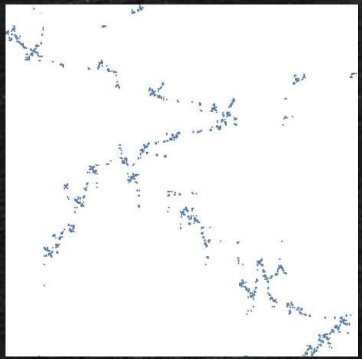
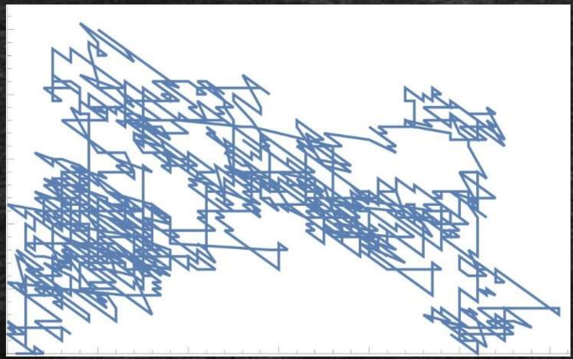
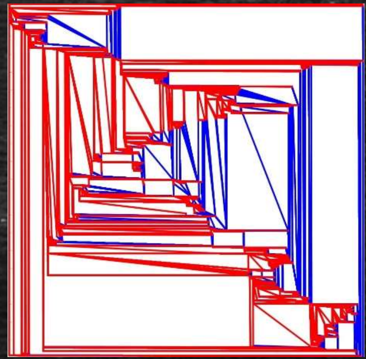
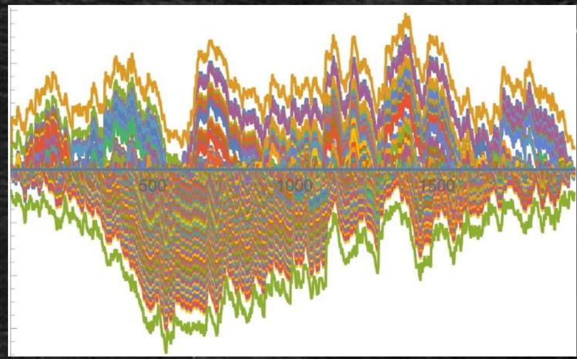
   

Scaling and local limits of Baxter permutations and bipolar orientations through coalescent-walk processes

Jacopo Borga

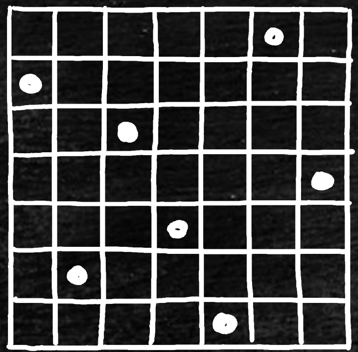
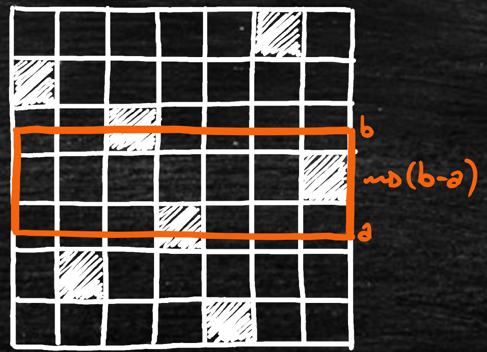
UZH ZÜRICH

(joint work with M. Maazoun)

Permuton limits

We look at permutations from a geometric perspective:

Consider the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 2 & 5 & 3 & 1 & 7 & 4 \end{pmatrix}$

$\sigma =$  $\rightsquigarrow$ $\mu_\sigma =$ 

Probability measure on the unit-square with uniform marginals

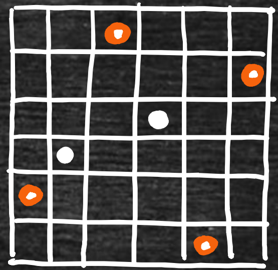
Def: A **PERMUTON** is a probability measure on the square $[0,1]^2$ with uniform marginals.

