Two arithmetical sources and their associated tries

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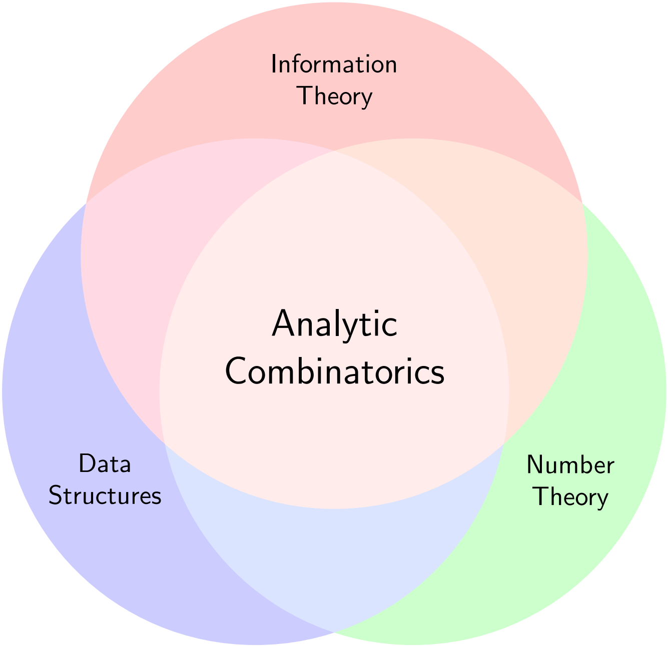
Introduction



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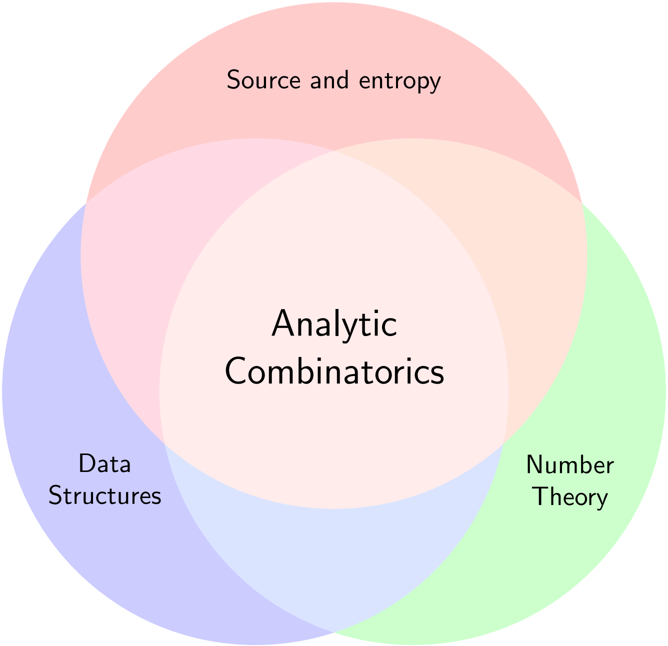

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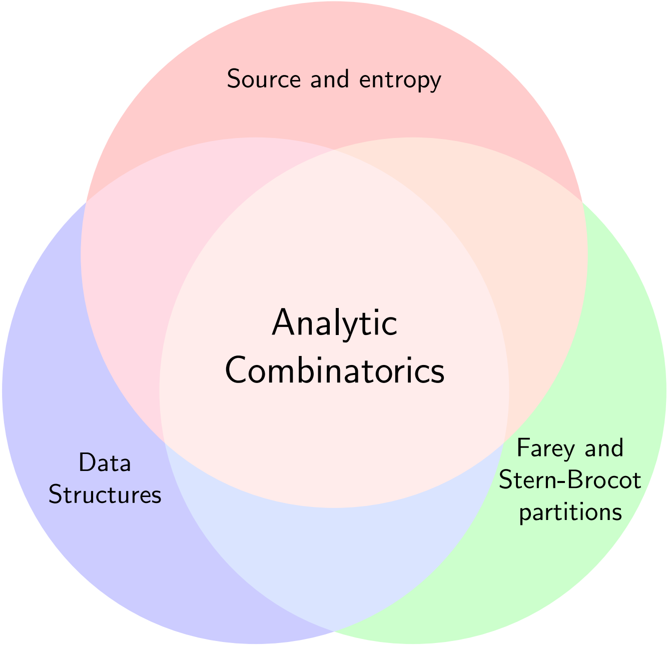

