# More Models of Walks Avoiding a Quadrant

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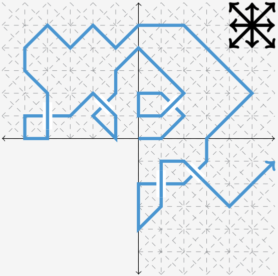
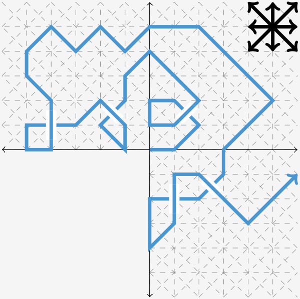


# The problem

**Question**

How many walks of length $n$ starting from $(0,0)$ avoid the quadrant?



- We fix the starting point $(0,0)$,
- a step set $\mathcal{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$ of small steps, and
- the three-quadrant cone $\mathcal{C} = \{(i,j) : i \geq 0 \text{ or } j \geq 0\}$.

# Real-life applications

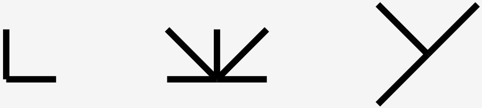

## More seriously ...

it is a model for many discrete objects in

- combinatorics, statistical physics
- probability theory, queueing theory
- ...

# How many interesting models are there?

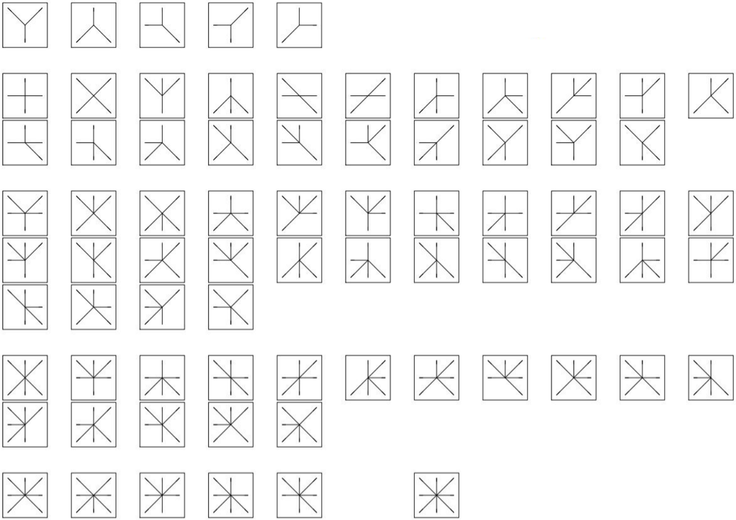
- $\mathcal{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\} \quad \Rightarrow \quad 2^8 = 256$ models
- However, some are equivalent
  - to a model of walks in the full or half-space ($\Rightarrow$ algebraic)



  - to another model in the collection (diagonal symmetry)

- We are left with 74 interesting models (79 in the quarter plane)

# The 74 interesting models in the three-quadrant cone



# Interesting questions

- Closed form/asymptotics for the number $c(n)$ of walks of length n?
- Closed form/asymptotics for the number $c_{i,j}(n)$ of walks ending at $(i,j)$?
- The generating functions and their nature?

$$C(t) = \sum_{n\geq 0} c(n)t^n, \qquad C(x,y;t) = \sum_{(i,j)\in\mathcal{C}} \sum_{n\geq 0} c_{i,j}(n) t^n x^i y^j$$

  - Can we express these series?
  - Are they rational/algebraic/D-finite?