Remark: We have a natural notion of convergence of such objects: the **WEAK CONVERGENCE**. This defines a nice compact Polish space.

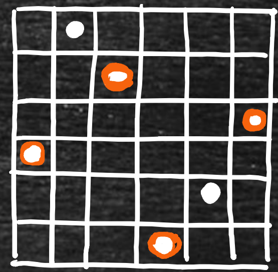
$\Longrightarrow$ limits of permutons are permutons, i.e., potential limits of sequences of permutations also have uniform marginals.

Def: Baxter permutations are permutations avoiding the patterns

2413 & 3142.



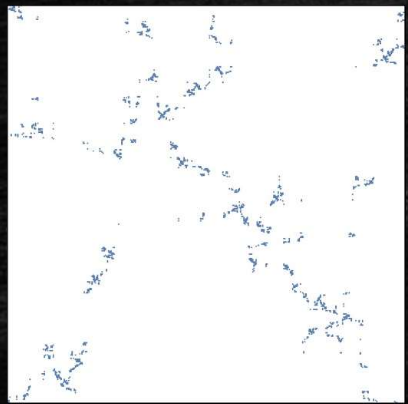
BAXTER

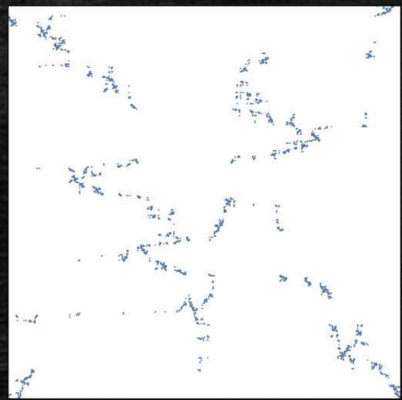


NOT BAXTER

Dokos & Pak (2014) explored the expected shape of doubly alternating Baxter permutations, i.e. Baxter perm. σ s.t. σ and σ^{-1} are alternating and they claimed that

"IT WOULD BE NICE TO COMPUTE THE LIMIT SHAPE OF BAXTER PERMUTATIONS"

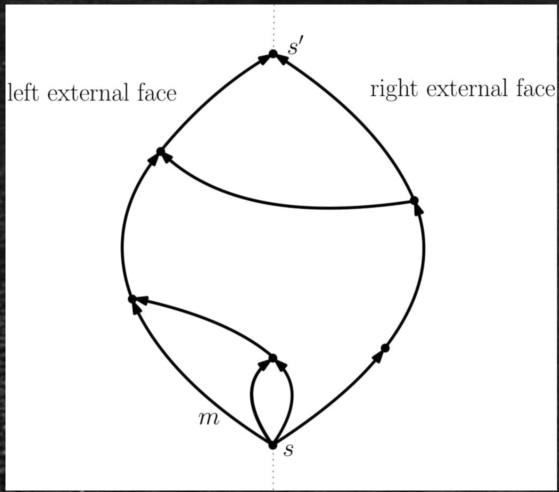




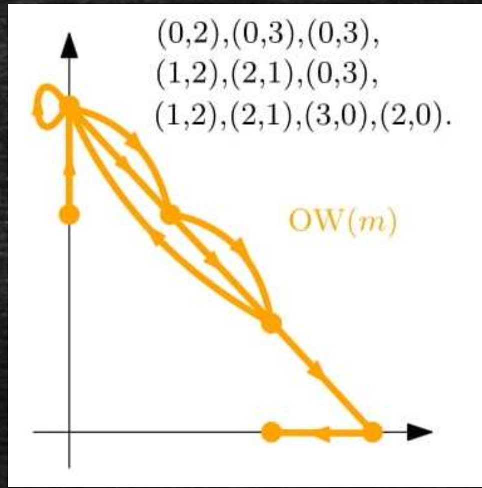
Bipolar orientations and walks in cones

Bonichon, Bousquet-Mélou & Fusy (2011) showed that Baxter permutations are in bijection with plane bipolar orientations.

Def: A PLANE BIPOLAR ORIENTATION is a planar map (connected graphs properly embedded in the plane up to continuous deformations) equipped with an acyclic orientation of the edges with exactly one source (a vertex with only outgoing edges) and one sink (a vertex with only incoming edges) both on the outer face.