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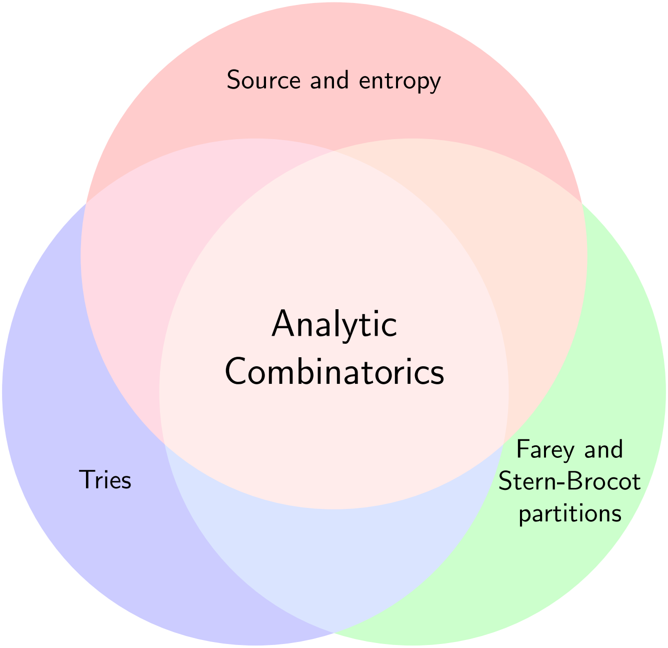

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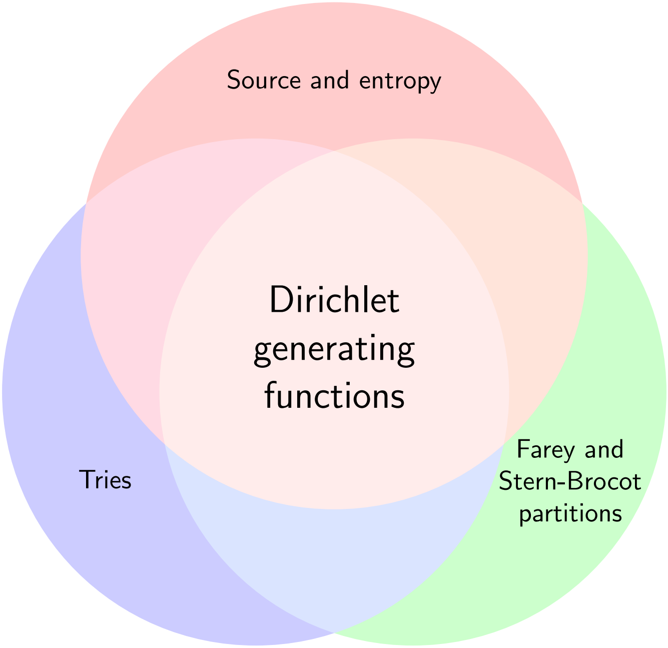

Introduction

Sources:

- We consider sources defined by two classical related partitions:
  - Sturm source (Farey partitions)
  - Stern-Brocot source (Stern-Brocot partitions).
- The Sturm source produces characteristic Sturmian words.
- These two partitions have zero Shannon entropy ($h = 0$).

Tries:

- We use tries to differentiate these two sources.
- For good sources, with entropy $h > 0$,

$$\text{the average trie depth is} \sim \frac{\log n}{h}.$$

Our main result is the following:

Theorem

The average trie depths for these sources are:

$$\text{Sturm source: } \sim \frac{24}{\pi^{3/2}}\sqrt{n}$$

$$\text{Stern-Brocot source: } \sim \frac{3}{\pi^2}\log^2 n$$

Overview

Sources, partitions, and expressions for DGFs

Tries, average depth, and the Rice method

Analytical study of the DGFs

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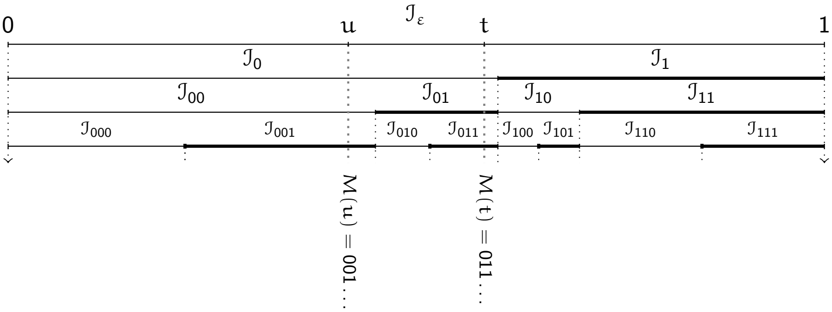
Sources defined by partitions