# A hierarchy of formal power series

The formal power series $C(t)$ is

- rational if it can be written as

$$C(t) = \frac{P(t)}{Q(t)},$$

  where $P(t)$ and $Q(t)$ are polynomials in $t$.
- algebraic (over $\mathbb{Q}(t)$) if it satisfies a (non-trivial) polynomial equation

$$P(t, C(t)) = 0.$$

- D-finite if it satisfies a (non-trivial) linear differential equation with polynomial coefficients:

$$p_k(t)C^{(k)}(t) + \cdots + p_0(t)C(t) = 0.$$

## Why is it important to be D-finite?

- Nice and effective closure properties (sum, product, differentiation, ...)
- Fast algorithms to compute coefficients
- Asymptotics of coefficients

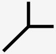
# Solved cases

**D-finite** $C(x, y; t)$

1  [Bousquet-Mélou 16]

2  [Bousquet-Mélou 16]

3  [Raschel, Trotignon 19]

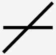
4  [Raschel, Trotignon 19]

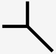
5  [Raschel, Trotignon 19]

6  King [This talk!]

**D-finite excursions** $\sum_{n\geq 0} c_{0,0}(n)t^n$

7  [Budd 20]

8  [Budd 20]

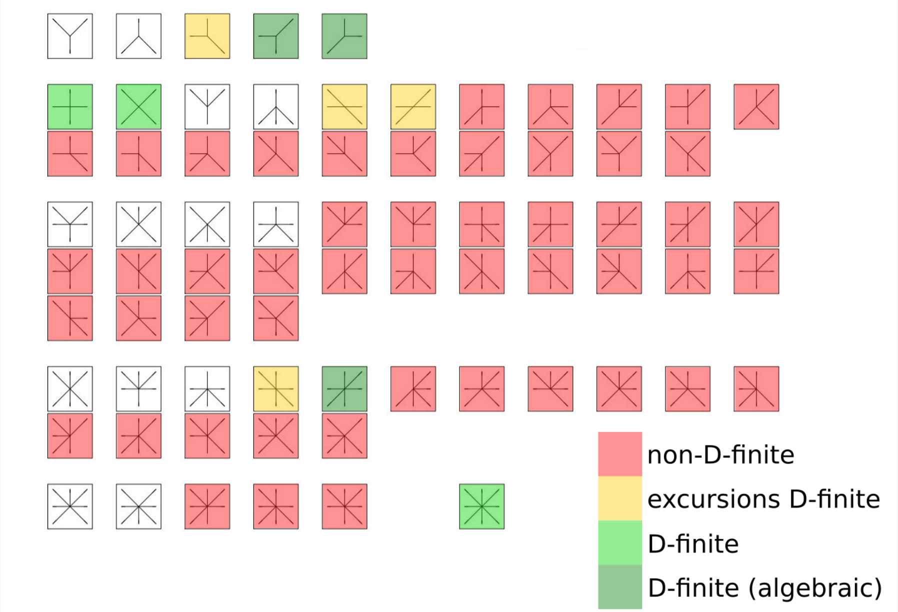
9  [Elvey-Price 20]

10  [Elvey-Price 20]

**Non-D-finite**

- 51 models [Mustapha 19]

# The taxonomy so far



# The group of the walk for the king

- From now on we use $\bar{x} := \frac{1}{x}$ and $\bar{y} := \frac{1}{y}$
- The step polynomial encodes the possible steps

$$S(x, y) = x + xy + y + \bar{x}y + \bar{x} + \bar{x}\bar{y} + \bar{y} + x\bar{y}.$$

- $S(x, y)$ is left unchanged by the rational transformations

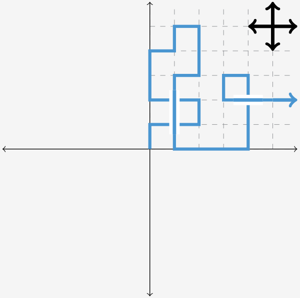
$$\Phi : (x, y) \mapsto (\bar{x}, y) \qquad \text{and} \qquad \Psi : (x, y) \mapsto (x, \bar{y}).$$

- They are involutions and generate a finite dihedral group $G$:

$$(x, y) \overset{\Phi}{\nearrow} (\bar{x}, y) \overset{\Psi}{\searrow} (\bar{x}, \bar{y}), \qquad (x, y) \overset{\Psi}{\searrow} (x, \bar{y}) \overset{\Phi}{\nearrow} (\bar{x}, \bar{y})$$

- The group can be defined for any model with small steps!