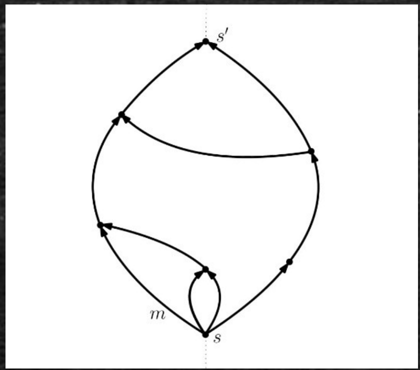
$\overset{OW}{\longmapsto}$



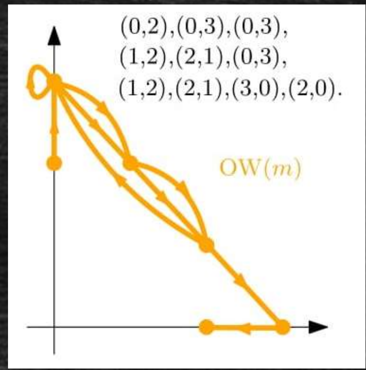
Kenyon, Miller, Sheffield & Wilson (2019) constructed a bijection OW from bipolar orientations to a specific family of two-dimensional walks in the non-negative quadrant, called TANDEM WALKS.

THEOREM: (Gwynne, Holden, Sun 2016)
The pairs of height functions for an infinite-volume random bipolar triangulation and its dual converge jointly in law to the two Brownian motions which encode the same $\sqrt{4/3}$-LQG surface decorated by both an SLE_{12} and the "dual" SLE_{12} which travels in a perpendicular direction.

So far...

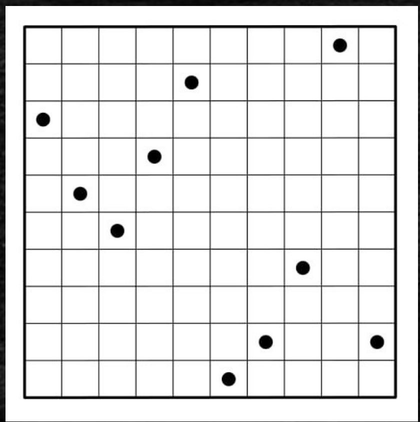


OW



OP

$OP \circ OW^{-1}$



WE WANT "TO READ" THE PATTERNS OF A PERMUTATION IN THE CORRESPONDING WALK

Coalescent-walk processes

Let $W_t = (X_t, Y_t)$ be a tandem walk & $\sigma = OP \circ OW^{-1}(W)$ be the corresponding Baxter permutation.

IDEA: Given $i<j$, we want to find a way in order to "read" in W_t

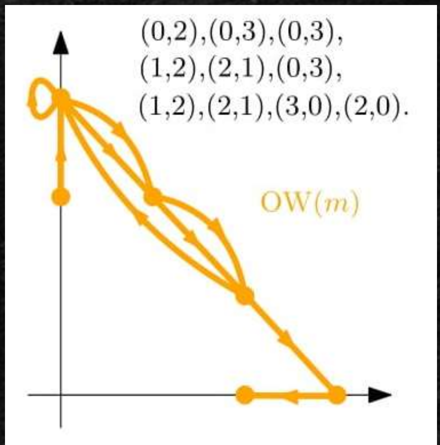
if $\sigma(i) < \sigma(j)$ or $\sigma(j) < \sigma(i)$.