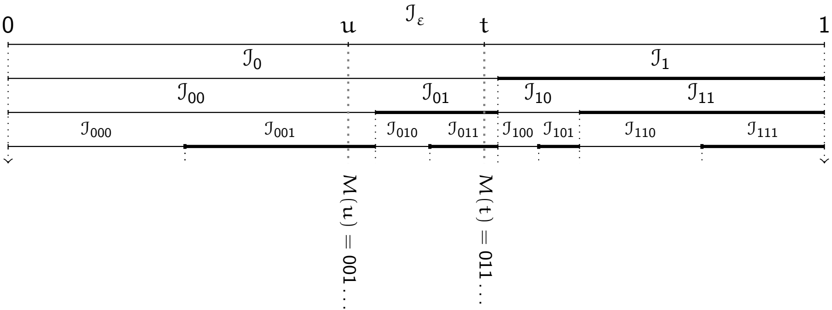
Let $\Sigma = \{0,1\}$ and let $\{\mathcal{P}_k\}_{k \geqslant 0}$ be a family of partitions of $[0,1]$

$$\mathcal{P}_k = \{\mathcal{I}_w : w \in \Sigma^k\} \quad \text{where} \quad \max_{w \in \Sigma^k} |\mathcal{I}_w| \to 0 \ (k \to \infty)$$

consisting of closed intervals such that:

- $\mathcal{I}_\epsilon = [0,1]$
- For $k \geqslant 0$ and $w \in \Sigma^k$, $\mathcal{I}_{w0}$ and $\mathcal{I}_{w1}$ partition $\mathcal{I}_w$ (left to right).

Dirichlet generating functions (DGFs)



Such a family of partition defines a probabilistic source where

$$p_w := |\mathcal{I}_w|$$

is the probability of emitting prefix w.

The Dirichlet generating function (DGF) of this source is defined as:

$$\Lambda(s) := \sum_{w \in \Sigma^\star} p_w^s = \sum_{k \geqslant 0} \sum_{w \in \Sigma^k} p_w^s.$$

Sturm and Stern-Brocot sources

Sturm source is defined by
Farey partitions $\mathcal{S}_k$

$\mathcal{S}_0 := [0/1, 1/1]$

$\mathcal{S}_k$ arises from $\mathcal{S}_{k-1}$ by

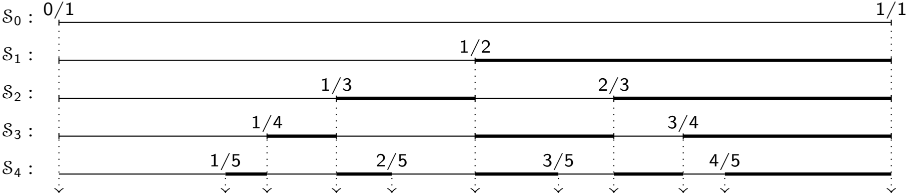
dividing each interval $[a/c, b/d]$ by its mediant $(a+b)/(c+d)$ only if $c+d \leqslant k+1$

Stern-Brocot source is defined by
Stern-Brocot partitions $\mathcal{B}_k$

$\mathcal{B}_0 := [0/1, 1/1]$

$\mathcal{B}_k$ arises from $\mathcal{B}_{k-1}$ by

dividing each interval $[a/c, b/d]$ by its mediant $(a+b)/(c+d)$ (always)



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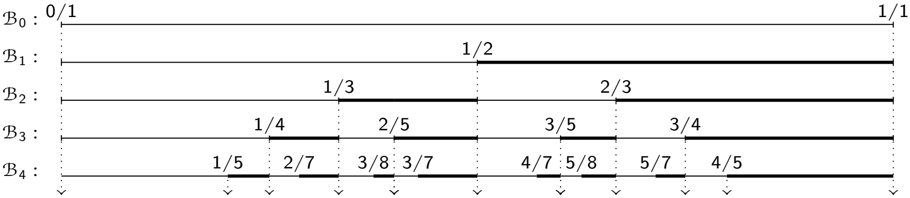