# The quarter plane



- Quarter plane
  $$\mathcal{Q} = \{(i,j) : i \geq 0 \text{ and } j \geq 0\}.$$
- Generating function
  $$Q(x,y;t) = \sum_{i,j\geq 0} \sum_{n \geq 0} q_{i,j}(n) t^n.$$

**Theorem** [Bousquet-Mélou, Mishna 10], [Bostan, Kauers 10], [Kurkova, Raschel 12], [Mishna, Rechnitzer 07], [Melczer, Mishna 13], [and more!]

The series $Q(x,y;t)$ is D-finite if and only if $G$ is finite.

# An algebraicity phenomenon for the king 

## Theorem

*The generating function* $C(x, y; t) \equiv C(x, y)$, *of king walks starting from* $(0, 0)$ *that are confined to* $\mathcal{C}$, *satisfies*

$$C(x, y) = A(x, y) + \frac{1}{3}\left(Q(x, y) - \bar{x}^2 Q(\bar{x}, y) - \bar{y}^2 Q(x, \bar{y})\right),$$

*where* $A(x, y)$ *is algebraic of degree* 216 *over* $\mathbb{Q}(x, y, t)$.

*This series satisfies*

$$\begin{aligned} K(x, y)A(x, y) = \frac{2 + \bar{x}^2 + \bar{y}^2}{3} &- t\bar{y}(x + 1 + \bar{x})A_-(\bar{x}) \\ &- t\bar{x}(y + 1 + \bar{y})A_-(\bar{y}) - t\bar{x}\bar{y}A_{0,0}, \end{aligned}$$

*where* $A_-(x) \in \mathbb{Q}[x][[t]]$ *is algebraic of degree* 72 *over* $\mathbb{Q}(x, t)$ *and* $A_{0,0} \in \mathbb{Q}[[t]]$ *is algebraic of degree* 24 *over* $\mathbb{Q}(t)$.

Such a phenomenon already proved for  and  in [Bousquet-Mélou 16].
Proof: Follow [Bousquet-Mélou 16] + guess-and-check + neat algebraic extensions (see next).

# Asymptotics

## Corollary

*The number $c_{0,0}(n)$ of n-step king walks confined to $\mathcal{C}$ and ending at the origin, and the number $c(n)$ of walks of $\mathcal{C}$ ending anywhere satisfy for $n \to \infty$:*

$$c_{0,0}(n) \sim \left(\frac{2^{29} K}{3^7}\right)^{1/3} \frac{\Gamma(2/3)}{\pi} \frac{8^n}{n^{5/3}},$$

$$c(n) \sim \left(\frac{2^{32} K}{3^7}\right)^{1/6} \frac{1}{\Gamma(2/3)} \frac{8^n}{n^{1/3}},$$

*where $K$ is the unique real root of*

$$101^6 K^3 - 601275603\, K^2 + 92811\, K - 1.$$

- Refines results of [Denisov, Wachtel 15] and [Mustapha 19] by the precise multiplicative constant
- Lower order terms are easily computable

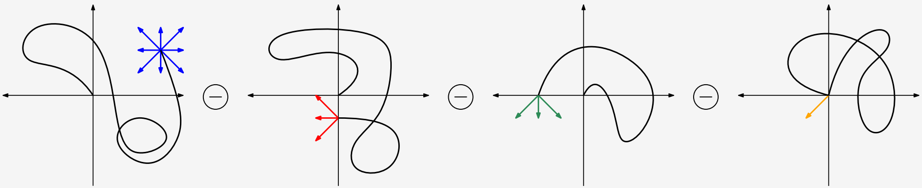
# A functional equation

Step by step construction ($S(x,y) = x + xy + y + \bar{x}y + \bar{x} + \bar{x}\bar{y} + \bar{y} + x\bar{y}$):

$$C(x,y) = 1 + tS(x,y)C(x,y) - t\bar{x}(\bar{y}+1+y)C_-(\bar{y}) - t\bar{y}(\bar{x}+1+x)C_-(\bar{x}) - t\bar{x}\bar{y}C_{0,0}$$

with

$$C_-(x) = \sum_{\substack{i>0\\ n\geq 0}} c_{-i,0}(n)x^i t^n \in \bar{x} \qquad \text{and} \qquad C_{0,0} = \sum_{n\geq 0} c_{0,0}(n)x^i t^n \in \bar{x}.$$