SOLUTION: **COALESCENT - WALK PROCESSES**

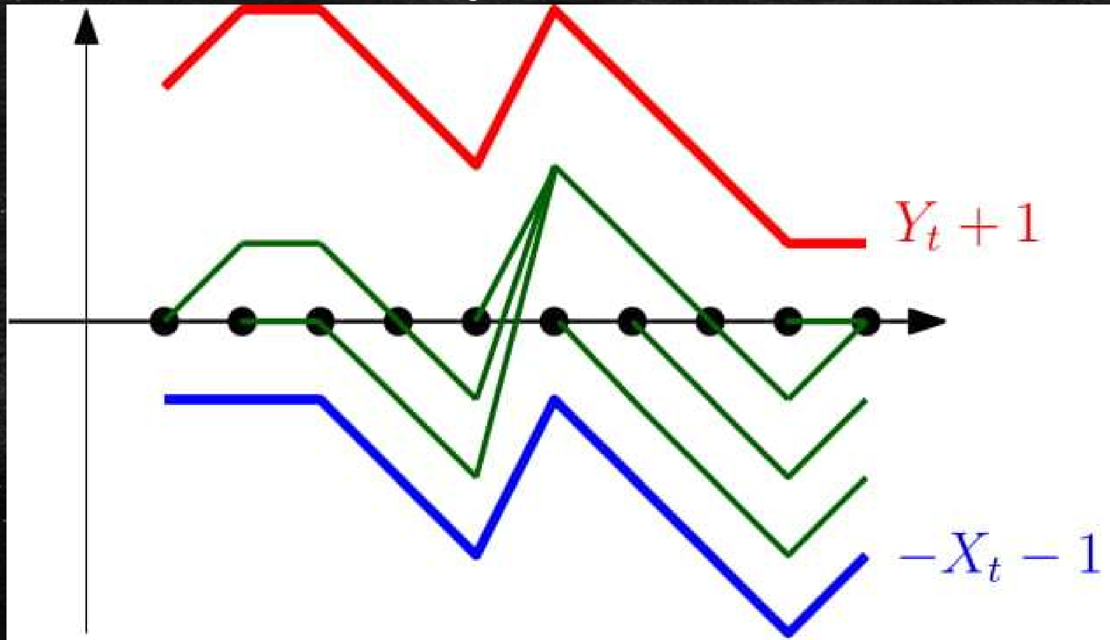
i.e. a collection of walks $\left(Z^{(t)}_{i \geq t}\right)_t$ that "follow" Y_t when they are positive and $-X_t$ when they are negative.

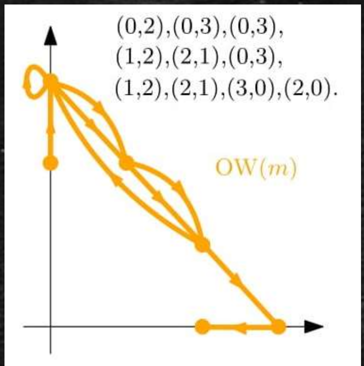
Def: Let $(W_t)_{t\in[n]} = (X_t, Y_t)_{t\in[n]}$ be a tandem walk of length $n\in\mathbb{N}$. The COALESCENT-WALK PROCESS associated to $(W_t)_{t\in[n]}$ is a collection of n one-dimensional walks $(Z^{(t)})_{t\in[n]} =: WC(W)$ defined for every $t\in[n]$ by:

- $Z^{(t)}_t = 0$
- $Z^{(t)}_k = \begin{cases} Z^{(t)}_{k-1} + (Y_k - Y_{k-1}) & \text{if } Z^{(t)}_{k-1} \geq 0 \\ Z^{(t)}_{k-1} - (X_k - X_{k-1}) & \text{if } Z^{(t)}_{k-1} < 0 \ \& \ Z^{(t)}_{k-1} - (X_k - X_{k-1}) < 0 \\ Y_k - Y_{k-1} & \text{if } Z^{(t)}_{k-1} < 0 \ \& \ Z^{(t)}_{k-1} - (X_k - X_{k-1}) \geq 0 \end{cases}$