DGF for the Sturm source

- If $I_w = \left[\frac{a}{c}, \frac{b}{d}\right] \in \mathcal{S}_k$, then $|I_w| = \frac{1}{cd}$ and $\gcd(c, d) = 1$.
- The set of pairs (c, d) for which some $\left[\frac{a}{c}, \frac{b}{d}\right] \in \mathcal{S}_k$ is

  $\mathcal{C}_k = \{(c, d) : \max(c, d) \leqslant k + 1,\ c + d > k + 1,\ \text{and}\ \gcd(c, d) = 1\}$
- Since $|\{k \geqslant 0 : \ (c, d) \in \mathcal{C}_k\}| = \min(c, d)$ whenever $\gcd(c, d) = 1$,

$$\Lambda(s) = \sum_{k \geqslant 0} \sum_{(c,d) \in \mathcal{C}_k} \left(\frac{1}{cd}\right)^s = \sum_{\substack{c \geqslant 1, d \geqslant 1 \\ \gcd(c,d)=1}} \frac{\min(c, d)}{(cd)^s}.$$

Proposition

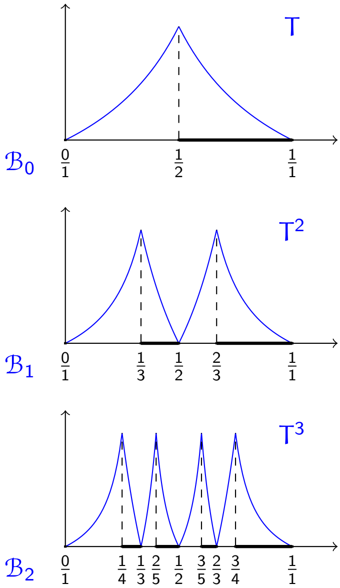
The DGF Λ for the Sturm source is:

$$\Lambda(s) = 1 + 2\frac{\zeta(s, s-1)}{\zeta(2s-1)} \quad \text{where } \zeta(\alpha, \beta) := \sum_{c \geqslant 1} \frac{1}{c^\beta} \sum_{d:\, d > c} \frac{1}{d^\alpha}.$$

Here:

- the denominator $\zeta(2s-1)$ arises from the $\gcd(c, d) = 1$ condition,
- the numerator $\zeta(s, s-1)$ arises from the $\min(c, d)$ term.

DGF for the Stern-Brocot source



The dynamical system associated with the

Farey map $T : [0, 1] \to [0, 1]$

$$x \mapsto \begin{cases} x/(1-x) & \text{if } x \in [0, 1/2] \\ (1-x)/x & \text{if } x \in [1/2, 1] \end{cases}$$

generates the Stern-Brocot partitions $\mathcal{B}_k$.

Let:
a be the inverse of the left branch of T,
b be the inverse of the right branch of T,
and $\mathcal{H} = \{a, b\}$.

$\mathcal{H}^k := \{a, b\}^k$ is the set of inverses branches of T^k and generates $\mathcal{B}_k$:

$$\mathcal{B}_k = \{h([0, 1]) : h \in \mathcal{H}^k\}.$$

DGF for the Stern-Brocot source

Since

$$\mathcal{B}_k = \{h([0,1]) : h \in \mathcal{H}^k\},$$

for each $k \geqslant 0$ we have that

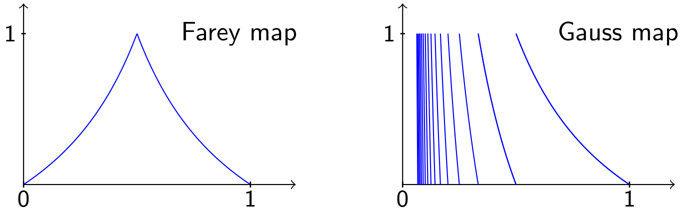
$$\Lambda_k(s) = \sum_{w \in \Sigma^k} p_w^s = \sum_{h \in \mathcal{H}^k} |h(1) - h(0)|^s = \mathbf{H}_s^k[\mathbf{1}](0,1)$$

where $\mathbf{H}_s$ is a variant of the transfer operator for the Farey map.

Hence, if $\mathcal{H}^\star = \bigcup_{k \geqslant 0} \mathcal{H}^k$, then

$$\Lambda(s) = \sum_{w \in \Sigma^\star} p_w^s = \sum_{h \in \mathcal{H}^\star} |h(1) - h(0)|^s = (I - \mathbf{H}_s)^{-1}[\mathbf{1}](0,1).$$

DGF for the Stern-Brocot source



The induced map of the Farey map (associated with Stern-Brocot partitions) is the Gauss map (associated with continued fractions).