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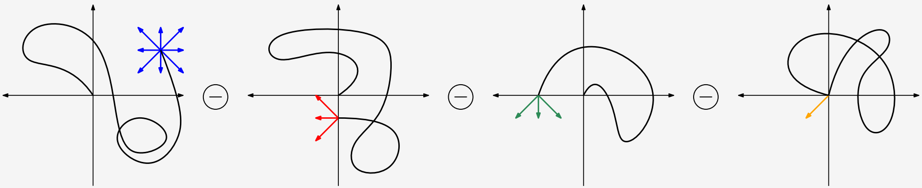

**The kernel equation for $C(x,y)$**

$$K(x,y)xyC(x,y) = xy - t(1+y+y^2)C_-(\bar{y}) - t(1+x+x^2)C_-(\bar{x}) - tC_{0,0}$$

$$K(x,y) := 1 - tS(x,y)$$

# An important observation

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**The kernel equation of $Q(x, y)$ is very similar**

$$K(x,y)xyQ(x,y) = xy - t(1+y+y^2)Q(0,y) - t(1+x+x^2)Q(x,0) + tQ(0,0)$$

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Hence, they have the same orbit sum:

$$xyQ(x,y) - \bar{x}yQ(\bar{x},y) + \bar{x}\bar{y}Q(\bar{x},\bar{y}) - x\bar{y}Q(x,\bar{y}) =$$

$$xyC(x,y) - \bar{x}yC(\bar{x},y) + \bar{x}\bar{y}C(\bar{x},\bar{y}) - x\bar{y}C(x,\bar{y}) = \frac{(x-\bar{x})(y-\bar{y})}{K(x,y)}.$$

(As well as $-\bar{x}^2Q(\bar{x},y)$, $-\bar{y}^2Q(x,\bar{y})$, and $\bar{x}^2\bar{y}^2Q(\bar{x},\bar{y})$.)

# Idea: create a zero orbit sum (hence an algebraic GF?!)

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(As well as $-\bar{x}^2Q(\bar{x},y)$, $-\bar{y}^2Q(x,\bar{y})$, and $\bar{x}^2\bar{y}^2Q(\bar{x},\bar{y})$.)

We introduce the formal power series

$$A(x,y) := C(x,y) - \frac{1}{3}\left(Q(x,y) - \bar{x}^2Q(\bar{x},y) - \bar{y}^2Q(x,\bar{y})\right).$$

**A (lattice path) functional equation for $A(x,y)$ and orbit sum 0**

$$K(x,y)A(x,y) = \frac{2+\bar{x}^2+\bar{y}^2}{3} - t\bar{y}(x+1+\bar{x})A_-(\bar{x}) - t\bar{x}(y+1+\bar{y})A_-(\bar{y}) - t\bar{x}\bar{y}A_{0,0},$$

and

$$xyA(x,y) - \bar{x}yA(\bar{x},y) + \bar{x}\bar{y}A(\bar{x},\bar{y}) - x\bar{y}A(x,\bar{y}) = 0.$$

# Continue with $A(x,y)$

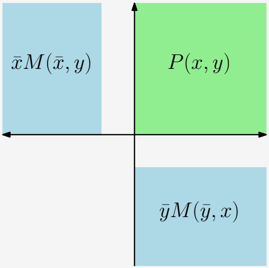
In order to characterize $C(x,y)$ it suffices to solve for $A(x,y)$:

$$K(x,y)A(x,y) = \frac{2+\bar{x}^2+\bar{y}^2}{3} - t\bar{y}(x+1+\bar{x})A_-(\bar{x}) - t\bar{x}(y+1+\bar{y})A_-(\bar{y}) - t\bar{x}\bar{y}A_{0,0}.$$

We want to cancel the kernel, BUT $A(x,y)$ contains negative powers of $x$ and $y$. Hence, we split it into 3 parts:

$$A(x,y) = P(x,y) + \bar{x}M(\bar{x},y) + \bar{y}M(\bar{y},x),$$

where now $P(x,y), M(x,y) \in \mathbb{Q}[x,y][[t]]$.