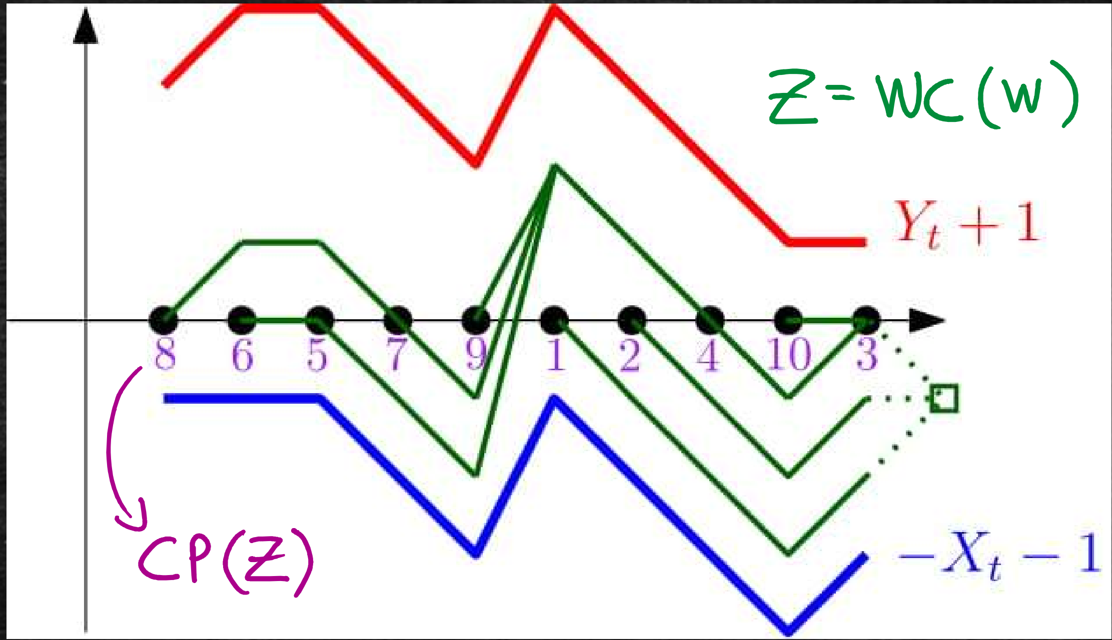
$\xmapsto{WC}$

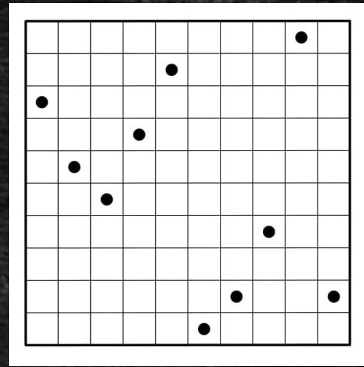




$W = (W_t)_t$

$\xmapsto{\text{wc}}$





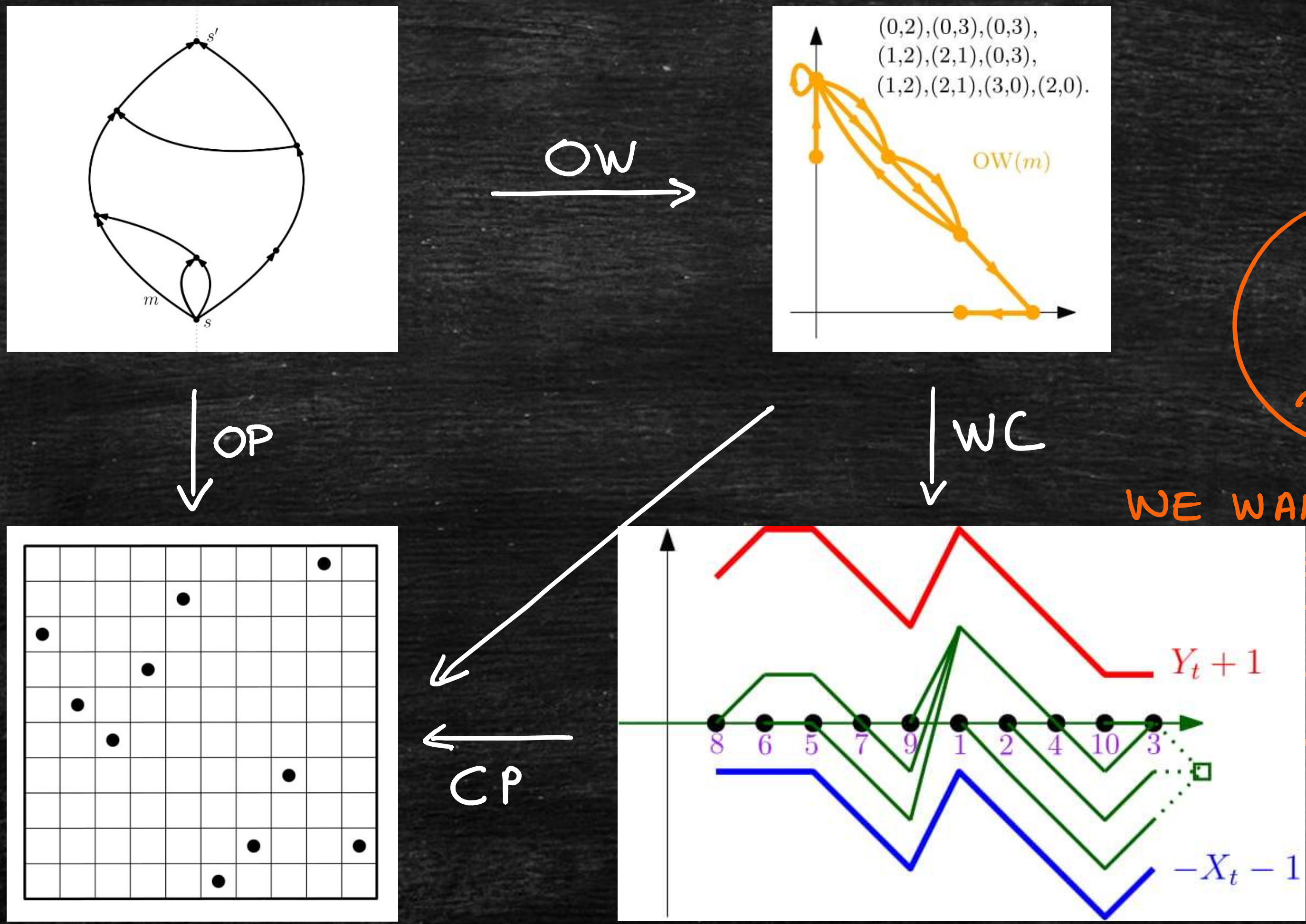
$\sigma = OP \circ OW^{-1}(W)$

Theorem: Let $W = (W_t)_{t\in[n]}$ be a tandem walk and $\sigma = OP \circ OW^{-1}(W)$ the corresponding Baxter permutation. Then $CP \circ WC(W) = \sigma$.

