{and} \quad C_{0, 0} = \sum_{n \geq 0} c_{0, 0}(n)x^i t^n \in \bar{x}. \]

The kernel equation for \(C(x, y)\)

\[ K(x, y)xyC(x, y) = xy - t(1 + y + y^2)C_-(\bar{y}) - t(1 + x + x^2)C_-(\bar{x}) - tC_{0, 0} \]

\[ K(x, y) := 1 - tS(x, y) \]
An important observation

The kernel equation for \( C(x, y) \)

\[
K(x, y)xyC(x, y) = xy - t(1+y+y^2)C_-(\bar{y}) - t(1+x+x^2)C_-(\bar{x}) - tC_{0,0}
\]

The kernel equation of \( Q(x, y) \) is very similar

\[
K(x, y)xyQ(x, y) = xy - t(1+y+y^2)Q(0, y) - t(1+x+x^2)Q(x, 0) + tQ(0, 0)
\]
An important observation

The kernel equation for $C(x, y)$

$$K(x, y)xyC(x, y) = xy - t(1+y+y^2)C_-(\bar{y}) - t(1+x+x^2)C_-(\bar{x}) - tC_{0,0}$$

The kernel equation of $Q(x, y)$ is very similar

$$K(x, y)xyQ(x, y) = xy - t(1+y+y^2)Q(0, y) - t(1+x+x^2)Q(x, 0) + tQ(0, 0)$$

Hence, they have the same orbit sum:

$$xyQ(x, y) - \bar{x}yQ(\bar{x}, y) + \bar{x}\bar{y}Q(\bar{x}, \bar{y}) - x\bar{y}Q(x, \bar{y}) =$$

$$xyC(x, y) - \bar{x}yC(\bar{x}, y) + \bar{x}\bar{y}C(\bar{x}, \bar{y}) - x\bar{y}C(x, \bar{y}) = \frac{(x - \bar{x})(y - \bar{y})}{K(x, y)}.$$  

(As well as $-\bar{x}^2Q(\bar{x}, y)$, $-\bar{y}^2Q(x, \bar{y})$, and $\bar{x}^2\bar{y}^2Q(\bar{x}, \bar{y}).$)
**Idea: create a zero orbit sum (hence an algebraic GF?!)**

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\]

(As well as \(-\bar{x}^2 Q(\bar{x}, y), -\bar{y}^2 Q(x, \bar{y}), \text{ and } \bar{x}^2 \bar{y}^2 Q(\bar{x}, \bar{y}).\))
Idea: create a zero orbit sum (hence an algebraic GF?!) 

Hence, they have the same orbit sum:

\[ xyQ(x, y) - \bar{x}yQ(\bar{x}, y) + x\bar{y}Q(x, \bar{y}) = \]
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(As well as \(-\bar{x}^2 Q(\bar{x}, y), -\bar{y}^2 Q(x, \bar{y}), \) and \(\bar{x}^2 \bar{y}^2 Q(\bar{x}, \bar{y})\).)

We introduce the formal power series

\[ A(x, y) := C(x, y) - \frac{1}{3} (Q(x, y) - \bar{x}^2 Q(\bar{x}, y) - \bar{y}^2 Q(x, \bar{y})). \]

A (lattice path) functional equation for \(A(x, y)\) and orbit sum 0

\[ K(x, y)A(x, y) = \frac{2 + \bar{x}^2 + \bar{y}^2}{3} - t\bar{y}(x+1+\bar{x})A_{-}(\bar{x}) - t\bar{x}(y+1+\bar{y})A_{-}(\bar{y}) - t\bar{x}\bar{y}A_{0,0}, \]

and

\[ xyA(x, y) - \bar{x}yA(\bar{x}, y) + x\bar{y}A(x, \bar{y}) - x\bar{y}A(x, \bar{y}) = 0. \]
Continue with $A(x, y)$

In order to characterize $C(x, y)$ it suffices to solve for $A(x, y)$:

$$K(x, y)A(x, y) = \frac{2 + \bar{x}^2 + \bar{y}^2}{3} - t\bar{y}(x+1+\bar{x})A_-(\bar{x}) - t\bar{x}(y+1+\bar{y})A_-(\bar{y}) - t\bar{x}\bar{y}A_{0,0}.$$  

We want to cancel the kernel, BUT $A(x, y)$ contains negative powers of $x$ and $y$. Hence, we split it into 3 parts:

$$A(x, y) = P(x, y) + \bar{x}M(\bar{x}, y) + \bar{y}M(\bar{y}, x),$$

where now $P(x, y), M(x, y) \in \mathbb{Q}[x, y][[t]]$.  