In fact, $(a^{m-1} \circ b)(x) = \dfrac{1}{m+x}$ (an inverse branch of the Gauss map).

Note that $\mathcal{H}^\star = \{a, b\}^\star = (a^\star \circ b)^\star \circ a^\star$.

Proposition

For the DGF of the Stern-Brocot source,

$$\Lambda(s) = (I - \mathbf{H}_s)^{-1}[\mathbf{1}](0,1) = (I - \mathbf{A}_s)^{-1}(I - \mathbf{G}_s)^{-1}[\mathbf{1}](0,1)$$

where

- $\mathbf{A}_s$ is the part of $\mathbf{H}_s$ corresponding to inverse branch a, and
- $\mathbf{G}_s$ is a variant of the transfer operator of the Gauss map.

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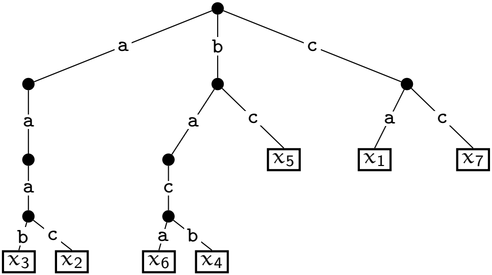
Tries, average depth, and the Rice method

Analytical study of the DGFs

Tries from words emitted by a source

$x_1 = \texttt{cacccaaac}\ldots$
$x_2 = \texttt{aaacacbac}\ldots$
$x_3 = \texttt{aaabcbbca}\ldots$
$x_4 = \texttt{bacbaabcc}\ldots$
$x_5 = \texttt{bcabbbbbc}\ldots$
$x_6 = \texttt{bacaabbba}\ldots$
$x_7 = \texttt{ccbacbcbb}\ldots$



- Let $\mathcal{T}$ be a trie on n words independently drawn from a source.
- We perform a probabilistic analysis of the shape of $\mathcal{T}$.
- We focus on trie depth D_n, the depth of a random branch.
- If $D_n^{(i)}$ is the depth of the branch corresponding to the i-th word,

$$\Pr[D_n \geqslant k+1] = \frac{1}{n}\sum_{i=1}^{n} \Pr[D_n^{(i)} \geqslant k+1] = \sum_{w\in\Sigma^k} p_w[1-(1-p_w)^{n-1}].$$

Average trie depth and DGFs

Proposition

(i) If $\Lambda(s)$ is well-defined for $s \geqslant 2$, then

$$\mathbb{E}[D_n] = \frac{1}{n} \sum_{\ell=2}^{n} (-1)^{\ell} \binom{n}{\ell} \ell \Lambda(\ell).$$

(ii) If $\exists a, A > 0$ such that $\forall k \geqslant 1$, $\exists w \in \Sigma^k$ such that $p_w \geqslant A k^{-a}$,

$$\mathbb{E}[D_n^r] = \infty, \quad \forall r \geqslant 2.$$

Remarks

- Both statements (i) and (ii) apply to our two sources.
- Statement (ii) applies to our sources with $w = 000 \dots 0$ (k times).
- In order to find estimates for $\mathbb{E}[D_n]$, we will study

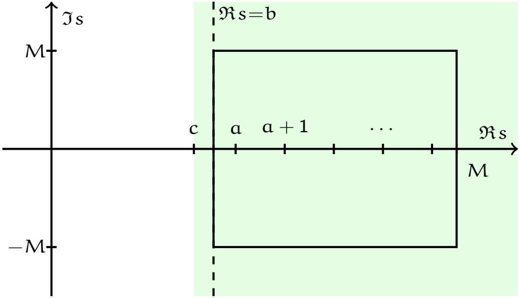
$$f(n) := n\mathbb{E}[D_n] = \sum_{\ell=2}^{n} (-1)^{\ell} \binom{n}{\ell} \underbrace{\ell \Lambda(\ell)}_{=:\, p(\ell)}.$$

Rice method: Step 1 – Integral representation

It applies to binomial sums

$$f(n) = \sum_{\ell=a}^{\infty} (-1)^{\ell} \binom{n}{\ell} p(\ell)$$

for some $\ell \mapsto p(\ell)$ for $\ell \geqslant a$ (where $a \geqslant 0$).