Proposition: Let σ be a Baxter permutation of size $n \in \mathbb{N}$ corresponding to a coalescent-walk process $(Z^{(t)})_{t\in[n]}$. Then for $i<j$

$$\sigma(i) < \sigma(j) \iff Z^{(i)}_j < 0$$

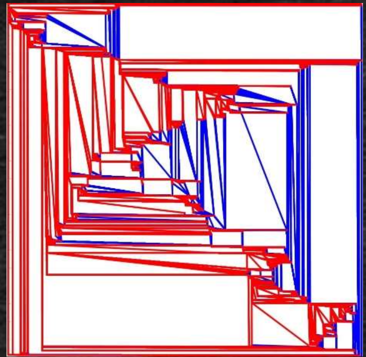
THEOREM : (B.-Maazoun 2020+)



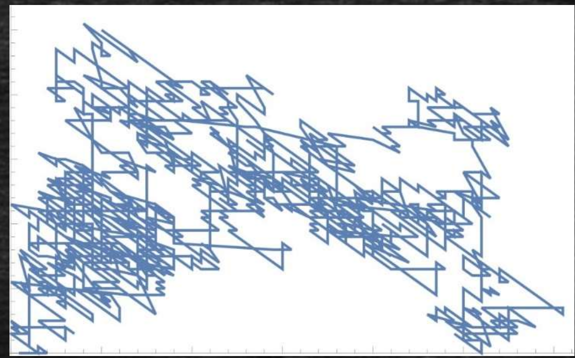
This diagram commutes.