# A quadrant-like problem for $M(x, y)$

. . . we get

$$P(x,y) = \bar{x}\,(M(x,y) - M(0,y)) + \bar{y}\,(M(y,x) - M(0,x)),$$

and (after some work)

$$K(x,y)(2M(x,y) - M(0,y)) = \frac{2x}{3} - 2t\bar{y}(x+1+\bar{x})M(x,0) + t\bar{y}(y+1+\bar{y})M(y,0)$$
$$+t(x-\bar{x})(y+1+\bar{y})M(0,y) - t\left(1+\bar{y}^2-2\bar{x}\bar{y}\right)M(0,0) - t\bar{y}M_x(0,0).$$

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Now it is legitimate to cancel the kernel. After more work we arrive at an equation for $M(0,x)$ with one catalytic variable only:

$$\mathrm{Pol}(M(0,x), B_1, B_2, B_3, B_4, t, x) = 0,$$

where $B_i \in \mathbb{Q}(t)$ are half-known power series; e.g., $B_4 = M(0,0)$.

- This is a (big !) polynomial equation with one catalytic variable $x$, in theory solvable using [Bousquet-Mélou, Jehanne 06].
- Unfortunately, the polynomial system was too big for our computers.
- Hence, we used a *guess-and-check* approach.

# Guessing

We guessed polynomial equations using $c_{i,j}(n)$ for $0 \leq n \leq 2000$:

| GF | Deg. GF | Deg. $t$ | # terms |
|---|---|---|---|
| $B_1$ | 12 | 26 | 229 |
| $B_2$ | 24 | 60 | 477 |
| $B_3$ | 24 | 12 | 323 |
| $B_4$ | 24 | 32 | 823 |

Hence, these equations define algebraic power series. In order to prove that they are the ones involved in

$$\mathrm{Pol}(M(0,x), B_1, B_2, B_3, B_4, t, x) = 0,$$

we needed to investigate their algebraic relations.

# The algebraic structure of the $B_i$'s

$\mathbb{Q}(t)$

$\hookdownarrow$ ④

1. Let $u = t + t^2 + \mathcal{O}(t^3)$ be the only series satisfying the
$$(1-3u)^3(1+u)t^2 + (1+18u^2-27u^4)t - u = 0.$$

$\mathbb{Q}(t,u)$

$\hookdownarrow$ ③

2. Let $v = t + 3t^2 + \mathcal{O}(t^3)$ be the only series satisfying
$$(1+3v-v^3)u - v(v^2+v+1) = 0.$$

$\mathbb{Q}(t,v)$

$\hookdownarrow$ ②

3. Define
$$w = \sqrt{1+4v-4v^3-4v^4} = 1 + 2t + 4t^2 + \mathcal{O}(t^3).$$

$\mathbb{Q}(t,w)$

We get
$$B_1 \in \mathbb{Q}(t,v) \qquad \text{and} \qquad B_2, B_3, B_4 \in \mathbb{Q}(t,w).$$

This gives
$$B_4 = M(0,0) = C_{-1,0} = \frac{1}{2t}\left(\frac{w(1+2v)}{1+4v-2v^3} - 1\right)$$
$$= t + 2t^2 + 17t^3 + 80t^4 + 536t^5 + \mathcal{O}(t^6).$$

# The final result

The precise knowledge of $v$ and $w$ allows us to prove that the guesses are correct and finishes the proof.

## Theorem

*The generating function* $C(x,y;t) \equiv C(x,y)$, *of walks starting from* $(0,0)$ *that are confined to* $\mathcal{C}$, *satisfies*

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# More models

- For each of the following 7 models we can define $A(x, y)$ with orbit sum 0

- First 3 models are now solved
- Methods of this presentation applicable
- For last 3 models: guessed equations of degree 24 for $A_{-1,0}$ (resp. $A_{-2,0}$) (For the first 3: degree 4, 8, 24, respectively)

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Thank you!