Rice method: Step 1

If $\psi(s)$ is a lifting of $p(\ell)$ and $\exists c \in \,]a-1, a[$ such that:

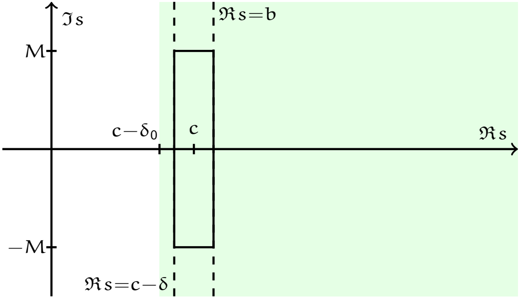
$\psi(s)$ is analytic and of polynomial growth in the half-plane $\Re s > c$,

then $\forall b \in \,]c, a[$ and sufficiently large n:

$$f(n) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} L_n(s) \cdot \psi(s)\, ds \quad \text{where } L_n(s) = \frac{\Gamma(n+1)\Gamma(-s)}{\Gamma(n+1-s)}.$$

Rice method: Step 2 – shifting to the left

The integral representation is shifted to the left, using the *tameness* of ψ to the left (meromorphic and of polynomial growth).



Rice method: Step 2

If for some $\delta_0 > 0$, the lifting ψ satisfies that:

- it is meromorphic on $\Re s > c - \delta_0$,
- its only pole on $\Re s > c - \delta_0$ is at $s = c$, and
- $\forall \delta < \delta_0$, $\psi(s)$ is of polynomial growth on $\Re s \geqslant c - \delta$ as $|\Im s| \to \infty$,

then $\forall \delta < \delta_0$

$$f(n) = \text{Res}[L_n(s) \cdot \psi(s); s = c] + O(n^{c-\delta}) \quad (n \to \infty).$$

Rice method: Step 3 – estimating the residue

Rice method: Step 3

The residue

$$\text{Res}[L_n(s) \cdot \psi(s); s = c] = n^c \cdot P(\log n)\,[1 + O(1/n)]$$

where P is a polynomial determined by the singular expression

$$\psi(s) \asymp a_d \frac{1}{(s-c)^d} + \cdots + a_1 \frac{1}{s-c} + a_0,$$

according to the following two cases:

1. If c is not an integer, then

   P has degree $d-1$ and leading coefficient $\dfrac{\Gamma(-c)}{\Gamma(d)}|a_d|$.

2. If c is an integer, then

   P has degre d and leading coefficient $\dfrac{1}{\Gamma(d+1)}|a_d|$.

   If $c = 1$, the factor $[1 + O(1/n)]$ equals 1.

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Singularity analysis for the Sturm source

Recall that for Sturm source: $\Lambda(s) = 1 + 2\dfrac{\zeta(s, s-1)}{\zeta(2s-1)}$.

Because of well-known properties of the zeta function:

Proposition

For any $a_0 > 0$, the DGF $\Lambda(s)$ of the Sturm source satisfies:

- it is meromorphic on $\Re s > 1 + a_0$,
- its only pole is a simple pole at $s = 3/2$,
- $\forall a > a_0$, it is of polynomial growth on $\Re s \geqslant 1 + a$.

Moreover, as $s \to 3/2$,

$$s\Lambda(s) \sim \frac{36}{\pi^2}\left(\frac{1}{2s-3}\right) \quad \text{and} \quad \Gamma(-s)\cdot s\Lambda(s) \sim \frac{36}{\pi^2}\Gamma\left(\frac{-3}{2}\right)\left(\frac{1}{2s-3}\right).$$

Since $\Gamma(-3/2) = (4/3)\sqrt{\pi}$, in Step 3 of the Rice method we have:

The DGF $\Lambda(s)$ of the Sturm source satisfies:

$$\text{Res}\left[L_n(s)\cdot s\Lambda(s); s = \frac{3}{2}\right] = \frac{24}{\pi^{3/2}} n^{3/2}\left(1 + O\left(\frac{1}{n}\right)\right).$$

Singularity analysis for the Stern-Brocot source

Recall that for Stern-Brocot: $\Lambda(s) = (\mathrm{I} - \mathbf{A}_s)^{-1}(\mathrm{I} - \mathbf{G}_s)^{-1}[\mathbf{1}](0,1)$.