WE WANT "TO READ"
...ERNS IN A
...MUTATION
... CORRESPONDING
...WALK

THEOREM : (B.-Maazoun 2020+)

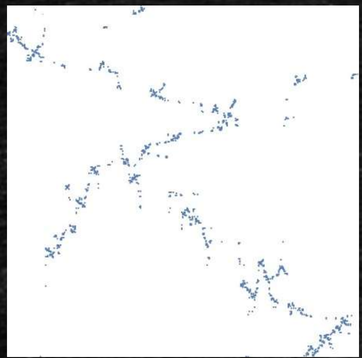


OW

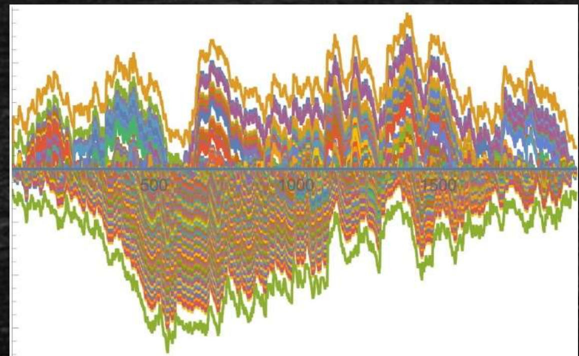


OP

WC



CP



This diagram commutes.

Scaling limits of coalescent-walk processes

The continuous coalescent-walk process

Consider a two dimensional process $W(t)=(X(t),Y(t))_{t\in I}$ and the following family of stochastic differential equations (SDEs) indexed by $u\in I$

$$(*)\quad \begin{cases} dZ^{(u)}(t) = \mathbb{1}_{\{Z^{(u)}(t)>0\}}\,dY(t) - \mathbb{1}_{\{Z^{(u)}(t)\leq 0\}}\,dX(t), & t\in(u,\infty)\cap I, \\ Z^{(u)}(t)=0, & t\in(-\infty,u]\cap I. \end{cases}$$

THEOREM (Prokaj 2013, Çağlar-Hajri-Karakuş 2018)

Let $(W(t))_{t\in I}$ be a two-dimensional Brownian motion with covariance matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ for some $\rho\in(-1,1)$. Fix $u\in I$. We have path-wise uniqueness and existence of a strong solution for the SDE $(*)$ driven by $W(t)$.

$$\begin{cases} dZ^{(u)}(t) = \mathbb{1}_{\{Z^{(u)}(t)>0\}}\, dY(t) - \mathbb{1}_{\{Z^{(u)}(t)\le 0\}}\, dX(t) & t>u \\ Z^{(u)}(t) = 0, & t\le u \end{cases}, \quad u\in\mathbb{R}$$

& $W(t) = (X(t), Y(t))$ a two-dim. BM with cov. $\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$

The previous theorem + Fubini-Tonelli imply that:

For almost every ω, $Z^{(u)}$ is a solution for almost every u.

Def: We call <u>CONTINUOUS COALESCENT-WALK PROCESS</u> (driven by W) the collection of solutions $\{Z^{(u)}\}_{u\in\mathbb{R}}$ where properly defined.

Let $\overline{W} = (\bar{X}, \bar{Y}) = (\bar{X}_k, \bar{Y}_k)_{k\geq 0}$ be a two dimensional random walk having value $(0,0)$ at time 0 and step distribution

$$\nu = \frac{1}{2}\delta_{(+1,-1)} + \sum_{i,j\geq 0} 2^{-i-j-3}\,\delta_{(-i,j)}$$

Proposition: The following is a uniform tandem walk of lenght n:

$$W_n := \left((\overline{W}_t)_{1\leq t\leq n} \,\middle|\, \overline{W}_0 = (0,0),\ \overline{W}_{n+1} = (0,0),\ (\overline{W}_t)_{0\leq t\leq n+1} \in (\mathbb{Z}^2_{\geq 0})^{n+1} \right)$$

Let $\mathcal{W}_n$ be the associated rescaled continuous process that interpolates the steps of W_n.

THEOREM (B.-Maazoun 2020+): Let $u \in (0,1)$. We have the following joint convergence in $\mathcal{C}([0,1],\mathbb{R})^3$

$$\left(\mathcal{W}_n, \mathcal{Z}_n^{(u)}\right) \xrightarrow[n\to\infty]{d} \left(\mathcal{W}_e, \mathcal{Z}_e^{(u)}\right)$$

continuous interpolation of the walk starting at time u in the discrete coal-walk process associated to W_n.