A quadrant-like problem for $M(x, y)$

...we get

\[ P(x, y) = \bar{x} (M(x, y) - M(0, y)) + \bar{y} (M(y, x) - M(0, x)), \]

and (after some work)

\[
K(x, y)(2M(x, y) - M(0, y)) = \frac{2x}{3} - 2t\bar{y}(x+1+\bar{x})M(x, 0) + t\bar{y}(y+1+\bar{y})M(y, 0) + t(x-\bar{x})(y+1+\bar{y})M(0, y) - t(1+\bar{y}^2 - 2\bar{x}\bar{y})M(0, 0) - t\bar{y}M_x(0, 0).
\]
A quadrant-like problem for $M(x, y)$

... we get

$$ P(x, y) = \bar{x}(M(x, y) - M(0, y)) + \bar{y}(M(y, x) - M(0, x)),$$

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$$ + t(x-\bar{x})(y+1+\bar{y})M(0, y) - t(1+\bar{y}^2 - 2\bar{x}\bar{y})M(0, 0) - t\bar{y}M_x(0, 0).$$

Now it is legitimate to cancel the kernel. After more work we arrive at an equation for $M(0, x)$ with one catalytic variable only:

$$ \text{Pol}(M(0, x), B_1, B_2, B_3, B_4, t, x) = 0,$$

where $B_i \in \mathbb{Q}(t)$ are half-known power series; e.g., $B_4 = M(0, 0)$.

- This is a (big !) polynomial equation with one catalytic variable $x$, in theory solvable using [Bousquet-Mélou, Jehanne 06].
- Unfortunately, the polynomial system was too big for our computers.
- Hence, we used a guess-and-check approach.
Guessing

We guessed polynomial equations using $c_{i,j}(n)$ for $0 \leq n \leq 2000$:

<table>
<thead>
<tr>
<th>GF</th>
<th>Deg. GF</th>
<th>Deg. $t$</th>
<th># terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_1$</td>
<td>12</td>
<td>26</td>
<td>229</td>
</tr>
<tr>
<td>$B_2$</td>
<td>24</td>
<td>60</td>
<td>477</td>
</tr>
<tr>
<td>$B_3$</td>
<td>24</td>
<td>12</td>
<td>323</td>
</tr>
<tr>
<td>$B_4$</td>
<td>24</td>
<td>32</td>
<td>823</td>
</tr>
</tbody>
</table>

Hence, these equations define algebraic power series. In order to prove that they are the ones involved in

$$\text{Pol}(M(0, x), B_1, B_2, B_3, B_4, t, x) = 0,$$

we needed to investigate their algebraic relations.
The algebraic structure of the $B_i$’s

1. Let $u = t + t^2 + O(t^3)$ be the only series satisfying the equation
   \[(1 - 3u)^3(1 + u)t^2 + (1 + 18u^2 - 27u^4)t - u = 0.\]

2. Let $v = t + 3t^2 + O(t^3)$ be the only series satisfying
   \[(1 + 3v - v^3)u - v(v^2 + v + 1) = 0.\]

3. Define
   \[w = \sqrt{1 + 4v - 4v^3 - 4v^4} = 1 + 2t + 4t^2 + O(t^3).\]

We get

\[B_1 \in \mathbb{Q}(t, v) \quad \text{and} \quad B_2, B_3, B_4 \in \mathbb{Q}(t, w).\]

This gives

\[B_4 = M(0, 0) = C_{-1,0} = \frac{1}{2t} \left( \frac{w(1 + 2v)}{1 + 4v - 2v^3} - 1 \right)\]
\[= t + 2t^2 + 17t^3 + 80t^4 + 536t^5 + O(t^6).\]
The final result

The precise knowledge of $v$ and $w$ allows us to prove that the guesses are correct and finishes the proof.

Theorem

The generating function $C(x, y; t) \equiv C(x, y)$, of walks starting from $(0, 0)$ that are confined to $C$, satisfies

$$C(x, y) = A(x, y) + \frac{1}{3} \left( Q(x, y) - \bar{x}^2 Q(\bar{x}, y) - \bar{y}^2 Q(x, \bar{y}) \right),$$

where $A(x, y)$ is algebraic of degree 216 over $\mathbb{Q}(x, y, t)$.

This series satisfies

$$K(x, y)A(x, y) = \frac{2 + \bar{x}^2 + \bar{y}^2}{3} - t\bar{y}(x + 1 + \bar{x})A_-(\bar{x}) - t\bar{x}(y + 1 + \bar{y})A_-(\bar{y}) - t\bar{x}\bar{y}A_{0,0},$$

where $A_-(x) \in \mathbb{Q}[x][[t]]$ is algebraic of degree 72 over $\mathbb{Q}(x, t)$ and $A_{0,0} \in \mathbb{Q}[[t]]$ is algebraic of degree 24 over $\mathbb{Q}(t)$. 
More models

- For each of the following 7 models we can define $A(x, y)$ with orbit sum 0

- First 3 models are now solved

- Methods of this presentation applicable

- For last 3 models: guessed equations of degree 24 for $A_{-1,0}$ (resp. $A_{-2,0}$) (For the first 3: degree 4, 8, 24, respectively)
More models

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Thank you!