Because of deep properties of the quasi-inverse of $\mathbf{G}_s$:

Proposition

The DGF $\Lambda(s)$ of the Stern-Brocot source satisfies:

- it is meromorphic on $\Re s > 1 - \delta_0$ (for some $\delta_0 > 0$),
- its only pole is at $s = 1$ and is of order 2,
- $\forall \delta < \delta_0$, it is of polynomial growth on $\Re s \geqslant 1 - \delta$.

Moreover, as $s \to 1$,

$$s\Lambda(s) \sim \frac{1}{\zeta(2)} \left(\frac{1}{s-1} \right)^2 \quad \text{and} \quad \Gamma(-s) \cdot s\Lambda(s) \sim \frac{6}{\pi^2} \left(\frac{1}{s-1} \right)^3 .$$

Hence, for Step 3 of the Rice method we have:

The DGF $\Lambda(s)$ of the Stern-Brocot source satisfies:

$$\mathrm{Res}[L_n(s) \cdot s\Lambda(s); s = 1] = n \left(\frac{3}{\pi^2} \log^2 n + b_1 \log n + b_0 \right)$$

for some constants b_1 and b_0.

Main results

We are ready to apply the Rice method to

$$f(n) := n\mathbb{E}[D_n] = \sum_{\ell=2}^{n} (-1)^{\ell} \binom{n}{\ell} \ell \Lambda(\ell)$$

for our sources. We show that the two sources behave very differently.

Theorem

For each source, consider a trie built on n words independently drawn from the source. Then, the mean values of the trie depths are:

- For the Sturm source:

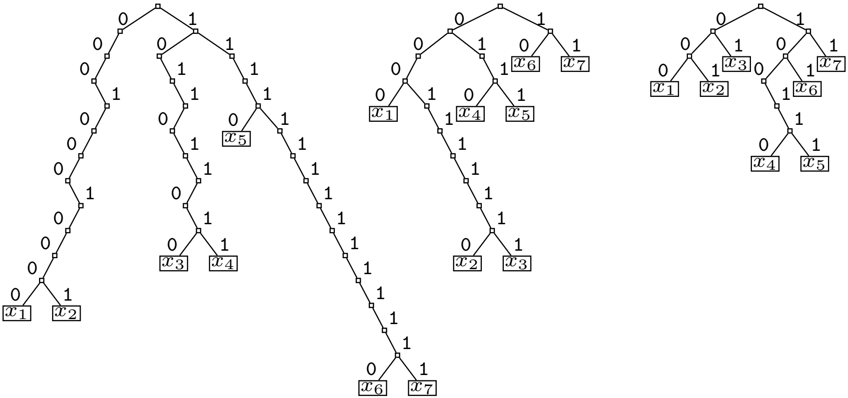
$$\mathbb{E}[D_n] = \frac{24}{\pi^{3/2}} n^{1/2} + O(n^{a}) \quad \text{for any } a > 0.$$

- For the Stern-Brocot source:

$$\mathbb{E}[D_n] = \frac{3}{\pi^2} \log^2 n + b_1 \log n + b_0 + O(n^{-\delta}) \quad \text{for some } \delta > 0$$

and some constants b_1 and b_0.

Concluding remarks and future work



- These are instances of tries built on seven words emitted from the Sturm source (on the left), the Stern-Brocot source (in the middle), and the Bernoulli source with $p = 1/2$ (on the right)
- As the value $n = 7$ is small, and the moments $\mathbb{E}[D_n^2]$ for the Sturm source and the Stern-Brocot are infinite, there does not really exist a "typical trie" for these sources.

Concluding remarks and future work

This work appears as (one of) the first study on sources of zero Shannon entropy via Analytic Combinatorics tools.

1. Rényi entropy:
   - The Rényi entropy for our two sources are very similar
   - Known via Number Theory arguments
   - TO DO: Derive these results using Analytic Combinatorics.
2. The VLMC (Variable Length Markov Chain):
   - Are the simplest source where dependency from the past is unbounded
   - The depth of the associated suffix tries has been studied before on a special class of VLMC sources
   - TO DO: Analyze the trie depth in an intermittent subclass
3. Trie built on the Farey dynamical source
   - Its invariant measure is $1/t$ which has infinite mass
   - Not clearly related fundamental probabilities with Stern-Brocot source
   - Strongly different from absolutely continuous invariant measure
   - TO DO: We wish to analyze its trie depth

Thank you very much
for your attention!