2-dim. Brownian excursion in the quadrant with cov. $\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$

associated continuous coal-walk process

THEOREM (B.–Maazoun 2020+): Let σ_n be a uniform Baxter permutation of size n. We have the following convergence in the space of permutons

$$\mu_{\sigma_n} \xrightarrow{d} \underbrace{\mu_{\mathcal{Z}_e}}_{\text{BAXTER PERMUTON}} := \phi\left(\{\mathcal{Z}_e^{u}\}_{u\in[0,1]}\right)$$

↳ EXPLICIT FUNCTION!

Proof based on:

- **PROPOSITION**: Let σ be a Baxter permutation of size $n \in \mathbb{N}$ corresponding to a coalescent-walk process $(Z^{(t)})_{t\in[n]}$. Then for $i<j$

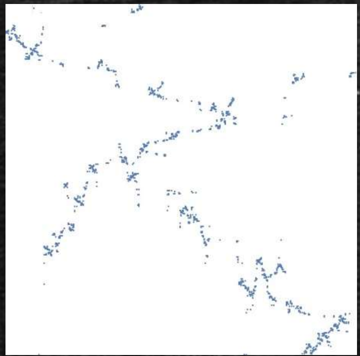
$$\sigma(i) < \sigma(j) \iff Z_j^{(i)} < 0$$

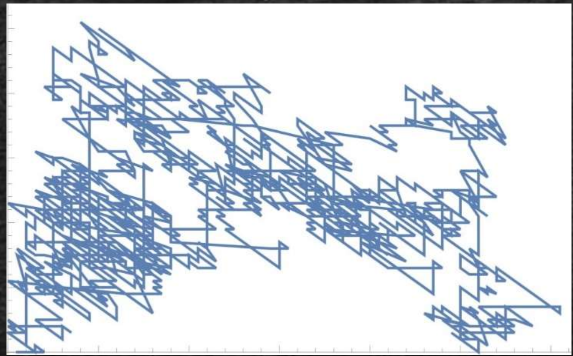
- **THEOREM**: Let $(u_i)_{i\geq 1}$ be a sequence of iid uniform random variables on $[0,1]$ independent of all other variables. Then

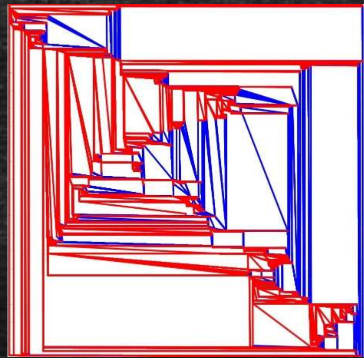
$$\left(W_n, \left(\mathcal{Z}_n^{(u_i)}\right)_{i\geq 1}\right) \xrightarrow{d} \left(W_e, \left(\mathcal{Z}_e^{(u_i)}\right)_{i\geq 1}\right)$$

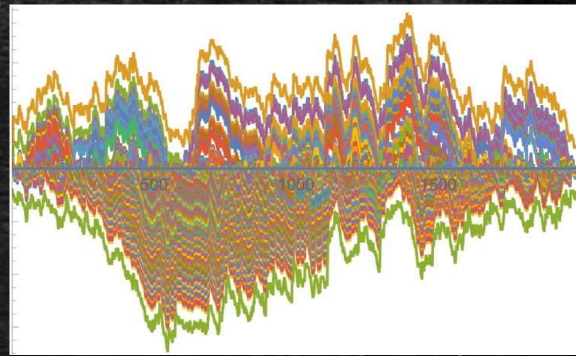
Final comments

- Our results imply convergence of finite-volume bip-orientations to a $\sqrt{4/3}$-LQG. What is the connection between our approach and the LQG approach?
- The converge of all the objects (walks, permutations, map, coal-walk proc.) holds jointly.
- We also proved joint Benjamini-Schramm local limits (both in the ANNEALED & QUENCHED sense) for all the objects involved in the commutative diagram.
- We believe that our techniques are rather general: we would like to consider other families of permutations (and maps?) encoded by two-dimensional walks.
- We would also like to investigate better the Baxter permuton. For instance, what is $\mathbb{E}[\mu_{z_e}] = ?$.









THANK YOU!

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